On bounded velocity/bounded vorticity solutions to the incompressible 2D Euler equations... or: How to live without the Biot-Savart law

Helena J. Nussenzveig Lopes

IM-Universidade Federal do Rio de Janeiro (UFRJ) BRASIL

Isaac Newton Institute

Cambridge, UK

Collaborators:

Collaborators:

David Ambrose (Drexel Univ)

Collaborators:

David Ambrose (Drexel Univ)

Jim Kelliher (UC Riverside)

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David Ambrose (Drexel Univ)

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Milton C. Lopes Filho (IM-UFRJ, Rio de Janeiro, Brazil)

Introduction

Equations for incompressible, non-viscous (ideal) fluid flow in $2D$ are given by:

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These are the Euler equations – they should be supplemented by initial data – $u(t = 0) = u_0$ and – if there is a boundary – by slip boundary conditions $\to u \cdot \hat{n} = 0$ on the finite boundary, together with conditions at infinity.

Vorticity is defined by $\omega \equiv \partial_{x_1} u^2 - \partial_{x_2} u^1 \equiv \nabla^\perp \cdot u$, where $\nabla^\perp = (-\partial_{x_2}, \partial_{x_1})$. It is the only component of curl $(u^1, u^2, 0) \equiv (0, 0, \omega)$.

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is elliptic and can, in principle, be explicitly inverted, writing $u \equiv K[\omega] = \nabla^{\perp}(\Delta)^{-1}\omega$. Here, K depends on the fluid domain and is given by a Biot-Savart kernel; $K[\omega]$ is the Biot-Savart law, integration against $K = K(x,y) = \nabla_x^{\perp} G(x,y)$.

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Now, $K \in L^p_{loc}(\mathbb{R}^2)$, $1 \leq p < 2$ and K is q-th power integrable at infinity, with $q > 2$. Hence, to calculate $K * \omega$ need $\omega \in L^{p'} \cap L^{q'}$ with $p' > 2$ and $q' < 2$, e.g. $\omega \in L^{\infty} \cap L^1$.

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- If ω_0 *does not* decay at infinity: $\exists$ and ! of periodic smooth flows.

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- u_0 smooth (C^1_b) and the vorticity has some decay ($\omega_0 \in L^1$, $|x|^r \omega_0 \in L^1$, some $r > 0$) and regularity ($\nabla \omega_0 \in L^\infty \cap L^1$) then well-posedness (Kikuchi 1981).

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- u_0 smooth (C_b^1) and the vorticity has some decay ($\omega_0 \in L^1$, $|x|^r \omega_0 \in L^1$, some $r > 0$) and regularity ($\nabla \omega_0 \in L^\infty \cap L^1$) then well-posedness (Kikuchi 1981).

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This talk: discussion of Serfati's work, extension to continuous dependence on initial data and to flow domains exterior to a connected obstacle.

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Fundamental questions: what is a 'solution'? What happens to the Biot-Savart law? Where do we get 'uniqueness'? What perturbations in initial data are allowed?

Related results

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- Cozzi (2009, 2010) proves the vanishing viscosity limit of "viscous Serfati" solutions to inviscid ones in full plane.

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- Local versus non-local;
- Need new idea to substitute Biot-Savart law;
- Broader potential applications in Serfati's key idea (new representation formula).

Serfati's representation formula

Key new idea: consider a smooth cutoff a_ε of the origin ($a_\varepsilon(z) = 1$ if $z \in \mathbb{R}^2$ and $|z|$ small, and vanishes if $|z|$ large) and establish a formula like

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$$u(t,x) = u_0(x) + \int a_\varepsilon(x-y)K(x-y)(\omega(t,y) - \omega_0(y))\, dy +$$

$$- \int_0^t \int \left(u(s,y) \cdot \nabla_y \nabla_y [(1 - a_\varepsilon(x-y))K(x-y)]^\perp \right) \cdot u(s,y)\, dyds.$$

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Indeed: $a_\varepsilon K \in L^1$ and $\omega \in L^\infty$ so the first integral converges; $\nabla_y \nabla_y [(1 - a_\varepsilon)K] \in L^1$ because of the extra decay at infinity coming from taking two derivatives, hence, if $u \in L^\infty$, then the second integral also converges.

Where does this formula come from?

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Hence,

$$\partial_t u = \partial_t \int K(x-y)\omega(t,y)\,dy$$

$$= \partial_t \int a_\varepsilon(x-y)K(x-y)\omega(t,y)\,dy + \int (1-a_\varepsilon(x-y))K(x-y)\partial_t\omega\,dy.$$

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Therefore, integrating in time yields

$$u(t,x) = u_0(x) + \int a_\varepsilon(x-y)K(x-y)[\omega(t,y)-\omega_0(y)]\,dy$$

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Therefore, integrating by parts carefully, using div $u=0$, get

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1. Construct a sequence $\{u_{0,N}\}$ such that $u_{0,N} \in C_c^\infty$, $u_{0,N} = K * \omega_{0,N}$, $u_{0,N} \to u_0$ uniformly on compact sets, $\omega_{0,N} \to \omega_0$ in L^p on compact sets, for some $C > 0$, $\|u_{0,N}\|_{L^\infty} + \|\omega_{0,N}\|_{L^\infty} \leq C < \infty$ for all N.

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3. Use L^∞ estimate for u_N and ω_N to prove that u_N is uniformly log-Lipschitz. From this get, in standard way, passing to subsequences as needed, uniform convergence (on compacts) of particle trajectories $X_N = X_N(t, \alpha)$ ($dX_N/dt = u_N(t, X_N)$ and $X_N(0, \alpha) = \alpha$) to $X = X(t, \alpha)$. Also, easy to show X is measure-preserving.

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4. Define limit vorticity as $\omega(t,x) \equiv \omega_0(X^{-1}(t,x))$. Show $\omega_N \to \omega$ in $L^\infty_{loc}(dt, L^p_{loc}(dx))$, any $p < \infty$. Show $u_N \to u$ uniformly in compacts in space and time.

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5. Conclude u, ω satisfy incompressible 2D Euler in distributions, ω constant on particle paths and, also, representation formula remains valid. Also, (limit) u is log-Lipschitz.

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$$+ \int_0^t \sup_x \left| \int \left(u(s,y)\cdot\nabla_y\nabla_y[(1-a_\varepsilon(x-y))K(x-y)]^\perp \right)\cdot u(s,y)\,dy \right|$$

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The desired L^∞-estimate for u follows, hence, by Gronwall's lemma.

Extension to exterior domains

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Hindsight from previous exterior domain work: know

$$K_\Omega(x,y) + K(x) \sim K(x-y)$$

$(K(x) = x^\perp/(2\pi|x|^2))$, not $K_\Omega \sim K(x-y)$.

Denote $J_\Omega = J_\Omega(x, y) = K_\Omega(x, y) + K(x)$.

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Still, arrive at new – and just as useful – Serfati identity; note boundary integral:

$$\begin{aligned} u(t,x) &= u_0(x) + \int_\Omega a_\varepsilon(x-y) J_\Omega(x,y)(\omega(t,y)-\omega_0(y))\,dy \\ &\quad - \int_0^t \int_\Omega (u(s,y)\cdot\nabla_y)\,\nabla_y^\perp\left[(1-a_\varepsilon(x-y))J_\Omega(x,y)\right]\cdot u(s,y)\,dy\,ds \\ &\quad + \frac{K(x)}{2}\int_0^t\int_\Gamma |u(s,y)|^2\,\nabla a_\varepsilon(x-y)\cdot d\sigma(y)\,ds. \end{aligned}$$

Uniqueness

Serfati's strategy for uniqueness: assume two solutions u_1, u_2, same initial data. Let X_1 and X_2 be respective flow maps. Show $X_1 = X_2$. This implies $u_1 = u_2$.

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Start by estimating, using the Serfati identity,

$$\left| \frac{dX_1}{dt} - \frac{dX_2}{dt} \right| = |u_1(t, X_1) - u_2(t, X_2)|$$

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$P(t) = \|u_1(t, X_1(t, \cdot)) - u_2(t, X_2(t, \cdot))\|_{L^\infty}$, and $M(t) = \int_0^t P(s)\, ds$.

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where X_j is short for $X_j(t,x)$.

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$$I_2 = \int_0^t \int_\Omega |\nabla_y \nabla_y ((1 - a(X_1(s,x) - y)) K_\Omega(X_1(s,x), y))| \, |u_1 \otimes u_1 - u_2 \otimes u_2| (s,y)\, dy\, ds.$$

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- Change variables Lagrangianly, use measure-preserving property of flow maps, and properties of K_Ω to show that

$$I_1 \leq -C \|\omega^0\|_{L^\infty} h \log h = C\mu(h).$$

This is the most difficult step, particularly in an exterior domain.

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$$\leq C \int_0^t \mu(h(s))\, ds + C \int_0^t \|u_2(s, X_2(s,\cdot)) - u_1(s, X_1(s,\cdot))\|_{L^\infty}.$$

- Putting these bounds together, what we have shown is that

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and taking the supremum over all x in Ω, we conclude that

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- Now, ν is Osgood-continuous ($\int_0^1 ds/\nu(s) = \infty$).
- $\Longrightarrow M(t) \equiv 0$ by Osgood's lemma $\Longrightarrow P \equiv 0 \Longrightarrow X_1 \equiv X_2$.

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Introduce the Serfati space S, of divergence-free vector fields tangent to the boundary with the norm $\|u\|_S = \|u\|_{L^\infty} + \|\text{curl } u\|_{L^\infty}$.

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Note: For such weak solutions, velocity has a spatial log-Lipschitz MOC uniformly over finite time.

Main result

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Theorem

Let Ω be a smooth domain exterior to a connected and bounded set. Let $u_0 \in S$. Then there exists one and at most one weak solution of Euler in Ω with initial velocity u_0.

Continuous dependence

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Theorem

Suppose initial velocities, u_1^0 and u_2^0, with vorticities, ω_1^0 and ω_2^0, such that $u_1^0 - u_2^0$ lies in

$$S_p := \{u \in (L^\infty(\Omega))^2 : \ div\, u = 0, u \cdot \mathbf{n} = 0, \omega \in L^p(\Omega)\}$$

for some p in $(2, \infty]$, with $\| \cdot \|_{S^p} = \| \cdot \|_{L^\infty} + \|\omega(\cdot)\|_{L^p}$. Then, for all sufficiently small $s_0 = \|u_1^0 - u_2^0\|_{S^p}$,

$$\|u_1(t) - u_2(t)\|_{L^\infty} \leq Cs_0 e^{Ct} - C_t e^{Ct}(s_0 t)^{e^{-Ct(1+t)}} \log(Cs_0 t)$$
$$= o(1) \text{ as } s_0 \to 0.$$

where C and C_t depend on the initial data and on p, with C_t a continuous function of time.

Concluding remarks

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2. Should try to improve continuous dependence result so as to have same norm comparison, not velocity stable in L^∞ in terms of initial perturbation in S. Recall Taniuchi et alli, continuous dependence in $B^0_{\infty,1}$. But only for full plane and only by Littlewood-Paley...

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(iii) Serfati vorticity with no decay: vorticity is characteristic function of a strip lying outside/”above" obstacle, velocity interpolates linearly from 0 ”below" strip to constant parallel to and ”above" strip. For example, if the strip is $\{2 < x_2 < 3\}$ then

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$$u = u(x) = \begin{cases} (1,0) & \text{if } x_2 > 3, \\ (x_2 - 2, 0) & \text{if } 2 < x_2 < 3, \\ (0,0) & \text{if } x_2 < 2, \end{cases}$$

is Serfati in the exterior of the unit disk.

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