On the vortex-wave system

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Introduction

Consider 2D incompressible, ideal flow with velocity $u = (u^1, u^2)$ and vorticity

$$\omega = \text{curl}\, u = \partial_{x_1} u^2 - \partial_{x_2} u^1.$$

Fluid domain is full plane $\mathbb{R}^2$, so flow in vortex dynamics form given by

$$\begin{cases} \partial_t \omega + u \cdot \nabla \omega = 0 \\ u = u(t,x) = \displaystyle\int_{\mathbb{R}^2} \frac{(x-y)^\perp}{2\pi |x-y|^2} \omega(t,y)\, dy, \\ \omega(0,x) = \omega_0(x). \end{cases} \tag{1}$$

Here, $(a,b)^\perp = (-b, a)$.

Notation: $K = K(\xi) = \dfrac{1}{2\pi} \dfrac{\xi^\perp}{|\xi|^2}$.

Theorem

(Yudovich, 1963) *If* $\omega_0 \in L^\infty(\mathbb{R}^2) \cap L^1(\mathbb{R}^2)$ then $\exists!$ *weak solution of the Euler equations.*

Some singular flow examples:

(1) *Contour dynamics* ω_0 is the characteristic function of a bounded, smooth, region.
(2) *Point-vortex dynamics* $\omega_0 = \sum_i^N m_i \delta(\cdot - P_i)$, where $m_i \in \mathbb{R}$, $P_i \in \mathbb{R}^2$, $i = 1, \dots, N$ and δ is the Dirac delta.
Velocity induced by point-vortices:

$$u = u(x) = \sum_i^N m_i K(x - P_i).$$

Vortex dynamics $\longleftrightarrow$ point-vortex system (*ignore self-induced velocity*)

$$\frac{d}{dt} P_i = \sum_{j \neq i}^N m_j K(P_i - P_j), \quad i = 1, \dots, N.$$

The vortex-wave system

Introduced by C. Marchioro and M. Pulvirenti, 1991.

Assume vorticity = bounded part + point-vortices (for simplicity take only one point-vortex)

Evolve by coupling vorticity equation with point-vortex system:

$$
\begin{cases}
\partial_t \omega + (u + H) \cdot \nabla \omega = 0, \\
\dot{P} = u(t, P) \\
u(t,x) = \int_{\mathbb{R}^2} K(x-y)\omega(t,y)\, dy, \\
H(t,x) = mK(x - P(t)), \\
P(0) = P_0, \quad \omega(0,x) = \omega_0(x).
\end{cases}
\tag{2}
$$

Measures as weak solutions

If $\omega_0 \in \mathcal{BM}_{+,c}(\mathbb{R}^2) \cap H^{-1}(\mathbb{R}^2)$ $\exists$ weak solution to (1) (J.-M. Delort, 1991).

Problem: Diracs $\notin H^{-1}(\mathbb{R}^2)$. (Point-vortices are not $L^2_{\text{loc}}(\mathbb{R}^2)$)

Cannot make sense of $K \cdot \nabla \delta$ (nor $K \cdot \nabla K$), even in weak sense.

Concentration of vorticity $\longleftrightarrow$ approximations with Diracs: lead to *diagonal defect measures* (treated by F. Poupaud, 2002). Not a solution of Euler!

Ignore self-induced velocity – key to the point-vortex system. Hard to justify rigorously – many attempts. See Marchioro and Pulvirenti, 1993 – *Vortices and localization in Euler flows*.

Results for the vortex-wave system

Theorem

(Marchioro and Pulvirenti, 1991) *If* $\omega_0 \in L^\infty_c(\mathbb{R}^2)$ *then* $\exists$ *a globally defined solution of the vortex-wave system with one point-vortex* $P = P(t)$. *If* $P(0) = P_0 \notin$ *supp* ω_0 *then solution is unique.*

Essential point of the (uniqueness) proof is to show that the distance between $P(t)$ and supp $\omega(t,\cdot)$ is bdd away from 0 over finite time intervals. Other important piece: log-Lipschitz regularity of u.

Conjecture: uniqueness still true if ω_0 constant near P_0, but not necessarily zero.

Verified if:

1. ω_0 Lipschitz (Starovoitov 94),
2. ω_0 bounded (Lacave and Miot 09).

First result

Theorem

(L., Miot and Nussenzveig Lopes, 2011) Assume $\omega_0 \in L^p_c(\mathbb{R}^2)$, $p > 2$. *Then there exists a globally defined solution of the vortex-wave system.*

Key point: new notion of weak solution - Lagrangian vs. Eulerian solutions.

Remark: Existence is global for one point-vortex, or for several point vortices with the same sign; existence is only up to vortex collision if point vortices are allowed to change sign.

Lagrangian solutions

Definition

Fix $T > 0$. Let $\omega_0 \in L^1 \cap L^\infty$ and let $P_0 \in \mathbb{R}^2$. Say $t \mapsto (\omega(t,\cdot), P(t))$ is a Lagrangian solution to the vortex-wave system in $(0,T)$ with initial data (ω_0, P_0) if
$\omega \in L^\infty(0,T; L^1 \cap L^\infty)$, $\quad u = K * \omega \in C([0,T] \times \mathbb{R}^2)$, $\quad t \mapsto P \in C^1$ and there exists a flow $\Phi : [0,T] \times \mathbb{R}^2 \setminus \{P(t)\} \to \mathbb{R}^2$, $\Phi \in C^1([0,T] \times \mathbb{R}^2)$ such that:

$$\begin{cases} \omega(t, \Phi(t,x)) = \omega_0(x) \\ \dot{P} = u(t,P) \\ \partial_t \Phi(t,x) = u(t, \Phi(t,x)) + mK(\Phi(t,x) - P(t)) \\ \omega(t=0) = \omega_0, \quad P(t=0) = P_0, \quad \Phi(0,x) = x, x \neq P_0. \end{cases} \tag{3}$$

Additionally, $\Phi(t,\cdot)$ is a volume-preserving homeomorphism from $\mathbb{R}^2 \setminus \{P\}$ to itself.

Eulerian solutions

Definition

Fix $T > 0$. Let $\omega \in L^1 \cap L^\infty(\mathbb{R}^2)$ and let $P_0 \in \mathbb{R}^2$. Say (ω, P) is an Eulerian solution to the vortex-wave system in $(0, T)$ with initial data (ω_0, P_0) if $\omega \in L^\infty(0, T; L^1 \cap L^\infty)$, $P = P(t) \in L^\infty(0, T)$ and if we have, in the sense of distributions,

$$\begin{cases} \partial_t \omega + \operatorname{div}[(u + H)\omega] = 0, \\ u = K * \omega, \quad H = H(t, x) = mK(x - P(t)), \\ P(t) = P_0 + \displaystyle\int_0^t u(s, P(s))\, ds, \\ \omega(0) = \omega_0. \end{cases}$$

Proof of Lacave and Miot

Main point in their analysis is that when the vorticity is bounded, and initially constant near the point-vortex, it will stay constant near the point vortex for all time.

Main steps:

1. Lagrangian implies Eulerian. Existence of Eulerian weak solution;
2. Solution is *renormalized solution* of the transport equation in the sense of DiPerna-Lions;
3. There is a disk with constant vorticity around each point vortex, at all times in a finite interval;
4. Uniqueness and solution is actually Lagrangian.

Proof of Lopes Filho, Miot and NL

1. $\omega_0 \in L^p$, $p > 2$. Mollify continuous part, use existence by Marchioro and Pulvirenti, get approximate solution sequence with uniform L^p bound;
2. Use elliptic regularity to get approximate (continuous part) velocity is uniformly continuous;
3. Peano existence argument for ODE's with continuous velocities + standard compactness argument of existence for the vorticity equation;
4. If more than one point-vortex then uniform control of the distance between vortices if mass is positive - energy based argument.

$p < 2$: Velocity no longer *a priori* continuous. Only ODE theory that applies is DiPerna-Lions. Flows which only exist almost everywhere. Need to propagate points, which have zero Lebesgue measure...

Approximation using Euler-α

Joint work with Eleonora Moura

$$\begin{cases} \partial_t \eta + v \cdot \nabla \eta = 0, \\ v = K^\alpha * \eta, \\ K^\alpha = K * G^\alpha, \quad G^\alpha = (\mathbb{I} - \alpha^2 \Delta)^{-1}, \\ \eta(0,x) = \eta_0(x). \end{cases}$$

This is a special case of the *vortex blob method* – has physical interest and deep geometric meaning.

Global existence and uniqueness of a weak solution with η_0 arbitrary finite measure was proved by M. Oliver and S. Shkoller in 2001.

Problem: Start with $\eta_0 = \omega_0 + m\delta_{P_0}$ and obtain the vortex-wave dynamics as limit $\alpha \to 0$ of the Oliver-Shkoller weak solution. (More generally: $\eta_0 = \omega_0 + \sum_i m_i \delta_{P_{i,0}}$.)

Theorem

Let $p > 2$, $\omega_0 \in L^p_c(\mathbb{R}^2)$. Let $\{P_{i,0}\}$ be N points in $\mathbb{R}^2$ and $m_i \in \mathbb{R}_+$, $i = 1, \ldots, N$. Then there exists a subsequence $\alpha \to 0$ of weak solutions of the Euler-α equation with initial data $\eta_0 = \omega_0 + \sum_i m_i \delta_{P_{i,0}}$ which converges to an Eulerian weak solution of the vortex-wave system.

Remarks:

- Sign condition on the vortices can be eliminated, with convergence guaranteed until first collision.
- Related work by C. Bjorland (2011), convergence of the vortex blob method - smooth blob, a single point vortex, different technique.

Weak transport and Lagrangian solutions with L^p-vorticity

(joint with G. Crippa, E. Miot and H. Nussenzveig Lopes.)

We have existence of Eulerian solutions - what about particle trajectories/flow map?

Required: extension of DiPerna-Lions/Ambrosio transport theory to vector fields satisfying: divergence-free, curl = L^p + diracs in motion.

Note that transporting velocity consisting of $K * \omega(\cdot, t) + \sum m_i \delta(\cdot - P(t))$ is not BV

Regular Lagrangian flow

Following DiPerna and Lions and Ambrosio, here is the definition of regular Lagrangian flow. We denote by $\mathcal{L}^2$ the Lebesgue measure on $\mathbb{R}^2$.

Definition

We say that a map $X : [0,T] \times \mathbb{R}^2 \to \mathbb{R}^2$ is a regular Lagrangian flow for the vector field b if:
(i) There exists an $\mathcal{L}^2$-negligible set $S \subset \mathbb{R}^2$ such that for all $x \in \mathbb{R}^2 \setminus S$ the map $t \mapsto b(t, X(t,x))$ belongs to $L^1([0,T])$, and

$$X(t,x) = x + \int_0^t b(s, X(s,x))\, ds, \quad \forall t \in [0,T].$$

(ii) For all $R > 0$ there exists $L_R > 0$ such that

$$X(t,\cdot)_{\#}(\mathcal{L}^2 \llcorner B_R) \leq L_R \mathcal{L}^2, \quad \forall t \in [0,T],$$

i.e. $\mathcal{L}^2 \left(X(t,\cdot)^{-1}(A) \cap B_R \right) \leq L_R \mathcal{L}^2(A)$ for every Borel set $A \subset \mathbb{R}^2$.

Hypothesis on v

Let $b(x,t) = v(x,t) + H(x,t)$ with v satisfying:

$$(H_1) \quad \frac{v}{1+|x|} \in L^1\left([0,T], L^1(mathbbR^2)\right) + L^1\left([0,T], L^\infty(\mathbb{R}^2)\right),$$

$$(H_2) \quad v \in L^1\left([0,T], \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^2)\right),$$

$$(H_3) \quad \text{div } v) \in L^1\left([0,T], L^\infty(\mathbb{R}^2)\right).$$

Anbrosio's Theorem ensures existence and uniqueness of the regular Lagrangian flow associated to such fields. In addition, in our context, we require the following assumption:

$$(H_4) \quad v \in L^\infty\left([0,T], L^q(\mathbb{R}^2)\right) \quad \text{for some } 2 < q \leq +\infty.$$

Hypothesis on H

Next, we define our singular part H as follows. We consider a given Lipschitz trajectory $z \in W^{1,\infty}([0,T],\mathbb{R}^2)$. We introduce the map

$$K : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2, \quad K(x) = \frac{x^\perp}{|x|^2} = \frac{(-x_2, x_1)}{|(x_1,x_2)|^2}$$

and we define

$$H(t,x) = K\left(x - z(t)\right). \tag{4}$$

Then H satisfies (H_1) and (H_3): actually, it is divergence free. It does not satisfy (H_2) therefore such a field is not covered by Ambrosio's Theorem . However note that H is smooth off of the set $\{(t,z(t)),\ \ t \in [0,T]\}$.

Lagrangian map

Theorem

(Crippa,L. Miot, Nussenzveig Lopes, 2013) Let $b = v + H$, where v satisfies $(H_1) - (H_2) - (H_3) - (H_4)$ and where H is given as defined. Then there exists a unique regular Lagrangian flow. Moreover, for $\mathcal{L}^2$-a.e. $x \in \mathbb{R}^2$ we have

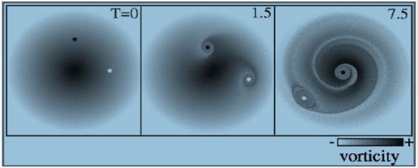
$$X(t, x) \neq z(t), \quad \forall t \in [0, T].$$

The proof comes with an explicit construction of approximate Lagrangian trajectories and an error estimate, following ideas by Lacave and Miot and by De Lellis and Crippa.

Should not expect uniqueness for the vortex-wave system with vorticity in L^p, $2 < p < \infty$, because the velocity in this case is $C^{1,\alpha}$ with $\alpha < 1$ and no better, and standard nonuniqueness examples for ODE come to mind.

However, for $p = \infty$, it is not unreasonable to expect uniqueness. As we have seen, this works when the background vorticity is constant near the vortex, an interesting and delicate set of results.

Removing this last assumption would leave us with a substantial extension of Yudovich's result, perhaps the first such result since 1963, and in a situation of well-established physical interest.



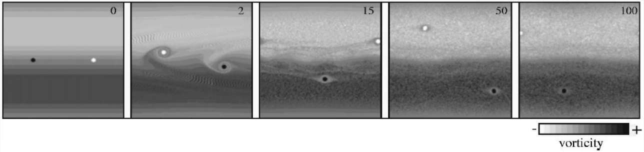


Figure : Vortices on a varying vorticity background; Schecter and Dubin, Phys. of Fluids v. 13, 2001

Conclusions and open problems

- Recast vortex-wave theory on the surface of a rotating sphere.
- The *stuck vortex*. Limiting behavior of 2D ideal flow in the exterior of a small obstacle (Iftimie, L. and Nussenzveig Lopes, 2003). Compactness result. Convergence (and error estimates) depend on uniqueness for the limiting system. Lacave and Miot also proved uniqueness if the vorticity is constant near the stuck vortex.
- Point vortices with vortex sheets
- Vanishing viscosity limit

Thank you!