Exponential Decay of the Power Spectrum and Finite Dimensionality for Solutions of the Three Dimensional Primitive Equations

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Outline of the talk

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1. The Primitive Equations
2. The Variational Formulation of the Problem
3. Finite Dimensionality
4. Exponential Decay of the Power Spectrum of the Solutions

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The Primitive Equations

The Primitive Equations are obtained from the fundamental laws of physics using the hydrostatic and Boussinesq approximations:

$$\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\nabla)\boldsymbol{v} + w\frac{\partial \boldsymbol{v}}{\partial z} + f k \times \boldsymbol{v} + \frac{1}{\rho_0}\nabla p = \nu_{\boldsymbol{v}}\Delta\boldsymbol{v} + S_{\boldsymbol{v}}, \tag{1a}$$

$$\frac{\partial p}{\partial z} = -\rho g, \tag{1b}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0, \tag{1c}$$

$$\frac{\partial T}{\partial t} + (\boldsymbol{v}\cdot\nabla)T + w\frac{\partial T}{\partial z} = \nu_T\Delta T + S_T, \tag{1d}$$

$$\frac{\partial S}{\partial t} + (\boldsymbol{v}\cdot\nabla)S + w\frac{\partial S}{\partial z} = \nu_S\Delta S, \tag{1e}$$

$$\rho_{\mathrm{full}} = \rho_0\left[1 - \beta_T(T_{\mathrm{full}} - T_0) + \beta_S(S_{\mathrm{full}} - S_0)\right]. \tag{1f}$$

Function spaces

$$\Omega = (0, L_1) \times (0, L_2) \times (-L_3/2, L_3/2).$$

Assuming that S_u, S_v are even in x_3 and S_T odd in x_3, we search for periodic solutions of PEs with initial condition

$$(u(0),\ v(0),\ T(0)) = (u_0,\ v_0,\ T_0),$$

such that $u,\ v$ are even in x_3 and T is odd in x_3.

The functional spaces for this problem are:

$$\begin{aligned}
\boldsymbol{V} =& \Big\{ (u, v, T) \in (\dot{H}^1_{\mathrm{per}}(\Omega))^3,\ u, v \text{ even in } x_3,\ T \text{ odd in } x_3, \\
& \textstyle\int_{-L_3/2}^{L_3/2} (u_{x_1} + v_{x_2})\, \mathrm{d}x_3 = 0 \Big\} \\
\boldsymbol{H} =& \text{the closure of } \boldsymbol{V} \text{ in } \dot{L}^2(\Omega)^3, \\
\boldsymbol{V}_2 =& \text{the closure of } \boldsymbol{V} \cap \dot{H}^2_{\mathrm{per}}(\Omega)^3 \text{ in } \dot{H}^2_{\mathrm{per}}(\Omega)^3.
\end{aligned}$$

Prognostic and diagnostic variables

The prognostic variables for this problem are u, v and T, while the diagnostic variables are w and p.

For each $U = (u,\ v,\ T)$, we uniquely find $w = w(U)$ as:

$$w(U) = -\int_0^{x_3} (u_{x_1} + v_{x_2})\,\mathrm{d}x_3'.$$

The pressure can be determined uniquely, up to the surface pressure:

$$p(x_1, x_2, x_3, t) = p_s(x_1, x_2, t) + \beta_T \rho_0 \int_0^{x_3} T(x_1, x_2, x_3', t)\,\mathrm{d}x_3'.$$

The variational formulation of the problem

We search $U : [0, t_0] \to \boldsymbol{V}$ such that:

$$\frac{\mathrm{d}}{\mathrm{d}t}(U, \tilde{U})_{\boldsymbol{H}} + b(U, U, \tilde{U}) + a(U, \tilde{U}) + e(U, \tilde{U}) = (S, \tilde{U})_{\boldsymbol{H}}, \forall \tilde{U} \in \boldsymbol{V}, \tag{2}$$

where:

$$a(U, \tilde{U}) = \nu_{\boldsymbol{v}}((u, \tilde{u})) + \nu_{\boldsymbol{v}}((v, \tilde{v})) + \kappa\nu_T((T, \tilde{T})),$$

$$e(U, \tilde{U}) = \int_\Omega f(u\tilde{v} - v\tilde{u})\,\mathrm{d}\Omega - g\beta_T \int_\Omega Tw(\tilde{U})\,\mathrm{d}\Omega,$$

$$\begin{aligned} b(U, U^\sharp, \tilde{U}) = &\int_\Omega \left(u\frac{\partial u^\sharp}{\partial x_1} + v\frac{\partial u^\sharp}{\partial x_2} + w(U)\frac{\partial u^\sharp}{\partial x_3}\right)\tilde{u}\,\mathrm{d}\Omega \\ &+ \int_\Omega \left(u\frac{\partial v^\sharp}{\partial x_1} + v\frac{\partial v^\sharp}{\partial x_2} + w(U)\frac{\partial v^\sharp}{\partial x_3}\right)\tilde{v}\,\mathrm{d}\Omega \\ &+ \kappa\int_\Omega \left(u\frac{\partial T^\sharp}{\partial x_1} + v\frac{\partial T^\sharp}{\partial x_2} + w(U)\frac{\partial T^\sharp}{\partial z}\right)\tilde{T}\,\mathrm{d}\Omega. \end{aligned}$$

Properties for the forms a, e and b

$a,\ e : \boldsymbol{V} \times \boldsymbol{V} \to \mathbb{R}$ bilinear, continuous, and we choose $\kappa > 0$ such that:

$a + e$ is coercive on $\boldsymbol{V}$,

b is trilinear, continuous on $\boldsymbol{V} \times \boldsymbol{V}_2 \times \boldsymbol{V}$ with values in $\mathbb{R}$,

and on $\boldsymbol{V} \times \boldsymbol{V} \times \boldsymbol{V}_2$ with values in $\mathbb{R}$,

$$\begin{aligned} & b(U, \tilde{U}, U^\sharp) = -b(U, U^\sharp, \tilde{U}),\ b(U, \tilde{U}, \tilde{U}) = 0, \\ & |b(U, U^\sharp, \tilde{U})| \leqslant c_2 |U|_{L^2}^{1/2} \|U\|^{1/2} \|U^\sharp\| |\tilde{U}|_{L^2}^{1/2} \|\tilde{U}\|^{1/2} \\ & \qquad + c_2 \|U\| \|U^\sharp\|^{1/2} |U^\sharp|_{V_2}^{1/2} |\tilde{U}|_{L^2}^{1/2} \|\tilde{U}\|^{1/2}, \\ & \forall\, U,\ \tilde{U},\ U^\sharp \text{ in } \boldsymbol{V}, \text{ with } \tilde{U} \text{ or } U^\sharp \text{ in } \boldsymbol{V}_2. \end{aligned} \tag{3}$$

Equation (2) can be written as an operator evolution equation in $\boldsymbol{V}_2'$:

$$\begin{aligned} &\frac{\mathrm{d}U}{\mathrm{d}t} + AU + B(U,U) + EU = S, \quad \text{in } \boldsymbol{V}_2', \\ &U(0) = U_0, \end{aligned} \tag{4}$$

where

$$\begin{aligned} &< AU, U >_{\boldsymbol{V}', \boldsymbol{V}} + < EU, U >_{\boldsymbol{V}', \boldsymbol{V}} \geqslant c_0 \|U\|^2, \\ &< B(U,U), U >_{\boldsymbol{V}', \boldsymbol{V}} = 0. \end{aligned}$$

- Lions, Temam, Wang (1992): global existence of weak solutions
- Hu, Temam, Ziane (2003): global existence of strong solutions in thin domains, when the initial data depends inverse proportionally on the thickness of the domain
- Temam, Ziane (2004): local in time existence of strong solutions in domains with arbitrary, variable, depth
- Cao, Titi (2005): global existence of strong solutions in domains of arbitrary constrant depth
- Ju (2007): existence of global attractor for the strong solutions
- Kukavica, Ziane (2007): global existence of strong solutions in a domain with variable bottom and when Dirichlet boundary conditions are considered on the side and on the bottom
- M.P. (2006): global existence of very regular solutions for the periodic case

Sobolev regularity results

Given $m \in \mathbb{N}$, $m \geqslant 1$, $U_0 \in \boldsymbol{V} \cap (\dot{H}^m_{\mathrm{per}}(\Omega))^3$ and $S \in L^\infty(\mathbb{R}_+; \boldsymbol{V} \cap (\dot{H}^{m-1}_{\mathrm{per}}(\Omega))^3)$, there exists a unique solution U of (2) on $\mathbb{R}_+$ such that:

$$U \in \mathcal{C}(\mathbb{R}_+; (\dot{H}^m_{\mathrm{per}}(\Omega))^3) \cap L^2(0, t_0; (\dot{H}^{m+1}_{\mathrm{per}}(\Omega))^3), \quad \forall t_0 > 0. \tag{5}$$

Finite Dimensionality

In the context of Navier-Stokes equations:

- Foias and Prodi (1967)
- Foias and Temam (1984)
- Constantin, Foias, Manley and Temam (1986)
- Jones and Titi (1993)
- Doering and Gibbon (1995)
- Constantin, Doering and Titi (1996)

Finite Dimensionality

Let F, G be two forcing terms having the same asymptotic behavior:

$$\lim_{t\to\infty} \|F(x,t) - G(x,t)\| = 0. \tag{6}$$

We consider a set of N points, uniformly distributed into the domain, denoted by

$$\mathcal{E} = \{x^1, x^2, \dots, x^N\}. \tag{7}$$

We suppose that the corresponding solutions U_1 and U_2 have the same asymptotic behavior at the points of $\mathcal{E}$:

$$\max_{j=1,2,\dots,N} |U_1(x^j,t) - U_2(x^j,t)| \to 0 \quad \text{as}\, t \to \infty. \tag{8}$$

Determining nodes

In the above context, we say that $\mathcal{E}$ is a set of *determining nodes* if the solutions U_1 and U_2 have the same time-asymptotic behavior uniformly in space, meaning:

$$\lim_{t\to\infty} \int_\Omega |U_1(x,t) - U_2(x,t)|^2 \,\mathrm{d}\Omega = 0. \tag{9}$$

Technical results

Lemma

Let the domain Ω *be covered by N identical cubes of the type* $Q = (0, l) \times (0, l) \times (0, l)$, *with* $l > 0$ *and consider the set* $\mathcal{E} = \{x^1, ..., x^N\}$ *of points in* Ω, *distributed one in each cube. Then, for each* $f \in D((-\Delta)^{3/2})$ *the following inequality holds:*

$$|\Delta f|^2_{L^2(\Omega)} \leqslant c_1 l^{-4} \eta(f)^2 + c_2 l^2 |(-\Delta)^{3/2} f|^2_{L^2(\Omega)},$$

with $\eta(f) = \max_{1,..,N} |f(x^j)|$.

Lemma

(Generalized Gronwall, Jones&Titi1992) Let $\alpha = \alpha(t)$ *and* $\beta = \beta(t)$ *be locally integrable real-valued functions on* $[0, \infty)$ *satisfying, for some* $T > 0$ *the following conditions:*

$$\liminf_{t\to\infty} \frac{1}{T}\int_t^{t+T} \alpha(\tau)\,\mathrm{d}\tau > 0,$$

$$\limsup_{t\to\infty} \frac{1}{T}\int_t^{t+T} \alpha^-(\tau)\,\mathrm{d}\tau < \infty,$$

$$\lim_{t\to\infty} \frac{1}{T}\int_t^{t+T} \beta^+(\tau)\,\mathrm{d}\tau = 0,$$

where $\alpha^-(t) = \max(-\alpha(t), 0)$ *and* $\beta^+(t) = \max(\beta(t), 0)$. *Suppose that* $\xi = \xi(t)$ *is an absolutely continuous non-negative function on* $[0, \infty)$ *which satisfies the following inequality almost everywhere on* $[0, \infty)$*:*

$$\frac{\mathrm{d}\xi}{\mathrm{d}t} + \alpha\xi \leqslant \beta. \tag{10}$$

Then, $\xi(t) \to 0$ *as* $t \to \infty$.

We will actually prove a stronger result than (9), that is:

$$\lim_{t\to\infty}\int_\Omega |(-\Delta)U_1(x,t)-(-\Delta)U_2(x,t)|^2\,\mathrm{d}\Omega = 0. \tag{11}$$

We need to estimate $U = U_1 - U_2$, with U satisfying:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(U,\tilde{U}) + a(U,\tilde{U}) + e(U,\tilde{U}) - b(U,U,\tilde{U}) + b(U_1,U,\tilde{U})& \\ + b(U,U_1,\tilde{U}) = (F-G,\tilde{U}).&\end{aligned} \tag{12}$$

Taking $\tilde{U} = (-\Delta)^2 U$, we find:

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|\Delta U|^2 + c_0|(-\Delta)^{3/2}U|^2 \leqslant |(F-G,(-\Delta)^2U)| + |b(U_1,U,(-\Delta)^2U)|& \\ + |b(U,U_1,(-\Delta)^2U)| + |b(U,U,(-\Delta)^2U)|.&\end{aligned} \tag{13}$$

We need to estimate each term from the RHS of (13):

$$|b(U,U,(-\Delta)^2U)| \leqslant c|U|_2^{3/2}|U|_3^{3/2} \leqslant \frac{c_0}{8}|(-\Delta)^{3/2}U|_H^2 + \frac{c}{c_0^3}|U|_2^6. \tag{14}$$

We also have:

$$\begin{aligned}|b(U_1,U,(-\Delta)^2U)| &\leqslant c(|U_1|_2|U|_2^{1/2}|U|_3^{3/2} + |U_1|_2^{1/2}|U_1|_3^{1/2}|U|_1|U|_3 + |U_1|_3|U|_2^{3/2}|U|_3^{1/2}) \\ &\leqslant \frac{c_0}{8}|(-\Delta)^{3/2}U|_H^2 + \frac{c}{c_0^3}|U_1|_2^4|U|_2^2 \\ &\quad + \frac{c}{c_0^3}|U_1|_3^{4/3}|U|_2^2 + \frac{c}{c_0}|U_1|_2|U_1|_3|U|_1^2,\end{aligned} \tag{15}$$

and

$$\begin{aligned}|b(U,U_1,(-\Delta)^2U)| &\leqslant c(|U_1|_3|U|_2^{3/2}|U|_3^{1/2} + |U_1|_2|U|_2^{1/2}|U|_3^{3/2} + |U_1|_2^{1/2}|U_1|_3^{1/2}|U|_2|U|_3) \\ &\leqslant \frac{c_0}{8}|(-\Delta)^{3/2}U|_H^2 + \frac{c}{c_0^3}|U_1|_3^{4/3}|U|_2^2 \\ &\quad + \frac{c}{c_0^3}|U_1|_2^4|U|_2^2 + \frac{c}{c_0}|U_1|_2|U_1|_3|U|_2^2.\end{aligned} \tag{16}$$

The forcing term is easily bounded as:

$$\begin{aligned} |(F-G,(-\Delta)^2U)_H| &\leqslant c\|F-G\||(-\Delta)^{3/2}U|_H \\ &\leqslant \frac{c_0}{8}|(-\Delta)^{3/2}U|_H^2+\frac{c}{c_0}\|F-G\|^2. \end{aligned} \tag{17}$$

We thus find:

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}|\Delta U|_H^2+\frac{c_0}{2}|(-\Delta)^{3/2}U|_H^2 &\leqslant \frac{c}{c_0^3}|U|_2^6+\frac{c}{c_0^3}|U_1|_2^4|U|_2^2 \\ &+\frac{c}{c_0^3}|U_1|_3^{4/3}|U|_2^2+\frac{c}{c_0}|U_1|_2|U_1|_3|U|_2^2+\frac{c}{c_0}\|F-G\|^2. \end{aligned} \tag{18}$$

From the first technical lemma, we have:

$$|(-\Delta)^{3/2}f|^2\geqslant\frac{1}{c_2l^2}(|\Delta f|^2-c_1l^{-4}\eta(f)^2),\mathrm{w}ith\quad c_1,\ c_2\quad absolute\quad constants,$$

and inserting this property into (18), we find:

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}|\Delta U|_H^2+|\Delta U|_H^2(c_0c_3l^{-2}-c_4c_0{}^{-3}|U|_2^4-c_5c_0{}^{-3}|U_1|_2^4-c_6c_0^{-3}|U_1|_3^{4/3}-c_7c_0{}^{-1}|U_1|_3^2) \\ &\quad\leqslant c_8c_0^{-1}\|F-G\|^2+c_9c_0l^{-6}\eta(U)^2. \end{aligned} \tag{19}$$

We need to apply the generalized Gronwall lemma to (19):
$\beta(t) = c_8 c_0^{-1} \|F - G\|^2 + c_9 c_0 l^{-6} \eta(U)^2$ goes to zero from the assumptions on F, G and U_1, U_2.
We need to check the conditions on

$$\alpha(t) = c_0 c_3 l^{-2} - c_4 c_0{}^{-3} |U|_2^4 - c_5 c_0{}^{-3} |U_1|_2^4 - c_6 c_0^{-3} |U_1|_3^{4/3} - c_7 c_0{}^{-1} |U_1|_3^2.$$

From the Sobolev regularity of the solutions as well as the existence of absorbing sets in all H^m-spaces, we find:

$$\liminf_{t\to\infty} \frac{1}{s} \int_t^{t+s} \alpha(\tau)\,\mathrm{d}\tau \geqslant c_0 c_3 l^{-2} - C(F, G, c_0, \Omega),$$

and taking l small enough, the condition on α is assured.

Theorem

Let the domain Ω be covered by N identical cubes of volume l^3 and consider a set $\mathcal{E} = \{x^1, x^2, ..., x^N\}$ of points in Ω, distributed one in each cube. Let F and G be two forcing terms in $L^\infty(0, \infty; V)$ satisfying (6). Then, there exists a constant $C = C(F, G, \nu, \mu, \Omega)$ such that if

$$l^{-2} \geqslant C(F, G, \nu, \mu, \Omega),$$

the set $\mathcal{E}$ is a set of determining nodes.

Let F, G be two forcing terms having the same asymptotic behavior:

$$\lim_{t\to\infty}\int_\Omega |F(x,t)-G(x,t)|^2\,\mathrm{d}\Omega = 0. \tag{20}$$

Let U and V be two solutions, corresponding respectively to F and G, that can be expanded as:

$$U(t,x)=\sum_{k=1}^{\infty} U_k(t)W_k(x), \quad V(t,x)=\sum_{k=1}^{\infty} V_k(t)W_k(x),$$

with $\{W_k\}_k$ complete orthonormal basis in $\boldsymbol{H}$, $AW_k=\lambda_k W_k$.
Let P_m be the projection associated to the first m modes and $Q_m = I - P_m$.

Determining modes

A set of modes $\{W_j\}_{j=1}^m$ is called a set of *determining modes* if the condition

$$\int_\Omega |P_mU(t,x)-P_mV(t,x)|^2\,\mathrm{d}\Omega \to 0 \quad \text{as } t\to\infty, \tag{21}$$

implies

$$\int_\Omega |U(t,x)-V(t,x)|^2\,\mathrm{d}\Omega \to 0 \quad \text{as } t\to\infty. \tag{22}$$

Let $W = U - V$. We assume $|P_m W| \to 0$ as $t \to \infty$. The equation for W is:

$$\frac{\mathrm{d}W}{\mathrm{d}t} + AW + B(U, W) + B(W, U) - B(W, W) + EW = F(t) - G(t), \quad W(0) = 0. \tag{23}$$

Taking the scalar product in H of (23) with $Q_m W$, we find:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}|Q_m W|_H^2 + c_0\|Q_m W\|^2 \leqslant |(F - G, Q_m W)_H| + |b(U, W, Q_m W)| + |b(W, U, Q_m W)| + |b(W, W, Q_m W)|. \tag{24}$$

Estimating the RHS of (24) and using $\|P_m W\| \leqslant \lambda_m^{1/2}|P_m W|_H$ and $|P_m W|_2 \leqslant \lambda_m |P_m W|_H$, we find:

$$\frac{\mathrm{d}}{\mathrm{d}t}|Q_m W|_H^2 + c_0\|Q_m W\|^2 \leqslant \frac{c}{c_0^3}\lambda_m\|U\|^{4/3}|P_m W|_H^{4/3}|Q_m W|_H^{2/3} + \frac{c}{c_0^3}\|U\|^2|U|_2^2|Q_m W|_H^2 + \frac{c}{c_0^3}\lambda_m^{7/3}|P_m W|_H^{8/3}|Q_m W|_H^{2/3} + \frac{c}{c_0^3}\lambda_m^3|P_m W|_H^4|Q_m W|_H^2 + \frac{c}{c_0}|F - G|_H^2 \tag{25}$$

Using the fact that $\lambda_{m+1}^{1/2}|Q_m W| \leqslant \|Q_m W\|$, (25) becomes:

$$\frac{\mathrm{d}}{\mathrm{d}t}|Q_m W|_H^2 + \alpha(t)|Q_m W|_H^2 \leqslant \beta(t), \tag{26}$$

with

$$\alpha(t) = c_0\lambda_{m+1} - \frac{c}{c_0^3}\|U\|^2|U|_2^2,$$

and

$$\begin{aligned}\beta(t) = &\frac{c}{c_0^3}\lambda_m\|U\|^{4/3}|P_m W|_H^{4/3}|Q_m W|_H^{2/3} + \frac{c}{c_0^3}\lambda_m^{7/3}|P_m W|_H^{8/3}|Q_m W|_H^{2/3}\\ &+ \frac{c}{c_0^3}\lambda_m^3|P_m W|_H^4|Q_m W|_H^2 + \frac{c}{c_0}|F - G|^2.\end{aligned} \tag{27}$$

We have $\beta(t) \to 0$ as $t \to \infty$ from the initial assumption on $|P_m W|$ and $|F - G|$.

Using the Sobolev regularity results on the solutions, we know:

$$\begin{aligned} |U(t)|_1 &\leqslant K_0(F), \ \forall\, t \geqslant t_0(U_0, F, \Omega, \nu, \mu) \\ |U(s)|_2 &\leqslant K_1(F), \ \forall\, t \geqslant t_1(U_0, F, \Omega, \nu, \mu), \end{aligned} \tag{28}$$

so in order to have

$$\liminf_{t\to\infty} \frac{1}{T}\int_t^{t+T} \alpha(\tau)\,d\tau > 0,$$

we ask that:

$$\lambda_{m+1} \geqslant \frac{c}{c_0^4} K_0(F) K_1(F). \tag{29}$$

Theorem

Suppose that $m \in \mathbb{N}$ is such that:

$$\lambda_{m+1} \geqslant \frac{c}{c_0^4} K_0(F) K_1(F),$$

with c is a constant depending only on the shape of the domain. Then, the first m modes are determining.

Remark

Using for the trilinear terms an estimate of the type:

$$|b(U, U^\sharp, \tilde{U})| \leqslant c\|U\||\nabla U^\sharp|_\infty|\tilde{U}|,$$

we estimate:

$$|b(Q_m W, U, Q_m W)| \leqslant c|\nabla U|_\infty |Q_m W|\|Q_m W\| \leqslant \frac{c_0}{12}\|Q_m W\|^2 + \frac{c}{c_0}|\nabla U|^2_\infty |Q_m W|^2.$$

We apply the generalized Gronwall inequality for an modified α:

$$\alpha(t) = c_0\lambda_{m+1} - \frac{c}{c_0}|\nabla U|^2_\infty.$$

Thus, we define a modified mean rate of energy dissipation as:

$$\epsilon_\infty = c_0 \inf_{T>0}(\limsup_{t\to\infty} \frac{1}{T}\int_t^{t+T} |\nabla U(s)|^2_\infty \,\mathrm{d}s),$$

and we find that if $m \in \mathbb{N}$ is such that $\lambda_{m+1} \geqslant \frac{c}{{c_0}^3}\epsilon_\infty$, then the first m modes are determining.

Exponential Decay of the Power Spectrum of the Solutions

In the context of the Navier-Stokes equations:

- Foias, Temam (1989): Gevrey class regularity of sol. of the NS equations
- Doering, Titi (1996): Exponential decay rate of the power spectrum for sol. for NS

We recall that we work with periodic functions, so they write as:

$$f(x_1, x_2, x_3, t) = \sum_{k \in \mathbb{R}^3} f_k(t) e^{i(k_1' x_1 + k_2' x_2 + k_3' x_3)},$$

with $k_j' = 2\pi k_j / L_j$, for $j = 1, 2, 3$. We introduce the following notation:

$$[U_l]_\kappa^2 = |u_l|^2 + |v_l|^2 + \kappa |T_l|^2, \ \forall\, l \in \mathbb{Z}^3.$$

We define the Gevrey class $D(e^{\tau(-\Delta)^{1/2}})$, $\tau > 0$ as the set of functions U in H satisfying

$$|\Omega| \sum_{k \in \mathbb{Z}^3} e^{2\tau |k'|} [U_k]_\kappa^2 = |e^{\tau(-\Delta)^{1/2}} U|_H^2 < \infty. \tag{30}$$

The norm of the Hilbert space $D(e^{\tau(-\Delta)^{1/2}})$ is given by

$$|U|_\tau := |U|_{D(e^{\tau(-\Delta)^{1/2}})} = |e^{\tau(-\Delta)^{1/2}} U|_H, \ \text{for } U \in D(e^{\tau(-\Delta)^{1/2}}). \tag{31}$$

Another Gevrey space that we will use is $D((-\Delta)^{m/2}e^{\tau(-\Delta)^{1/2}})$, $m \geqslant 1$ integer, which is a Hilbert space when endowed with the inner product:

$$(U, V)_{D((-\Delta)^{m/2}e^{\tau(-\Delta)^{1/2}})} = ((-\Delta)^{m/2}e^{\tau(-\Delta)^{1/2}}U, (-\Delta)^{m/2}e^{\tau(-\Delta)^{1/2}}V)_H; \quad (32)$$

the norm of the space is given by

$$\begin{aligned} |U|^2_{D((-\Delta)^{m/2}e^{\tau(-\Delta)^{1/2}})} &= |(-\Delta)^{m/2}e^{\tau(-\Delta)^{1/2}}U|^2_H \\ &= |\Omega| \sum_{k\in\mathbb{Z}^3} |k'|^{2m}e^{2\tau|k'|}[U_k]^2_\kappa. \end{aligned} \quad (33)$$

Lemma

Let U, $U^\sharp$ and $U^\flat$ be given in $D(-\Delta e^{\tau(-\Delta)^{1/2}})$, for $\tau \geqslant 0$. Then the following inequality holds:

$$|(B(U, U^\sharp), \Delta U^\flat)_\tau| \leqslant c|(-\Delta)^{1/2}U|_\tau^{1/2}|\Delta U|_\tau^{1/2}|\Delta U^\flat|_\tau|(-\Delta)^{1/2}U^\sharp|_\tau^{1/2}|\Delta U^\sharp|_\tau^{1/2} \quad (34)$$

We need to derive a differential equation for $|e^{\tau t(-\Delta)^{1/2}}(-\Delta)^{1/2}U|_H$. We split the solution $U = U^* + \tilde{U}$, where $U^\star$ is the solution of the linear problem:

$$\begin{aligned}
&\frac{\mathrm{d}U^\star}{\mathrm{d}t} + AU^\star + EU^\star = 0, \\
&U^\star(0) = U_0,
\end{aligned} \tag{35}$$

and $\tilde{U}$ is the solution of the following nonlinear problem, in which $U^\star$ is now known:

$$\begin{aligned}
&\frac{\mathrm{d}\tilde{U}}{\mathrm{d}t} + A\tilde{U} + B(\tilde{U}, \tilde{U}) + B(\tilde{U}, U^\star) + B(U^\star, \tilde{U}) + E\tilde{U} = -B(U^\star, U^\star), \\
&\tilde{U}(0) = 0.
\end{aligned} \tag{36}$$

Estimates for the linear part: We apply the operator $e^{\tau t(-\Delta)^{1/2}}$ to (35) and we take the scalar product by $e^{\tau t(-\Delta)^{1/2}}\Delta U^\star$ in H:

$$\begin{aligned}
&(e^{\tau t(-\Delta)^{1/2}}\frac{\mathrm{d}}{\mathrm{d}t}U^\star, e^{\tau t(-\Delta)^{1/2}}(-\Delta)U^\star)_H + (e^{\tau t(-\Delta)^{1/2}}AU^\star, e^{\tau t(-\Delta)^{1/2}}(-\Delta)U^\star)_H \\
&+ (e^{\tau t(-\Delta)^{1/2}}EU^\star, e^{\tau t(-\Delta)^{1/2}}(-\Delta)U^\star)_H = 0.
\end{aligned} \tag{37}$$

We get:

$$\frac{\mathrm{d}}{\mathrm{d}t}|e^{\tau t(-\Delta)^{1/2}}U^\star|_V^2 + \frac{c_0}{2}|e^{\tau t(-\Delta)^{1/2}}\Delta U^\star|_H^2 \leqslant \frac{2\tau^2}{c_0}|e^{\tau t(-\Delta)^{1/2}}U^\star|_V^2. \tag{38}$$

We choose τ such that $2\tau^2/c_0 = c_0/4c'$, with c' the Poincaré constant. Thus, we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}t}|e^{\tau t(-\Delta)^{1/2}}U^\star|_V^2 + \frac{c_0}{4c'}|e^{\tau t(-\Delta)^{1/2}}U^\star|_V^2 \leqslant 0,$$

which implies, using the Gronwall lemma, that:

$$|e^{\tau t(-\Delta)^{1/2}}U^\star|_V^2 \leqslant e^{-\frac{c_0}{4c'}t}|U_0|_V^2. \tag{39}$$

By the same arguments, we can also obtain:

$$|e^{\tau t(-\Delta)^{1/2}}\Delta U^\star|_H^2 \leqslant e^{-\frac{c_0}{4c'}t}|\Delta U_0|_H^2. \tag{40}$$

Estimates for the non-linear part: Estimating the linear and nonlinear terms, we find:

$$\frac{\mathrm{d}}{\mathrm{d}t}|e^{\tau t(-\Delta)^{1/2}}\tilde{U}|_V^2 + \frac{c_0}{2}|e^{\tau t(-\Delta)^{1/2}}\Delta\tilde{U}|_H^2 \leqslant c|e^{\tau t(-\Delta)^{1/2}}\tilde{U}|_V|e^{\tau t(-\Delta)^{1/2}}\Delta\tilde{U}|_H^2 + f(t)|e^{\tau t(-\Delta)^{1/2}}\tilde{U}|_V^2 + g(t), \tag{41}$$

$$f(t) = \frac{c\tau^2}{c_0} + \frac{1}{c_0^2}g(t), \quad g(t) = \frac{c}{c_0}|e^{\tau t(-\Delta)^{1/2}}U^\star|_V^2|e^{\tau t(-\Delta)^{1/2}}\Delta U^\star|_H^2. \tag{42}$$

Since $\tilde{U}(0) = 0$, we assume $|e^{\tau t(-\Delta)^{1/2}}\tilde{U}|_V \leqslant \frac{c_0}{4c}$ for some finite interval $(0, t'_*)$. On $(0, t'_*)$ we find:

$$\frac{\mathrm{d}}{\mathrm{d}t}|e^{\tau t(-\Delta)^{1/2}}\tilde{U}|_V^2 + \frac{c_0}{4}|e^{\tau t(-\Delta)^{1/2}}\Delta\tilde{U}|_H^2 \leqslant f(t)|e^{\tau t(-\Delta)^{1/2}}\tilde{U}|_V^2 + g(t), \tag{43}$$

which implies by the classical Gronwall lemma:

$$|e^{\tau t(-\Delta)^{1/2}}\tilde{U}|_V^2 \leqslant \int_0^t g(s)\exp\Big(\int_s^t f(\xi)\,\mathrm{d}\xi\Big)\,\mathrm{d}s. \tag{44}$$

Since f and g are locally integrable, we can define t'_* as the first time for which we have:

$$\int_0^{t'_*} g(s)\exp\Big(\int_s^{t'_*} f(\xi)\,\mathrm{d}\xi\Big)\,\mathrm{d}s = \frac{c_0^2}{16c^2}. \tag{45}$$

Then, on the interval $(0, t'_*)$, we find:

$$|e^{\tau t(-\Delta)^{1/2}}\tilde{U}|_V \leqslant \frac{c_0}{4c}.$$

We have a rough estimate for t_* as $t_* = k_2/(c_0/8c' + |U_0|_V^2|\Delta U_0|_H^2/c_0^2)$, (k_2 const.)
Over $(0, t_*)$, we have:

$$\begin{aligned} |e^{\tau t(-\Delta)^{1/2}}U|_V^2 &\leqslant 2\{|e^{\tau t(-\Delta)^{1/2}}\tilde{U}|_V^2 + |e^{\tau t(-\Delta)^{1/2}}U^\star|_V^2\} \\ &\leqslant 2\{\frac{2c'}{c_0}|U_0|_V^2|\Delta U_0|_H^2(1 - e^{-\frac{c_0}{2c'}t}) + e^{-\frac{c_0}{4c'}t}|U_0|_V^2\}. \end{aligned} \tag{46}$$

This implies the following decay on the Fourier coefficients of the solution:

$$|U_k(t)|^2 \leqslant 2\frac{L_1L_2L_3}{|k'|^2}e^{-2\tau t|k'|}|U_0|_V^2\{\frac{2c'}{c_0}|\Delta U_0|_H^2(1 - e^{-\frac{c_0}{2c'}t}) + e^{-\frac{c_0}{4c'}t}\}, \ \forall t \leqslant t_*. \tag{47}$$

On t_* we thus have:

$$|U_k(t_*)|^2 \leqslant 2\frac{L_1L_2L_3}{|k'|^2}e^{-\frac{c_0}{\sqrt{2c'}}t_*|k'|}|U_0|_V^2\{\frac{2c'}{c_0}|\Delta U_0|_H^2(1-e^{-\frac{c_0}{2c'}t_*})+e^{-\frac{c_0}{4c'}t_*}\}, \quad (48)$$

and the exponential decay is:

$$\lambda_* = \frac{c_0}{\sqrt{2c'}}t_* = \frac{c_0}{\sqrt{2c'}}\frac{k_2}{c_0/8c' + |U_0|_V^2|\Delta U_0|_H^2/c_0^2}. \quad (49)$$

Since we know that $|U|_V^2$ and $|\Delta U|_H^2$ are bounded uniformly in time, we can reiterate the argument in order to obtain uniform bound on the decay of the Fourier spectrum for $t \geqslant t_*$. The decay after a transient time t_* is thus estimated as:

$$\lambda = \frac{c_0}{\sqrt{2c'}}\frac{k_2}{c_0/8c' + \sup_t|U|_V^2 \sup_t|\Delta U|_H^2/c_0^2}.$$