# Perturbations of Dispersing Billiards via Spectral Methods

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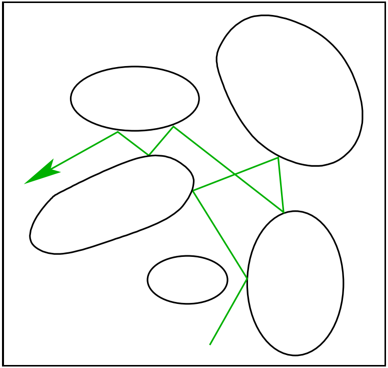
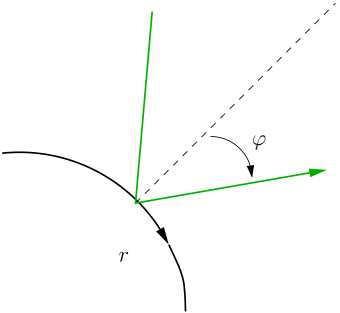
joint work with Hong-Kun Zhang

# A Periodic Lorentz Gas



- Finitely many scatterers on $\mathbb{T}^2$ whose boundaries are $C^3$ curves with strictly positive curvature
- Point mass moving with unit speed reflecting elastically off of scatterers
- Can have **finite** or **infinite horizon**

# The Billiard Map Associated with the Flow




- $r =$ arclength coordinate oriented clockwise on boundary of scatterer $\partial\Gamma_i$
- $\varphi =$ angle outgoing trajectory makes with normal to scatterer

$M = \bigcup_{i=1}^{d} \partial\Gamma_i \times [-\frac{\pi}{2}, \frac{\pi}{2}]$, the natural "collision" cross-section for the billiard flow. $M$ is a finite union of cylinders.

$T : (r, \varphi) \to (r', \varphi')$ is the first return map, i.e. the *billiard map*.

- a hyperbolic map with singularities

# Unified Framework for Perturbations

**Motivation:** We want to develop a framework that handles perturbations in a unified way, including:

- Movement and deformation of scatterers
- Billiards under external forces with thermostatting
- Twists or kicks at reflections, with 'slipping' along scatterers
- Random perturbations comprised of compositions of the above classes

**Main Idea:** Set up functional analytic framework centered on the transfer operators associated with such systems. Prove that the spectra and spectral projectors of these transfer operators vary (Hölder) continuously with these perturbations.

# Transfer Operator for the Map

For a smooth test function $\psi$, the transfer operator (Ruelle-Perron-Frobenius operator) $\mathcal{L}$ associated with $T$ acts on a distribution $\mu$ on $M$ by

$$\mathcal{L}\mu(\psi) = \mu(\psi \circ T).$$

The unperturbed billiard map preserves a smooth invariant measure $d\mu_0 = c \cos\varphi \, dr d\varphi$.
If $d\mu = f d\mu_0$ is a measure abs. cont. w.r.t. $\mu_0$, then

$$\mathcal{L}f(x) = \frac{f(T^{-1}x)}{J_{\mu_0}T(T^{-1}x)},$$

where $J_{\mu_0}T$ is the Jacobian of $T$ w.r.t. $\mu_0$.

We want to directly study the spectral properties of the transfer operator associated with the map on an appropriate Banach space without coding the dynamics.

# A Method to Show $\mathcal{L}$ is Quasi-Compact

Dynamical method to estimate the essential spectral radius [Hennion '93]. Essential ingredients:

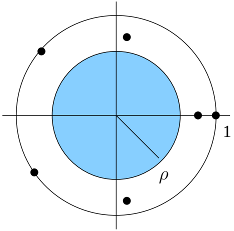
- Two Banach spaces $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ and $(\mathcal{B}_w, |\cdot|_w)$, with an embedding $\mathcal{B} \hookrightarrow \mathcal{B}_w$ such that $|f|_w \le \|f\|_{\mathcal{B}}$ for $f \in \mathcal{B}$
- The unit ball of $\mathcal{B}$ is compactly embedded in $\mathcal{B}_w$
- (Lasota-Yorke inequalities) There exist constants $C > 0$ and $\rho < 1$ such that for all $f \in \mathcal{B}$, $n \ge 0$,

$$\|\mathcal{L}^n f\|_{\mathcal{B}} \le C\rho^n \|f\|_{\mathcal{B}} + C|f|_w$$
$$|\mathcal{L}^n f|_w \le C|f|_w$$

Under these conditions, $\mathcal{L} : \mathcal{B} \circlearrowleft$ is quasi-compact with essential spectral radius $\le \rho$: for any $r > \rho$, spectrum of $\mathcal{L}$ outside disk of radius $r$ is finite-dimensional.
(Note: The above inequalities imply that the spectral radius is $\le 1$, but for reasonable choices of $\mathcal{B}$, it is actually 1.)

# Spectral Decomposition of $\mathcal{L}$ via Quasi-Compactness

- Eigenspace corresponding to $1 =$ invariant measures
  – must show separately that peripheral spectrum is comprised of measures
- Periodic behavior of $\mathcal{L}$ corresponds to eigenvalues other than 1 on the unit circle




- If 1 is a simple eigenvalue and we can eliminate periodicity, we can conclude that $\mathcal{L}$ has a spectral gap

This functional analytic framework gives a unified (and often simplified) approach to prove many statistical properties and limit theorems.

# Study of Transfer Operator in Specific Dynamical Systems

- Markov chains [Doeblin, Fortet '37], [Ionescu-Tulcea, Marinescu '50], [Nagaev '57]
- Systems admitting a Markov partition [Sinai '68], [Bowen '70], [Ruelle '76]
- Piecewise expanding maps [Lasota, Yorke '73], [Keller '84], [Baladi, Keller '90] [Saussol '00], [Buzzi, Keller '01], [Tsujii '01]
- Hyperbolic analytic maps [Rugh '92]
- Anosov diffeomorphisms [Blank, Keller, Liverani '01], [Baladi '05], [Gouëzel, Liverani '06], [Baladi, Tsujii '07]
- Piecewise hyperbolic maps (finitely many domains of differentiability and bounded derivative) [Demers, Liverani '08], [Baladi, Gouëzel '09, '10]
- Dispersing billiards [Demers, Zhang '11]

# Common Setting for a Class of Perturbed Billiard Maps

**Q:** How can we compare transfer operators for different billiard tables on a common space?

**Main Idea:** The union of cylinders $M = \bigcup_{i=1}^d I_i \times [-\frac{\pi}{2}, \frac{\pi}{2}]$ can correspond to many different arrangements of scatterers identifying $\partial\Gamma_i$ with $I_i$, and also to billiard flows under external forces and kicks at reflections.

(i) Fix $M = \bigcup_{i=1}^d I_i \times [-\frac{\pi}{2}, \frac{\pi}{2}]$

(ii) Consider a class of maps $\mathcal{F}$ on $M$ with uniform properties

(iii) Define a Banach space of distributions $\mathcal{B}$ on $M$ such that all transfer operators $\mathcal{L}_T$ for $T \in \mathcal{F}$ are quasi-compact on $\mathcal{B}$

Define $\mathcal{S}_0 = \{\varphi = \pm\frac{\pi}{2}\}$. Then all maps $T \in \mathcal{F}$ have singularity sets of the form $\mathcal{S}_1^T = \mathcal{S}_0 \cup T^{-1}\mathcal{S}_0$.

# Uniform Properties for $\mathcal{F}$ I

(H1) (Uniform hyperbolicity) All $T \in \mathcal{F}$ have a common family of stable and unstable cones, $C^s(x)$ and $C^u(x)$.

- Uniform transversality for all $T \in \mathcal{F}$.

(H2) Common class of stable and unstable curves, $\mathcal{W}^s$ and $\mathcal{W}^u$

- Invariant families with uniformly bounded curvature
- Curves lie in a single **homogeneity strip** $H_{\pm k}$

$$H_k = \left\{ \frac{\pi}{2} - \frac{1}{k^2} < \varphi < \frac{\pi}{2} - \frac{1}{(k+1)^2} \right\}$$

(H3) Bounded distortion estimates within homogeneity strips for

- $J_{\mu_0} T$ = Jacobian of $T$ w.r.t. $\mu_0 = c \cos\varphi \, dr d\varphi$;
- $J_W T$ = Jacobian of $T$ along $W \in \mathcal{W}^s$

(H4) Control on growth of Jacobian $J_{\mu_0} T^n \leq C(1+\eta)^n$ for $\eta \geq 0$ small compared to contraction appearing in Lasota-Yorke inequalities.

# Uniform Properties for $\mathcal{F}$ II

(H5) (One step expansion) Let $V_i$ be the maximal connected homogeneous components of $T^{-1}W$, $W \in \mathcal{W}^s$. Assume $\exists \delta_0 > 0, \theta_* < 1$ such that if $|W| < \delta_0$, then

$$\sum_i \frac{|TV_i|_*}{|V_i|_*} \leq \theta_*.$$

$|\cdot|_*$ denotes length in an adapted metric.

**Weakened one step expansion**: Assume $\exists \xi < 1$ and $C_0 > 0$ such that for $W \in \mathcal{W}^s$,

$$\sum_i \left( \frac{|TV_i|}{|V_i|} \right)^\xi \leq C_0.$$

This is a version of the common "hyperbolicity dominates cutting" condition.

# Definition of norms I

Work with $\mathcal{W}^s$, set of **homogeneous stable curves** $W$ satisfying,

- Tangent vectors to $W$ lie in stable cone
- $W$ lies in a single homogeneity strip
- Curvature of $W$ bounded by a fixed constant
- $|W| \le \delta_0$, fixed length scale from one-step expansion (H5)

Ingredients in the norms:

- Integrate along curves in $\mathcal{W}^s$ to average in stable direction
- Use expansion in unstable cone and contraction in stable cone to provide contraction in the strong norm
- Avoid stable and unstable foliations by using curves in $\mathcal{W}^s$ and $\mathcal{W}^u$

Norms are similar to those in [Demers, Liverani '08].

In spirit, dual to standard pairs of Dolgopyat along stable curves.

# Definition of norms II

For $\alpha, p > 0$, we define a class of test functions on $W \in \mathcal{W}^s$.

For $W \in \mathcal{W}^s$ and $\psi \in C^p(W, \mathbb{C})$, define

$$|\psi|_{W,\alpha,p} = |W|^\alpha \cdot |\psi|_{C^p(W)}.$$

For $f \in C^1(M)$ and $p \leq 1/3$, define the **weak norm** by

$$|f|_w = \sup_{W \in \mathcal{W}^s} \sup_{\substack{\psi \in C^p(W) \\ |\psi|_{W,0,p} \leq 1}} \int_W f\psi \, dm$$

$\mathcal{B}_w$ is the completion of $C^1(M)$ w.r.t. the weak norm $|\cdot|_w$.

# Definition of norms III

Choose, $q < p$, $\alpha \le 1 - \xi$ and $0 < \beta \le \min\{\alpha, p - q\}$.
Define the **strong stable norm** by

$$\|f\|_s = \sup_{W \in \mathcal{W}^s} \sup_{\substack{\psi \in C^q(W) \\ |\psi|_{W,\alpha,q} \le 1}} \int_W f \psi \, dm$$

Define the **strong unstable norm** by

$$\|f\|_u = \sup_{\varepsilon \le \varepsilon_0} \sup_{\substack{W_1, W_2 \in \mathcal{W}^s \\ d_{\mathcal{W}^s}(W_1, W_2) \le \varepsilon}} \sup_{\substack{|\psi_i|_{W,0,p} \le 1 \\ |\psi_1 - \psi_2|_q \le \varepsilon}} \varepsilon^{-\beta} \left| \int_{W_1} f \psi_1 \, dm - \int_{W_2} f \psi_2 \, dm \right|$$

Define $\|f\|_{\mathcal{B}} = \|f\|_s + b\|f\|_u$ for some $b > 0$ (to be determined by L-Y inequalities).

$\mathcal{B}$ is the completion of $C^1(M)$ w.r.t. the strong norm $\|\cdot\|_{\mathcal{B}}$.

Sequence of continuous embeddings: $C^p \subset \mathcal{B} \subset \mathcal{B}_w \subset (C^p)'$.

# Spectral Picture for Class of Maps $\mathcal{F}$

**Theorem**

*Let $\mathcal{F}$ be a class of maps satisfying (H1)-(H5) with uniform constants. For $T \in \mathcal{F}$, let $\mathcal{L}_T$ be the associated transfer operator.*

*Then $\mathcal{L}_T$ satisfies a uniform set of Lasota-Yorke inequalities for all $T \in \mathcal{F}$, i.e., $\exists \rho < 1$ and $C > 0$ s.t.*

$$\|\mathcal{L}_T^n f\|_B \leq C\rho^n \|f\|_{\mathcal{B}} + C(1+\eta)^n |f|_w \quad \forall n \in \mathbb{N}, f \in \mathcal{B}$$
$$|\mathcal{L}_T^n f|_w \leq C(1+\eta)^n |f|_w \quad \forall n \in \mathbb{N}, f \in \mathcal{B}_w.$$

- *For each $T \in \mathcal{F}$, $\mathcal{L}_T$ is quasi-compact with essential spectral radius $\leq \rho$ and spectral radius equal to 1.*
- *The elements of the peripheral spectrum are measures.*
- *All physical measures belong to $\mathcal{B}$ and they form a basis for the eigenspace corresponding to 1. (physical = ergodic, invariant with positive Lebesgue measure basin of attraction)*

# Metric in the Class of Maps $\mathcal{F}$

Next we want to prove that the spectral data of these transfer operators varies continuously in $\mathcal{F}$. For this, we define a notion of distance in $\mathcal{F}$.

Recall that $\mathcal{S}_0 = \{\varphi = \pm\frac{\pi}{2}\}$, and
$\mathcal{S}_{-1}^T = \mathcal{S}_0 \cup T\mathcal{S}_0 =$ singularity set for $T^{-1}$.

We say $d_{\mathcal{F}}(T_1, T_2) \leq \varepsilon$ if

For $x \notin N_\varepsilon(\mathcal{S}_{-1}^{T_1} \cup \mathcal{S}_{-1}^{T_2})$, we have

- $d(T_1^{-1}x, T_2^{-1}x) \leq \varepsilon$
- $\|DT_1^{-1}(x) - DT_2^{-1}(x)\| \leq \varepsilon$

Plus two similar conditions on $J_{\mu_0}T_i$ and $J_W T_i$.

Note: $\mathcal{S}_{-1}^{T_1}$ and $\mathcal{S}_{-1}^{T_2}$ are not assumed to be close to one another.

# Perturbative Framework [Keller, Liverani '99]

Define

$$|||\mathcal{L}_1 - \mathcal{L}_2||| := \sup\{|\mathcal{L}_1 f - \mathcal{L}_2 f|_w : \|f\|_{\mathcal{B}} \le 1\}.$$

Framework requires two elements:

- Uniform Lasota-Yorke inequalities for a family of maps $T_\varepsilon \in \mathcal{F}$, $\varepsilon \in [0, t]$,

$$\|\mathcal{L}_\varepsilon^n f\|_{\mathcal{B}} \le C\rho^n \|f\|_{\mathcal{B}} + C^n |f|_w, \quad |\mathcal{L}_\varepsilon^n f|_w \le C^n |f|_w.$$

- Small perturbation in the $|||\cdot|||$-norm

$$|||\mathcal{L}_{T_\varepsilon} - \mathcal{L}_{T_0}||| \to 0 \text{ as } \varepsilon \to 0.$$

In this scenario, the part of the spectrum outside of the disk of radius $\rho$ and the associated spectral projectors vary continuously with $\varepsilon$. So if $\mathcal{L}_{T_0}$ has a spectral gap, then so does $\mathcal{L}_{T_\varepsilon}$ for all $\varepsilon$ sufficiently small.

# Result on Smallness of Perturbation

**Theorem**

*For a fixed class of maps $\mathcal{F}$ satisfying (H1)-(H5), if $T_1, T_2 \in \mathcal{F}$ satisfy $d_{\mathcal{F}}(T_1, T_2) < \varepsilon$, then*

$$|||\mathcal{L}_{T_1} - \mathcal{L}_{T_2}||| < C\varepsilon^{\beta},$$

*for a uniform constant $C > 0$ and $\beta > 0$ is from the definition of the norms.*

Next we show a variety of concrete classes of perturbations fit into this framework - they have uniform properties (H1)-(H5) and also are small in the metric $d_{\mathcal{F}}(\cdot, \cdot)$.

Since the map corresponding to the unperturbed Lorentz gas has a spectral gap [Demers, Zhang '11], we will perturb off of such maps to conclude persistence of the spectral gap and Hölder continuity of spectra and spectral projectors.

# Movements and Deformations of Scatterers I

Fix a configuration $Q_0$ of $d$ scatterers on $\mathbb{T}^2$.

Define $\mathcal{F} = \mathcal{F}(\tau_*, \mathcal{K}_*, E_*)$, set of all maps corresponding to configurations $Q$ of $d$ scatterers such that

- $\tau_{\min}(Q) \geq \tau_* > 0$
- $\mathcal{K}_{\min}(Q) \geq \mathcal{K}_* > 0$,
  $\mathcal{K}_{\min}(Q) =$ minimum curvature of scatterers in $Q$
- $E_{\max}(Q) \leq E_* < \infty$.
  $E_{\max}(Q)$ is maximum of $C^3$ norm of boundaries of scatterers

**Theorem**

*Fix $\tau_*, \mathcal{K}_*$ and $E_*$. Then all maps $T \in \mathcal{F}(\tau_*, \mathcal{K}_*, E_*)$ satisfy (H1)-(H5) with uniform constants. Therefore $\mathcal{L}_T$ satisfies uniform Lasota-Yorke inequalities for all $T \in \mathcal{F}$ and is quasi-compact as an operator on $\mathcal{B}$.*

# Movements and Deformations of Scatterers II

Let $u_i(Q)$ be the curve (parametrized by arclength) describing the boundary of scatterer $\Gamma_i$ in configuration $Q$.

**Theorem**

*For any two configurations $Q_1$, $Q_2$ corresponding to maps in $\mathcal{F}$, if*

$$\sum_{i=1}^{d} \|u_i(Q_1) - u_i(Q_2)\|_{C^2} < \varepsilon, \text{ then } |||\mathcal{L}_{T_1} - \mathcal{L}_{T_2}||| \leq C\varepsilon^{\gamma}$$

*for some $\gamma > 0$ so that the spectra and spectral projectors of $\mathcal{L}_T$ vary Hölder continuously in $\mathcal{F}$.*

- This holds even as we move configurations from finite to infinite horizon
- Can string together small movements to obtain Hölder continuity of spectral data across large movements of scatterers since all maps in the class $\mathcal{F}$ enjoy a spectral gap.

# External Forces and Kicks at Reflections I

Fix a configuration $Q$ of $d$ scatterers with **finite horizon**.

Let $\mathbf{q} \in \mathbb{T}^2$, $\mathbf{p}$ velocity. Define a billiard flow by

$$\dot{\mathbf{q}} = \mathbf{p}, \quad \dot{\mathbf{p}} = F(\mathbf{q}, \mathbf{p})$$

At reflections, $\mathbf{p}$ changes according to the rule

$$(\mathbf{q}^+, \mathbf{p}^+) = (\mathbf{q}^-, \mathbf{p}^- - 2[n(\mathbf{q}^-) \cdot \mathbf{p}^-]n(\mathbf{q}^-)) + G(\mathbf{q}^-, \mathbf{p}^-).$$

- $F$ is a stationary external force, acting between collisions,
- $G$ is a twist or kick which acts only at reflections. $G$ allows "slips" along the scatterers.

$T_{F,G}$ is the corresponding billiard map.

# External Forces and Kicks at Reflections II

Various classes of systems have been studied that fit this general setup.

- Case $G = 0$ studied in [Chernov, Eyink, Lebowitz, Sinai '93], [Chernov '01, '08]
- Case $F = 0$, $G$ affecting only $\mathbf{p}$, studied in [Markarian, Pujals, Sambarino '10], [Zhang '11]
- Soft billiards where 'collisions' occur subject to a potential correspond to the case $F = 0$ and $G$ affects both $\mathbf{p}$ and $\mathbf{q}$, studied in [Sinai '63], [Kubo '76], [Markarian '92], [Donnay, Liverani '91], [Bálint, Tóth '03]
  (If the potential is rotationally symmetric, $G$ affects only $\mathbf{q}$.)

# External Forces and Kicks at Reflections III

Assumptions on $F$ and $G$.

- $F$ and $G$ are $C^2$ with $\|F\|_{C^1}, \|G\|_{C^1} \leq \varepsilon_0$.
- $F$ and $G$ admit an integral of motion such that $0 < p_{\min} \leq \|\mathbf{p}(t)\| \leq p_{\max} < \infty$ on each integral surface.

  For example, $G$ is a twist acting only on the angle (and possibly on position, but not on $\|\mathbf{p}\|$), and $F$ is:

  Ex 1. (potential force) $F = -\nabla U(\mathbf{q})$ for a potential $U$.

  Ex 2. (isokinetic force) $F \cdot \mathbf{p} = 0$ such as

  $$F = F_0 - \frac{F_0 \cdot \mathbf{p}}{\mathbf{p} \cdot \mathbf{p}} \mathbf{p}$$ for some force $F_0$ (**thermostatting**).

- The singularity sets for $T_{F,G}$ and $T_{F,G}^{-1}$ are the same as those for $T_{F,0}$ and $T_{F,0}^{-1}$, respectively.

  For example, $G(r, \pm\frac{\pi}{2}) = 0$ and the action of $G$ is reversible. (True for soft billiards with rotationally symmetric potential.)

# External Forces and Kicks at Reflections IV

- Define $\mathcal{F}(\varepsilon)$ to be the set of all $T_{F,G}$ such that $\|F\|_{C^1}, \|G\|_{C^1} \le \varepsilon$ and $C^2$ norms are uniformly bounded.
- Recall configuration $Q$ is fixed with finite horizon.

**Theorem**

*For $\varepsilon$ sufficiently small, all $T_{F,G} \in \mathcal{F}(\varepsilon)$ satisfy (H1)-(H5) with uniform constants.*

*In addition, $|||\mathcal{L}_{T_{F,G}} - \mathcal{L}_{T_{0,0}}||| \le C\varepsilon^\gamma$ for some $\gamma > 0$.*

*Thus all $T_{F,G}$ inherit a spectral gap from $T_{0,0}$ for $\varepsilon$ sufficiently small.*

# External Forces and Kicks at Reflections V

## Corollary

*A number of limit theorems hold for $T_{F,G}$ for piecewise Hölder observables with "reasonable" singularities:*

- *Central limit theorem*
- *Local large deviation estimates*
- *Almost-sure invariance principle*

*These are new results for $T_{F,G}$ and yield limit theorems with respect to both invariant and noninvariant measures as long as these measures are in $\mathcal{B}$.*

*Ex: The SRB measure for $T_{F,G}$ is singular with respect to Lebesgue, but we get the same rate function if we measure large deviations with respect to this SRB measure or with respect to Lebesgue.*

# Random Perturbations I

Following [Gouëzel, Liverani '06], [Demers, Liverani '08].

Fix a class of billiard maps $\mathcal{F}$ with uniform properties (H1)-(H5).

For $T_1 \in \mathcal{F}$, define $B_\varepsilon(T_1) = \{T \in \mathcal{F} : d_{\mathcal{F}}(T, T_1) < \varepsilon\}$.

$(\Omega, \nu)$ probability space and $g : \Omega \times M \to \mathbb{R}$ s.t. $\exists a, A > 0$ satisfying:

- $g(\omega, \cdot) \in C^1(M, \mathbb{R})$ and $|g(\omega, \cdot)|_{C^1} \leq A$, $\forall \omega \in \Omega$;
- $\int_\Omega g(\omega, x)\, d\nu(\omega) = 1$ for each $x \in M$;
- $g(\omega, x) \geq a$ for all $\omega \in \Omega$, $x \in M$.

Fix $\varepsilon > 0$ and $T_1 \in \mathcal{F}$ and define a random walk on $M$:

- To each $\omega \in \Omega$, assign $T_\omega \in B_\varepsilon(T_1)$;
- Start at $x \in M$, choose $T_\omega$ according to distribution $g(\cdot, x)d\nu$;
- Apply $T_\omega(x)$ and repeat process starting at new point.

# Random Perturbations II

We say the process defined this way has size $\Delta(\nu, g) \leq \varepsilon$.

Define transfer operator $\mathcal{L}_{(\nu,g)}$ associated with this process by,

$$\mathcal{L}_{(\nu,g)} f(x) = \int_\Omega \mathcal{L}_{T_\omega} f(x)\, g(\omega, T_\omega^{-1}(x))\, d\nu(\omega).$$

**Theorem**

*For $\varepsilon$ small, $\mathcal{L}_{(\nu,g)}$ satisfies uniform Lasota-Yorke inequalities.*

*If $\Delta(\nu, g) \leq \varepsilon$, then $|||\mathcal{L}_{(\nu,g)} - \mathcal{L}_{T_1}||| \leq C\varepsilon^\beta$ for a uniform $C > 0$ and $\beta > 0$ is from the definition of the norms.*

*Thus if $\mathcal{L}_{T_1}$ has a spectral gap, then so does $\mathcal{L}_{(\nu,g)}$. As a consequence, related statistical properties vary Hölder continuously with this type of perturbation: invariant measures, rate of decay of correlations, variance in CLT, etc.*

# Possible Applications and Extensions

- More general dispersing billiards such as,
  - billiards with corner points
  - return maps for some billiards with focusing boundaries
- Study transfer operators with more general potentials than $\log J_{\mu_0} T$.
  - Gibbs theory for billiards?
- Exponential decay of correlations for the billiard flow.