Analysis of some nonlinear PDEs from multi-scale geophysical applications

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Objects

Multi-time-scale PDE systems from geophysical applications, solutions fast oscillating in t. Nonlinearity generates resonances.

$$\partial_t U = \varepsilon^{-1}\mathcal{L}[U] + F$$

- Rotating shallow water eq in $\Omega \subset \mathbf{R}^2$; incompressible fluid dynamics on fast rotating $\mathbf{S}^2$.

Objectives

Estimate $\mathcal{P}$(*multiscale soln*) - (*limiting soln*) in terms of $\varepsilon \ll 1$.

- Multiscale solution is decomposed by $\mathcal{P}$, into slow component in $\ker\{\mathcal{L}\}$ and fast component.
- Initial data are ill-prepared (unfiltered), i.e. $O(1)$ fast component.

Methodology

- Time-averaging to suppress fast component.
- Estimate resonances using vorticity equation

vorticity dynamics $\equiv$ slow dynamics

Relation to numerical analysis

- Numerical methods can be validated by showing that they give the theoretically predicted rate of convergence to limiting solutions.

$\|\mathcal{P}u^{\varepsilon}_{\Delta x} - u^{0}_{\Delta x}\| \sim O(\varepsilon + (\Delta x)^n)$. Then,

$$\|\mathcal{P}(u^{\varepsilon}_{\Delta x} - u^{\varepsilon})\| \leq \|\mathcal{P}u^{\varepsilon}_{\Delta x} - u^{0}_{\Delta x}\| + \|u^{0}_{\Delta x} - u^{0}\| + \underbrace{\|u^{0} - \mathcal{P}u^{\varepsilon}\|}_{\textbf{this talk}}$$

$u^{\varepsilon}_{\Delta x}$: multiscale numerical soln. $\quad$ u^{ε}: multiscale exact soln.
$u^{0}_{\Delta x}$: limiting numerical soln. $\quad$ u^{0}: limiting exact soln.

(Direct error estimates: $\|\mathcal{P}u^{\varepsilon}_{\Delta x} - \mathcal{P}u^{\varepsilon}\| \sim (\Delta x)^m/\varepsilon$, not good if $\varepsilon \ll 1$.)

Singular perturbations

Multi-scale-ness is measured by parameters. As some parameters $\to 0/\infty$, reduced models *may* still capture relevant features.

Rotating shallow water (RSW) eq. as singular perturbation problem

$$\begin{cases} \partial_t H + \nabla\cdot(Hu) = 0, \\ \partial_t u + u\cdot\nabla u + \dfrac{\nabla H}{Fr^2} + \dfrac{u^\perp}{Ro} = 0, \end{cases} \quad \text{subject to} \quad u\cdot \mathsf{n}\big|_{\partial\Omega} = 0$$

H is total height. Parameters: 2D Froude and Rossby numbers.

With $Fr = Ro = \varepsilon$, $H = 1 + \varepsilon h$, system fits $\partial_t U = \varepsilon^{-1}\mathcal{L}[U] + F$.

But, Fr and Ro can scale differently.

- $Fr^2 = Ro \ll 1$: semi-geostrophic (SG) [Phillips 63'; Brenier, Cullen, Feldman, Roulstone, Oliver, etc etc 00s]
- $Fr^2 \ll Ro \ll 1$: quasi-geostrophic (QG) [Cheng 08', 14'].
- $Fr^2 \gg Ro$: near-inertia oscillation [Cheng & Tadmor 08']

Well-prepared vs ill-prepared data

From here on, use U for ε-dependent solution, $\widetilde{U}$ for limiting soln.

$$\varepsilon\big(\partial_t U + \mathit{nonlinear}\big) + \mathcal{L}[U] = 0.$$

Always assume U is smooth for $t \lesssim 1$ with spatial norms $\sim O(1)$.

	Well-prepared	**Ill-prepared**
Initial conditions	Impose $\partial_t U_0 \sim O(1)$ iff $\mathcal{L}[U_0] \sim O(\varepsilon)$	Allow $\partial_t U_0 \sim O(\varepsilon^{-1})$ iff $\mathcal{L}[U_0] \sim O(1)$
For local times t	Above persists	Above persists if w/o dispersion/dissipation
$\widetilde{U} \overset{?}{=} \lim_{\varepsilon \to 0} U$	Possible	Trouble!

Slow vs fast subspaces

System of slow/fast scales

$$\partial_t U = \varepsilon^{-1}\mathcal{L}[U] + F$$

$\mathcal{L}$: linear operator with purely imaginary spectrum (incl. 0).

F: nonlinear and source terms, with spatial norms $\sim O(1)$.

Lorentz

$$\mathcal{P}: \ \textit{some} \text{ projection onto } \ker\{\mathcal{L}\}, \quad U^s := \mathcal{P}U, \ \ U^f := U - U^s.$$

The solution space is decomposed into a slow subspace $\ker\{\mathcal{L}\} \ni U^s$, and a fast subspace $\ni U^f$.

- $\partial_t U^s \sim O(1)$, so it is possible to prove $\exists \lim_{\varepsilon\to 0} U^s$ by Arzelà-Ascoli
- $\ker\{\mathcal{L}\}$ is in general not invariant w.r.t. original, nonlinear dynamics.

RSW eq. subject to $u\cdot \mathsf{n}\big|_{\partial\Omega} = 0$

$$\text{For } U = \begin{pmatrix} h \\ u \end{pmatrix}, \mathcal{L}[U] = \begin{pmatrix} \nabla\cdot u \\ \nabla h + u^\perp \end{pmatrix}. \text{ So, } \mathcal{L}[U] = 0 \textit{ iff } u = \nabla^\perp h$$

$$U^s = \mathcal{P}U := \begin{pmatrix} \psi \\ \nabla^\perp \psi \end{pmatrix} \text{where } \psi := (\Delta - 1)^{-1}_{\text{QG}} (\nabla \times u - h)$$

- $(\Delta - 1)^{-1}_{\text{QG}} \psi$ enforces $\psi\big|_{\partial\Omega}$ =costant.
- $(\nabla \times u - h)$ is linear potential vorticity; $\mathcal{P}$ is linear *PV inversion*, analogue of Biot-Savart law.

For the "limiting" solution $\widetilde{U} = \lim_{\varepsilon\to 0} U^s$

WKB expansion in ε with *well-prepared* data $\Longrightarrow$

- Leading order. No dynamics; $\widetilde{U} \in \ker\{\mathcal{L}\}$, i.e. $\widetilde{u} = \nabla^\perp \widetilde{h}$
- Next order. Dynamics; 2D quasi-geostrophic (QG) eq.

2D RSW eq. with $u\cdot\mathsf{n}\big|_{\partial\Omega} = 0$, (fast+slow scales, $U = U^s + U^f$)

$$\begin{cases} \partial_t h + \nabla\cdot(hu) + \varepsilon^{-1}\nabla\cdot u = 0, \\ \partial_t u + u\cdot\nabla u + \varepsilon^{-1}u^\perp + \varepsilon^{-1}\nabla h = 0, \qquad (h_0, u_0) \text{ given.} \end{cases}$$

2D QG eq., (slow scale, $\widetilde{U} = \mathcal{P}\widetilde{U} = \widetilde{U}^s$)

$$\begin{cases} \partial_t \widetilde{\omega} + \widetilde{u}\cdot\nabla\widetilde{\omega} = 0 \qquad \widetilde{\omega}_0 = \nabla\times u_0 - h_0 \qquad (\widetilde{\omega} \text{ is PV}) \\ \widetilde{h} = (\Delta - 1)^{-1}_{\mathrm{QG}}\widetilde{\omega} \qquad \widetilde{u} = \nabla^\perp\widetilde{h} \end{cases}$$

[BC, SIMA 2013]

For **ill-prepared** $U_0 \in H^m(\Omega)$ and $t \lesssim 1$,

$$\|U^s - \widetilde{U}\|_{H^{m-3}} \lesssim \varepsilon, \quad \left\| \int_0^t (U - \widetilde{U}) \right\|_{H^{m-3}} \lesssim \varepsilon.$$

Time-averaging and its connection to $\mathcal{P}$

- Time-averaging is used in observations and simulations.

(time-average $\leftrightarrow$ time-integral), $\quad \partial_t U = \varepsilon^{-1}\mathcal{L}[U] + F \implies$

$$\varepsilon\Big(U(T) - U(0) - \int_0^T F\Big) = \int_0^T \mathcal{L}[U]\,dt = \mathcal{L}\Big[\int_0^T U^f\Big].$$

Bounded Inverse Theorem

Bounded, surjective, linear $\mathcal{L} : \mathrm{X}_1 \mapsto \mathrm{X}_2$ (X_i being Banach), and $\|(1-\mathcal{P})U\|_{\mathrm{X}_1} \approx \|U\|_{\mathrm{X}_1/\ker\{\mathcal{L}\}}$ where $\mathcal{P} =$ proj onto $\ker\{\mathcal{L}\}$. Then,

$$\Big\|(1-\mathcal{P})U\Big\|_{\mathrm{X}_1} \lesssim \big\|\mathcal{L}[U]\big\|_{\mathrm{X}_2} \quad i.e. \quad \|U^f\|_{\mathrm{X}_1} \lesssim \|\mathcal{L}[U^f]\|_{\mathrm{X}_2}$$

Well-known examples: $\mathcal{L} : \mathbf{R}^n \mapsto \mathbf{R}^m$ and $\mathcal{L} = \operatorname{div}$

Time-averaging suppresses fast, $\quad$ " $\int_0^T \approx \mathcal{P}$ "

$$\Big\| \int_0^T U^f \Big\|_{\mathrm{X}_1} \lesssim \Big\|\mathcal{L}\Big[\int_0^T U^f\Big]\Big\|_{\mathrm{X}_2} \lesssim \varepsilon.$$

Idea useful in later slides, but also interesting on its own.

$u, \widetilde{u}$: velocities. $s, \widetilde{s}$: (active) scalars so that $u = \mathsf{q}[s]$, $\widetilde{u} = \mathsf{q}[\widetilde{s}]$ and $\|u - \widetilde{u}\|_{Y_1} \lesssim \|s - \widetilde{s}\|_{Y_2}$,

$$\begin{aligned} \partial_t s + u \cdot \nabla s &= r, \\ \partial_t \widetilde{s} + \widetilde{u} \cdot \nabla \widetilde{s} &= \widetilde{r}, \end{aligned} \qquad s_0(\cdot) = \widetilde{s}_0(\cdot).$$

Only assume, $\displaystyle\int_0^t (r - \widetilde{r}) \sim O(\varepsilon)$,

but allow, $(r - \widetilde{r}) \sim O(1)$ for $t \in [0, O(1)]$.

Write an eq for $e := (s - \widetilde{s}) + \int_0^t (r - \widetilde{r})\, d\tau$ and apply energy method

$$\|e(t)\| \lesssim \varepsilon, \quad \text{thus} \quad \|s(t) - \widetilde{s}(t)\| \lesssim \varepsilon.$$

Literature

Results in relation to 2D RSW eq.

$$\begin{cases} \partial_t h + \nabla\cdot(hu) + \varepsilon^{-1}\nabla\cdot u = 0, \\ \partial_t u + u\cdot\nabla u + \varepsilon^{-1}u^\perp + \varepsilon^{-1}\nabla h = 0, \qquad u\cdot\mathsf{n}\big|_{\partial\Omega} = 0. \end{cases}$$

Well-posedness of IBV problem of hyperbolic PDEs

- Linear.
  - L^2 theory: Friedrichs (58'), Lax & Phillips (60').
  - H^m theory: Rauch & Massey (74', nonsingular boundary matrix), Rauch (85', singular boundary matrix, tangential regularity).
- Quasilinear.
  - Schochet (86'a, compressible Euler; 86'b, general case with vorticity equation) — full regularity.

More literature

Singular limit problem of multiscale Hyperbolic PDEs
(Most results: compressible Euler $\to$ incompressible Euler at zero-Mach-number limit)

- $\Omega = R^N$ or T^N
  - well-prepared : Ebin, Kreiss, Beirao de Veiga, Klainerman-Majda, Tadmor (80s) ...
  - ill-prepared : Ukai, Isozaki(80s), Schochet, Masmoudi (90s) ...
  - geophysical models: Embid-Majda, Beale-Bourgeois, Babin-Mahalov-Nicolaenko (90s), Chemin-Desjardins-Gallagher-Grenier (book 06') ...
- Solid Wall on $\partial\Omega$
  - Schochet (80s), Secchi, D. Jones (00s), ...
- Weak or strong convergence. No convergence rate.

- Link 1: extract slow dynamics.
  - Apply $\mathcal{P}$ on $\partial_t U + B(U,U) = -\varepsilon^{-1}\mathcal{L}[U]$ and use $\mathcal{P}\mathcal{L} \overset{?}{=} 0$

$$(\star) \quad \partial_t U^s = -\mathcal{P}B(U^s, U^s) - \mathcal{P}B(U^f, U^f) - \mathcal{P}B(U^s, U^f) - \mathcal{P}B(U^f, U^s)$$

  - Slow-slow interaction, i.e.

$$\partial_t \widetilde{U} = -\mathcal{P}B(\widetilde{U}, \widetilde{U}) \quad \text{with } \widetilde{U}_0 \in \ker\{\mathcal{L}\}$$

$\overset{?}{\Longleftrightarrow}$ The QG eq.

$$\partial_t \widetilde{\omega} = -\nabla\cdot(\widetilde{u}\,\widetilde{\omega}), \qquad \widetilde{u} = \nabla^\perp (1-\Delta)^{-1}_{\mathrm{QG}} \widetilde{\omega}$$

Note: need to show, with $u\cdot\mathsf{n}\big|_{\partial\Omega} = 0$, the ?'s are valid.
The nonlocal bound. cond. of $(1-\Delta)^{-1}_{\mathrm{QG}}$ is crucial.

- Link 2: **time-averaging**.
  - For the Bounded Inverse Theorem to work, need to verify $\mathcal{L}$ has closed graph (i.e. solvability). Or, directly use **elliptic estimates** from [S. Agmon, A. Douglis and L. Nirenberg, 1959 &1964]

    $$\text{With } u\cdot\mathsf{n}\big|_{\partial\Omega} = 0, \quad \left\|U^f\right\|_{H^m(\Omega)} \lesssim \left\|\mathcal{L}[U^f]\right\|_{H^{m-1}(\Omega)}$$

    More or less, $\|u\|_{H^m} \lesssim (\|\nabla \times u\|_{H^{m-1}} + \|\nabla\cdot u\|_{H^{m-1}} + \|u\|_{L^2})$.
  - So, $\int_0^T U^f \sim O(\varepsilon)$. Also, $(\star) \implies \partial_t U^s \sim O(1)$.

    Integrating by parts $\implies \int_0^T \mathcal{P}B(U^f, U^s) \sim O(\varepsilon)$.

Fast-fast interaction $\mathcal{P}B(U^f, U^f) \equiv 0$ by Calculus.

Physical reason?

- Link 3: Potential vorticity (PV)

$$\frac{\varepsilon^{-1} + \nabla \times u}{1 + \varepsilon h} = \varepsilon^{-1} + (\nabla \times u - h) + O(\varepsilon)$$

Define linear PV as $\omega = \mathcal{K}[U] := (\nabla \times u - h)$ so that $\mathcal{K}\mathcal{L} \equiv 0$ and

$$\partial_t \omega = -\nabla \cdot (u\, \omega) \quad \ldots \quad \text{(eq of linear PV)}$$

$\mathcal{L} = -\mathcal{L}^*, \quad \mathcal{P} = \mathcal{P}^* = \mathcal{K}^*(\mathcal{K}\mathcal{K}^*)^{-1}_{\mathrm{QG}}\mathcal{K}, \quad$ (least-square in frqncy space)

$\Longrightarrow\ U^s = \mathcal{P}[U]$ and $\omega = \mathcal{K}[U]$ are **1-to-1**.

$$\begin{aligned} \partial_t U^s = -\mathcal{P}B(U^s, U^s) - \cancel{\mathcal{P}B(U^f, U^f)} \quad &\longleftrightarrow \quad \partial_t \omega = -\nabla \cdot (u^s \omega) \\ -\mathcal{P}B(U^s, U^f) - \mathcal{P}B(U^f, U^s) \quad & \qquad\qquad -\nabla \cdot (u^f \omega) \end{aligned}$$

- Final step: apply energy estimates to compare slow solution of

$$\partial_t \omega = -\nabla\cdot(u^s \omega) - \nabla\cdot(u^f \omega), \qquad u^s = \nabla^\perp (1-\Delta)_{\mathrm{QG}}^{-1} \omega$$

and the QG solution of

$$\partial_t \widetilde{\omega} = -\nabla\cdot(\widetilde{u}\,\widetilde{\omega}), \qquad \widetilde{u} = \nabla^\perp (1-\Delta)_{\mathrm{QG}}^{-1} \widetilde{\omega}$$

knowing $\int_0^T \nabla\cdot(u^f \omega) \sim O(\varepsilon)$

(Back to the **active scalar transport** eq!)

Example 2: 2D fluid dynamics on a fast rotating sphere

Incompressible Euler equations on $\mathbf{S}^2$

$$\partial_t u + \nabla_u u + \nabla P = \frac{z}{\varepsilon} u^{\perp}, \qquad \operatorname{div} u = 0$$

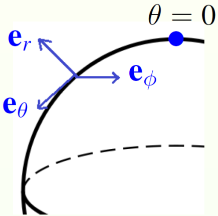
(admits Rossby-Haurwitz wave solutions)

θ: colatitude. ϕ: longitude

$z = \cos\theta$: north-south variation (a rugby will do)

$\ker\{\mathcal{L}\}$ are zonal flows

$\mathcal{P}u =$ zonal mean of u



Theorem [Cheng & Mahalov, E. J. Mech.-B/Fluids (2013)]

$$\left\| \int_0^T u(t,\cdot)\,dt - \int_0^T \mathcal{P}u(t,\cdot)\,dt \right\|_{H^{k-3}} \lesssim \varepsilon \quad \text{for} \quad T \lesssim 1$$

- Proofs are coordinate free.
- Also, estimates/constants must be independent of ε.

Zonal flows in observations

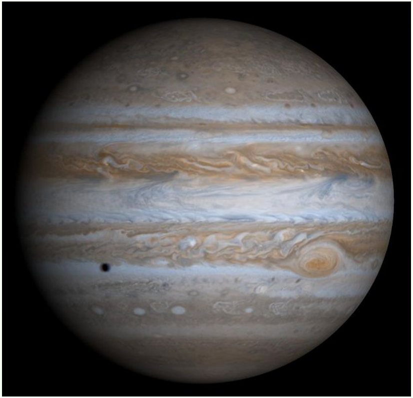
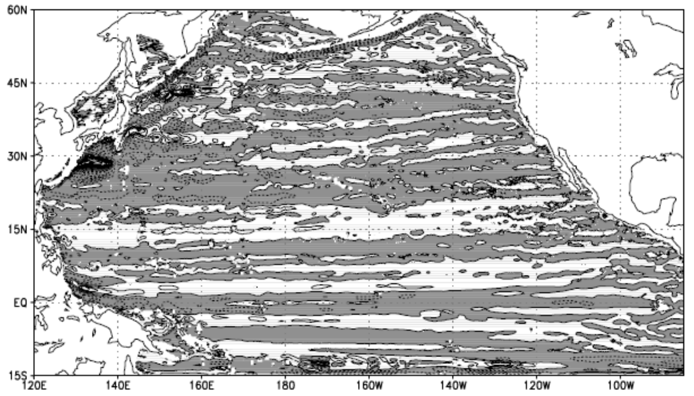


Figure : True-color view of Jupiter, composed of 4 images taken by NASA's Cassini spacecraft on December 7, 2000. Credit: NASA/JPL/U. of Arizona.

Zonal flows in simulations



Zonal jets at 1000 m depth in the North Pacific Ocean averaged over the last 5 years of a 58-year long computer simulation. Shaded and white areas are westward and eastward currents, respectively; the contour interval is 2 cm s^{-1}. [*H. Nakano and H. Hasumi* J. Phys. Oceanography (2005)]

Further discussion

Viscous flows:

- [Lion-Temam-Wang 90s] primitive equations in shell domain.
- [Wirosoetisno 2013] Navier-Stokes eq. in rotating $\mathbf{S}^2$. $\Longrightarrow$ Global attractor is “almost” zonal, and is 0-dim for small ε.

Inviscid flows:

- Incompressible Euler in rotating $\mathbf{S}^2$: governing dynamics of zonal flows $\mathcal{P}u$? [Schochet 94’ J. Diff. Eq.]: limiting dynamics *always* $\exists$.

PV eq. is $\partial_t(\operatorname{curl} u - \frac{z}{\varepsilon}) + u\cdot\nabla(\operatorname{curl} u - \frac{z}{\varepsilon}) = 0$
With wellprepared initial data, limiting dynamics is **stationary**, zonal.

- Compressible shallow water equations on rotating $\mathbf{S}^2$. Even H^s estimates uniformly in *Ro*, *Fr* are open problem.

To sum up ......

$$\partial_t U = \frac{1}{\varepsilon}\mathcal{L}[U] + F$$

- Decompose $U = U^s + U^f$ as well as the system. $U^s \in \ker\{\mathcal{L}\}$.
- Allow $\partial_t U \sim O(\varepsilon^{-1})$, namely, do not suppress fast U^f.
- Find vorticity operator $\mathcal{K}$ s.t. $\mathcal{K}\mathcal{L} = 0$ and establish correspondence between slow and vortical dynamics.
- Control of fast-fast interaction (?)
- Verify $\left\|U^f\right\|_{X_1} \lesssim \left\|\mathcal{L}[U^f]\right\|_{X_2}$, then, $\int_0^T (1-\mathcal{P})U\,dt \sim O(\varepsilon)$.
- Time-averaging and flows.
- **Numerical analysis** – the importance to have schemes preserving vortical dynamics (i.e. slow dynamics)

Thank you!