CHANGE OF PHASE FOR THE HUMID ATMOSPHERE

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[CT] M. Coti Zelati and R. Temam, The atmospheric equation of water vapor with saturation, *Bolletino Un. Mat. Ital.*, **9**, V, 2012, 309-336.

[CFTT] M. Coti Zelati, M. Frémond, R. Temam and J. Tribbia, Uniqueness, regularity and maximum principles for the equations of the atmosphere with humidity and saturation, *Physica D*, to appear.

1. THE EQUATIONS OF THE HUMID ATMOSPHERE

The equations of the atmosphere, also known as the Primitive Equations (PEs) govern the dynamics of the atmosphere, the exchange of heat, and the concentration of water vapor in the atmosphere.
Hence the unknown are

- $\mathbf{u} = (\mathbf{v}, \omega), \mathbf{v} = (u, v)$, the velocity and horizontal velocity of the wind, ω the vertical velocity in the x, y, p coordinate system
- $T =$ the temperature
- $q =$ the content of water vapor in the atmosphere, $0 < q < 1$.

Earlier works on the humid atmosphere, in particular by Lions, Temam and Wang [LTW] and by Guo and Huang [GH1,2] do not account for the possible saturation of water vapor in the atmosphere so that the equation for q is a mere transport equation.

But the condensation and evaporation of water govern rain and play a significant role everywhere in the atmosphere. They play a major role at the equator where the evaporation of water produces humidity which propagates to the mid-latitudes.

[LTW] J.L. Lions, R. Temam and S. Wang, New formulations of the primitive equations of the atmosphere and applications, *Nonlinearity,* **5**, 1992, 237-288.

[GH1] B. Guo and D. Huang, Existence of weak solutions and trajectory attractors for the moist atmospheric equations in geophysics, *J. Math. Phys.*, **47**, 2006.

[GH2] B. Guo and D. Huang, Existence of the universal attractor for the 3-D viscous primitive equations of large-scale moist atmosphere, *J. Differential Equations*, **251**, 2011, 457–491.

The present study, based on the articles [CT], [CFTT] is, to the best of our knowledge, the first mathematical study accounting for the possible saturation of water vapor.

The simplest model is considered, following e.g. [Hal], [RY]; liquid water, solid water (micro-ice) and (anthropogenic or natural) aerosols are not accounted for. Hence after condensation (rain) the water leaves the system.

[Hal] G.J. Haltiner, *Numerical weather prediction*, John Wiley & Sons, New York, 1971.

[RY] R.R. Rogers and M.K. Yau, A Short Course in Cloud Physics, Butterworth-Heinemann, 1989.

Simplifying further the mathematical study, we assume here (as in [CT], [CFTT]), that the dynamics (the velocity) of the atmosphere is known ($\mathbf{u} = (\mathbf{v}, \omega)$ is given) and we concentrate on the equations governing the temperature T and the relative concentration of water vapor q.

2. THE $q - T$ SYSTEM AND ITS MATHEMATICAL MODELING

$$(1) \qquad c_p \frac{\partial T}{\partial t} + c_p \mathbf{v} \cdot \nabla T + c_p \ \omega \frac{\partial T}{\partial p} + \mathcal{A}_T T - \frac{R}{p} \omega T = L\mathcal{F},$$

$$(2) \qquad \frac{\partial q}{\partial t} + \mathbf{v} \cdot \nabla q + \omega \frac{\partial q}{\partial p} + \mathcal{A}_q \ q = -\mathcal{F}.$$

Here $(x, y) \in \mathcal{M}'$ a bounded smooth domain in $\mathbb{R}^2, p \in (p_0, p_1)$ is the vertical (pressure) coordinate $0 < p_0 < p_1$; hence the "spatial" domain is $\mathcal{M} = \mathcal{M}' \times (p_0, p_1)$; $\mathbf{u} = (\mathbf{v}, \omega)$ is the *given* velocity, $\Delta = \partial_x^2 + \partial_y^2, \nabla = (\partial_x, \partial_y), \mathcal{A}_q$ and $\mathcal{A}_q$ are elliptic versions of the Laplacian in the (x, y, p) system (see below), L is a constant.

The term $\mathcal{F}$

We leave for the moment the boundary conditions and other aspects of equations (1), (2) and we concentrate on the term $\mathcal{F}$. We read in the literature ([Hal], [RY]):

$$\mathcal{F} = -\frac{1}{p}\omega F(T) \begin{cases} \text{if } \omega < 0 \text{ and } q > q_s \\ 0 \text{ otherwise,} \end{cases} \tag{3}$$

with

$$F = F(T) = q_s T \frac{LR - c_p R_v T}{C_p R_v T^2 + q_s L^2} \tag{4}$$

[Hal] G.J. Haltiner, *Numerical weather prediction*, John Wiley & Sons, New York, 1971.

[RY] R.R. Rogers and M.K. Yau, A Short Course in Cloud Physics, Butterworth-Heinemann, 1989.

Here q_s is the saturation concentration, $0 < q_s < 1$ which depends on T but is assumed to be constant for simplicity; L, c_p, R_v are constant described below.

The issue is now how to mathematically express (3). We do not worry about ω which is given and we replace $-\omega$ by $\omega^- = \max(-\omega, 0)$. Hence

$$\mathcal{F} = \begin{cases} \frac{1}{p}\omega^- F(T)^{\cdot} \text{ if } q > q_s, \\ 0 \text{ if } q < q_s. \end{cases} \tag{5}$$

As in all change of phase problems [Fre] we have to deal here with a discontinuous (nonlinear) function of q.

[Fre] M. Frémond, *Phase Change in Mechanics*, Lecture Notes of the Unione Matematica Italiana, Vol. 13, Springer Verlag, 2012.

Equations (5) does not say what is $\mathcal{F}$ when $q = q_s$. In fact describing $\mathcal{F}$ at $q = q_s$ would necessitate *a more refined model* accounting for the micro-physics occurring at condensation and evaporation.

For the mathematical modeling we replaced $\mathcal{F}$ by

$$\frac{1}{p}\omega^{-}F(T)H(q - q_s),$$

where H is the Heaviside function with $H(0) = [0, 1]$, so that (1), (2) become

$$
c_p \frac{\partial T}{\partial t} + \mathcal{A}_T T + c_p \mathbf{v}(t) \cdot \nabla T + c_p \;\omega \frac{\partial T}{\partial p} - \frac{R}{p} \omega T \in \frac{L}{p} \omega^- H(q - q_s) F(T), \tag{6}
$$

$$
\frac{\partial q}{\partial t} + \mathcal{A}_q \; q + \mathbf{v} \cdot \nabla q + \omega \frac{\partial q}{\partial p} \in -\frac{1}{p} \omega^- H(q - q_s) F(T). \tag{7}
$$

Alternatively we can write (6), (7) as

$$
\begin{aligned}
c_p \frac{\partial T}{\partial t} + \mathcal{A}_T T + c_p \mathbf{v} \cdot \nabla T \\
+ c_p \ \omega \frac{\partial T}{\partial p} - \frac{R}{p} \omega T = \frac{L}{p} \omega^- h_T F(T),
\end{aligned}
\tag{8}
$$

$$
\frac{\partial q}{\partial t} + \mathcal{A}_q \ q + \mathbf{v} \cdot \nabla q + \omega \frac{\partial q}{\partial p} = -\frac{1}{p} \omega^- h_q F(T),
\tag{9}
$$

with $h_T, h_q \in H(q - q_s)$, unspecified.

“Remarkably" we proved in [CT] an existence result for the boundary value problem associated with (8), (9), *with the same function* $h_T = h_q = h \in H(q - q_s)$.

This is fully satisfactory from the mathematical viewpoint but it raises, from the physical viewpoint, the question of the validity of this modeling in relation with *the uniqueness* of solution. We now address the question of uniqueness, which has been studied in [CFTT].

3. THE UNIQUENESS ISSUE

We can perceive the problem of the uniquess (addressed in [CFTT]), by assuming that the flow is spacially homogeneous, so that (6), (7) reduce to the following system of ordinary differential equations:

$$c_p \frac{dT}{dt} - \frac{RT}{p}\omega \in -f(T,q)LH(q-q_s), \tag{10}$$

$$\frac{dq}{dt} \in f(T,q)H(q-q_s), \tag{11}$$

with

$$f = f(T,q) = -\frac{F\omega^-}{p}. \tag{12}$$

We can already perceive the issue of uniqueness by considering the even simpler ODEs:

$$T' \in H(T), \quad T(0) = T_0, \tag{13}$$

$$q' \in -H(q), \quad q(0) = q_0. \tag{14}$$

It is well known/easy to verify, that the solution of (14) is unique and that the solution of (13) is not unique and even unstable, when the initial data T_0, q_0 are ≥ 0.

The difference comes from the sign in front of the function H, because H is a monotone (multivaled) function, but not $-H$.

With this in mind we see that the solution of (11) with $q(0) = q_0$ and T prescribed is unique; similarly the solution of (10) with $T(0) = T_0$ and q prescribed is unique.

However this reasoning does not provide the uniqueness for the *coupled* system (10), (11).

The remedy, suggested by the physicists, is to introduce the *moist static energy*:

$$e = c_p T + L_q,$$

and to replace (10) by

$$\text{(15)} \qquad c_p \frac{de}{dt} - \frac{R\omega}{pc_p}(e - Lq) = 0; \quad e_0 = cpT_0 + Lq_0.$$

Essential here is the fact that the same $h \in H(q - q_s)$ *appears in* (10) *and* (11).

The existence of solution for (10), (11) *with the same h* is proven as in the PDE case.

The uniqueness of solution for (11), (15) is then easy.

4. FORMULATION OF THE PROBLEM. THE MAIN RESULTS.

We return to the equations (1), (2) or (8), (9).

The operators $\mathcal{A}$ are an expression of the Laplacian in the (x, y, p) coordinates:

$$\mathcal{A}T = -\mu_T \Delta - \nu_T \frac{\partial}{\partial p}\left[\left(\frac{gp}{R\overline{\overline{T}}}\right)^2 \frac{\partial}{\partial p}\right], \tag{16}$$

$$\mathcal{A}q = -\mu_q \Delta - \nu_q \frac{\partial}{\partial p}\left[\left(\frac{gp}{R\overline{\overline{T}}}\right)^2 \frac{\partial}{\partial p}\right], \tag{17}$$

where $\mu_T, \nu_T, \mu_q, \nu_q, g, R, c_p$ are positive constants and $\overline{\overline{T}} = \overline{\overline{T}}(p)$ is the average temperature over the isobar with pressure p.

Boundary Conditions

We partition the boundary of $\mathcal{M}$ as

$$
\begin{aligned}
&\Gamma_i = \left\{(x, y, p) \in \overline{\mathcal{M}} : p = p_1\right\}, \\
(18) \qquad &\Gamma_u = \left\{(x, y, p) \in \overline{\mathcal{M}} : p = p_0\right\}, \\
&\Gamma_\ell = \left\{(x, y, p) \in \overline{\mathcal{M}} : (x, y) \in \partial\mathcal{M},\ p_0 \le p \le p_1\right\},
\end{aligned}
$$

and write the following physically suitable boundary conditions available in the literature (Gill, Haltiner, LTW):

$$
\begin{aligned}
&\text{on } \Gamma_i : \frac{\partial T}{\partial p} = \alpha_T(T_* - T), \quad \frac{\partial q}{\partial p} = \alpha_q(q_* - q), \\
(19) \qquad &\text{on } \Gamma_u : \frac{\partial T}{\partial p} = 0, \quad \frac{\partial q}{\partial p} = 0, \\
&\text{on } \Gamma_\ell : \frac{\partial T}{\partial n} = 0, \quad \frac{\partial q}{\partial n} = 0.
\end{aligned}
$$

Here, n is the unit outward normal vector to $\mathcal{M}$, the functions $T_* = T_*(x, y, t)$ and $q_* = q_*(x, y, t)$ are typical temperature and specific humidity distributions at the bottom of the atmosphere, and α_T, α_q are given positive constants. Finally, we supplement our system with the initial conditions

$$T(x,y,p,0) = T_0(x,y,p), \qquad q(x,y,p,0) = q_0(x,y,p), \quad (x,y,p) \in \mathcal{M}. \tag{20}$$

More about F

$$F(\xi) = q_s \xi \left(\frac{LR - c_p R_v \xi}{c_p R_v \xi^2 + q_s L^2} \right), \tag{21}$$

where R_v is equal to the gas constant for water vapor, L is the latent heat of condensation of water, R is the gas constant for air, c_p is the specific heat capacity and q_s is the saturation value of q; in general, L is a slightly varying function of T (see [RY]), which we here take constant. From direct calculations, we see that F is a globally Lipschitz bounded function, namely

$$|F(\xi_1) - F(\xi_2)| \leq c_F |\xi_1 - \xi_2|, \qquad \forall \xi_1, \xi_2 \in \mathbb{R}, \tag{22}$$

and

$$|F(\xi)| \leq C_F, \qquad \forall \xi \in \mathbb{R}. \tag{23}$$

Since $F(0) = 0$, from (22) we immediately see that, beside (23):

$$|F(\xi)| \leq c_F |\xi|, \qquad \forall \xi \in \mathbb{R}. \tag{24}$$

Notice that $F(\xi) \geq 0$ whenever $\xi \geq 0$ and $LR - c_p R_v \xi \geq 0$. Moreover, $F(0) = 0$ and

$$F(\xi_0) = 0 \quad \text{for} \quad \xi_0 = \frac{LR}{c_p R_v}. \tag{25}$$

Therefore,

$$F(\xi) \geq 0 \quad \Leftrightarrow \quad \xi \in [0, \xi_0]. \tag{26}$$

For usual values of the above constants, namely (see [Gill])

$$L = 2.5 \times 10^6 \, Jkg^{-1}, \quad R = 287 \, JK^{-1} kg^{-1},$$
$$R_v = 461.50 \, JK^{-1} kg^{-1}, \quad c_p = 1004 \, JK^{-1} kg^{-1},$$

it turns out that $\xi_0 \sim 1548K$. Hence, $F(\xi) \geq 0$ for $0 \leq \xi \leq 1548$, which is a temperature much higher than any atmospheric temperature (absolute temperature in Kelvin).

We recall the equations

$$c_p\frac{\partial T}{\partial t}+\mathcal{A}_T T + c_p\mathbf{v}\cdot\nabla T + c_p\ \omega\frac{\partial T}{\partial p} - \frac{R}{p}\omega T = h_q\frac{L}{p}\omega^- F(T), \tag{27}$$

$$\frac{\partial q}{\partial t} + \mathcal{A}_q\ q + \mathbf{v}\cdot\nabla q + \omega\frac{\partial q}{\partial p} = -h_q\frac{1}{p}\omega^- F(T), \tag{28}$$

$$T(x,y,p,0) = T_0(x,y,p), \quad q(x,y,p,0) = q_0(x,y,p), \quad (x,y,p)\in\mathcal{M}, \tag{29}$$

where

$$h_q \in H(q-q_s) \tag{30}$$

which can be expressed by the variational inequality,

$$([q^\flat - q_s]^+, 1) - ([q-q_s]^+, 1) \geq \langle h_q, q^\flat - q\rangle, \quad \forall q^\flat \in V, \text{ and for a.e. } t\in[0,t_1], \tag{31}$$

The boundary data T_*, q_* appearing in (19) will be assumed to satisfy

$$T_*, q_* \in L^2(0, t_1; L^2(\Gamma_i)). \tag{32}$$

Concerning the average temperature $\overline{T}(p)$ appearing in (16)-(17), we will require the existence of two positive constants $\overline{T}_*, \overline{T}^*$ such that

$$0 < \overline{T}_* \leq \overline{T}(p) \leq \overline{T}^*, \tag{33}$$

along with a boundedness assumption on its derivative needed at some point

$$|\partial_p \overline{T}(p)| \leq \overline{C}. \tag{34}$$

The velocity vector field $\mathbf{u}$ is given, time-dependent, and satisfying

$$\mathbf{u} = (\mathbf{v}, \omega) \in L^4(0, t_1; \mathbf{V}) \cap L^\infty(0, t_1, \mathbf{H}). \tag{35}$$

The main result

Theorem 1

The assumptions are those above. Then for any

$$(T_0, q_0) \in H \times H,$$

and any

$$t_1 > 0,$$

given, problem (27)-(31)*,* (16)-(21)*, admits a unique solution*

$$(T, q) \in L^2(0, t_1; V \times V) \cap C([0, t_1]; H \times H),$$

with

$$\partial_t (T, q) \in L^2(0, t_1; (V \times V)^*).$$

5. IDEA OF THE PROOF

The uniqueness is proven somehow as in the ODE case, introducing the *moist static energy*

$$e = c_p T + Lq.$$

However we need a careful monitoring of the boundary terms and certain additional terms in the equations.

The $e - q$ system reads:

$$\begin{aligned} c_p \frac{\partial e}{\partial t} + \mathcal{A}_T e + L(c_p \mathcal{A}_q - \mathcal{A}_T) q + c_p \mathbf{v} \cdot \nabla e \\ + c_p \ \omega \frac{\partial e}{\partial p} - \frac{R}{p} \omega (e - Lq) = 0, \end{aligned} \tag{36}$$

$$\frac{\partial q}{\partial t} + \mathcal{A}_q \ q + \mathbf{v} \cdot \nabla q + \omega \frac{\partial q}{\partial p} = -\frac{1}{p} \omega^- h_q F\left(\frac{e - Lq}{c_p}\right), \tag{37}$$

The boundary conditions read.

$$\text{(38)} \quad \begin{cases} \text{on } \Gamma_i : \frac{\partial e}{\partial p} = \alpha_T(e_* - e) + L\,(\alpha_q - \alpha_T)\,(q_* - q), \\ \qquad\qquad\qquad\qquad \frac{\partial q}{\partial p} = \alpha_q(q_* - q), \\ \text{on } \Gamma_u : \frac{\partial e}{\partial p} = 0, \quad \frac{\partial q}{\partial p} = 0, \\ \text{on } \Gamma_\ell : \frac{\partial e}{\partial n} = 0, \quad \frac{\partial q}{\partial n} = 0, \end{cases}$$

where $e_* = c_p T_* + L q_*$.

The initial conditions now read

$$\text{(39)} \quad \begin{aligned} e(x, y, p, 0) = e_0(x, y, p), \;\; q(x, y, p, 0) = q_0(x, y, p), \\ (x, y, p) \in \mathcal{M}, \end{aligned}$$

where $e_0 = c_p T_0 + L q_0$.

Proof of existence

We construct an approximation of the problem using an approximation (a regularization) of the Heaviside function, that is:

For $\varepsilon \in (0,1]$, we define the real functions

$$H_\varepsilon(r) = \begin{cases} 0, & r \leq 0, \\ r/\varepsilon, & r \in (0,\varepsilon], \\ 1, & r > \varepsilon, \end{cases} \qquad K_\varepsilon(r) = \begin{cases} 0, & r \leq 0, \\ r^2/2\varepsilon, & r \in (0,\varepsilon], \\ r - \varepsilon/2, & r > \varepsilon. \end{cases}$$

The regularized version of equations (27), (28) read

$$
\begin{aligned}
(40)\quad & c_p\frac{\partial T_\varepsilon}{\partial t} + \mathcal{A}_T T_\varepsilon + c_p\mathbf{v}\cdot\nabla T_\varepsilon + c_p\omega\frac{\partial T_\varepsilon}{\partial p} - \frac{R}{p}\omega T_\varepsilon \\
& \qquad = H'_\varepsilon(q_\varepsilon - q_s)\frac{L}{p}\omega^- F(T_\varepsilon),
\end{aligned}
$$

$$
(41)\quad \frac{\partial q_\varepsilon}{\partial t} + \mathcal{A}_q q_\varepsilon + \mathbf{v}\cdot\nabla q_\varepsilon + \omega\frac{\partial q_\varepsilon}{\partial p} = -H'_\varepsilon(q_\varepsilon - q_s)\frac{1}{p}\omega^- F(T_\varepsilon),
$$

(42)

$$
\begin{aligned}
T_\varepsilon(x,y,p,0) = T_0(x,y,p), \qquad & q_\varepsilon(x,y,p,0) = q_0(x,y,p), \\
& (x,y,p)\in\mathcal{M},
\end{aligned}
$$

Equations (40) - (42) supplemented with the same boundary and initial conditions form a classical system with a (relatively) smooth nonlinearity.

Existence and uniqueness of solutions for the regularized problem are standard.

A priori estimates

We obtain, for $T_\varepsilon, q_\varepsilon$, the following a priori estimates independent of ε:

(43) T_ε and q_ε remain bounded in $L^\infty(0, t_1; L^2(\mathcal{M}))$ and $L^2(0, t_1; H^1(\mathcal{M}))$

(44) $\dfrac{\partial T_\varepsilon}{\partial t}$ and $\dfrac{\partial q_\varepsilon}{\partial t}$ remain bounded in $L^2(0, t_1; H^{-1}(\mathcal{M}))$,

(45) $H(q_\varepsilon - q_s)$ remains bounded in $L^\infty((0, t_1) \times \mathcal{M})$.

We obtain the existence of solutions to the original problem by passing to the limit $\varepsilon \to 0$, using the relation (31) which expresses the fact that,

$$h_q = \text{ the weak star limit of } H(q_\varepsilon - q_s),$$

belongs to $\partial H(q - q_s)$, the sub differential of H at $q - q_s$.

6. ADDITIONAL RESULTS

Theorem 2 (Maximum principle for T)
With the above hypotheses, let $T \in L^2(0, t_1; V) \cap C([0, t_1]; H)$ be a solution of the problem with either F or F^+ as the nonlinear function. If $T_0 \in L^\infty(\mathcal{M})$, $T_* \in L^\infty(\Gamma_i \times [0, t_1])$ are positive and ω is such that $\omega^+ \in L^\infty(\mathcal{M} \times [0, t_1])$, then

$$0 \leq T \leq M_1 \mathrm{e}^{\varpi t_1}, \qquad \textit{a.e. in } \mathcal{M} \times [0, t_1],$$

where

$$M_1 = \max \left\{ \frac{LR}{c_p R_v}, \|T_0\|_{L^\infty(\mathcal{M})}, \|T_*\|_{L^\infty(\Gamma_i \times [0,t_1])} \right\}$$

and

$$\varpi = \frac{R}{p_0 c_p} \|\omega^+\|_{L^\infty(\mathcal{M} \times [0,t_1])}.$$

Theorem 3 (Maximum Principle for q)

With the hypotheses above, assume that the function $q \in L^2(0, t_1; V) \cap C([0, t_1]; H)$ is a solution of the problem with the nonlinear function F replaced by its positive part F^+. If $q_0 \in L^\infty(\mathcal{M})$ and $q_* \in L^\infty(\Gamma_i \times [0, t_1])$ are positive, then

$$0 \leq q \leq M_2 := \max\left\{\|q_*\|_{L^\infty(\Gamma_i\times[0,t_1])}, \|q_0\|_{L^\infty(\mathcal{M})}\right\}, \text{ a.e. in } \mathcal{M}\times[0, t_1].$$

In particular $0 \leq q \leq 1$ if $0 \leq q_*, q_0 \leq 1$, and $0 \leq q \leq q_s$, if $0 \leq q_*, q_0 \leq q_s$.

The proofs of the maximum principles are based on the *Stampacchia maximum principle*.

We now assume a stronger hypothesis on $\mathbf{v}, \omega$ namely

$$\mathbf{u} = (\mathbf{v}, \omega) \in \mathbf{L}^\infty(\mathcal{M} \times [0, t_1]). \tag{46}$$

Then we obtain

Theorem 4 (Additional Regularity for T)

The hypotheses are those above and we consider the system with either F or F^+. Assume that $T \in L^2(0, t_1; V) \cap C([0, t_1]; H)$ is a solution for some $h_q \in L^\infty(\mathcal{M} \times [0, t_1])$. If $T_0 \in V$ and the boundary data T_* satisfies

$$T_* \in L^2(0, t_1; H^1(\Gamma_i)), \partial_t T_* \in L^2(0, t_1; L^2(\Gamma_i)),$$

then

$$T \in L^2(0, t_1; H^2(\mathcal{M})) \cap C([0, t_1]; V)$$

and

$$\partial_t T \in L^2(0, t_1; H).$$

Theorem 5 (Additional regularity for q)

The hypotheses are those of above and we consider the system with either F or F^+. Assume that $q \in L^2(0, t_1; V) \cap C([0, t_1]; H)$ is a solution of the problem for some $h_q \in L^\infty(\mathcal{M} \times [0, t_1])$. If $q_0 \in V$ and the boundary data q_* satisfies

$$q_* \in L^2(0, t_1; H^1(\Gamma_i)),$$

$$\partial_t q_* \in L^2(0, t_1; L^2(\Gamma_i)),$$

then

$$q \in L^2(0, t_1; H^2(\mathcal{M})) \cap C(0, t_1; V)$$

and

$$\partial_t q \in L^2(0, t_1; H).$$

The proofs of these additional regularity results are based on the results of *P. Grisvard* and of *M. Ziane* on the regularity of the solutions of elliptic problems in domains with corners.

Thank you!

Humid atmosphere with saturation.tex