Slice Models

Darryl D Holm

Joint work with CJ Cotter and A Visram

Imperial College London

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Abstract

We'll propose a 2.5 D vertical slice model of fluid dynamics which conserves energy and PV, and also specialises to the Eady model. Some stability results will be discussed, too.

The key assumption in deriving the model is:

In taking components of flow velocity parallel and transverse to the slice,

$$u = u_S + u_T \, \hat{y}$$

the slice velocity u_S convects the transverse velocity u_T.

The terms proportional to transverse flow velocity u_T will model vortex stretching, by modifying the definition of circulation and PV.

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We consider Eady's problem in the following domain

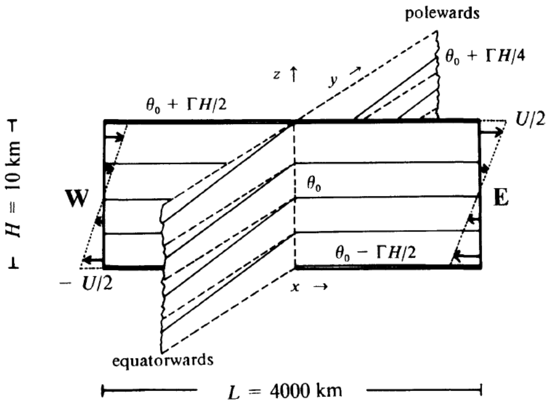


Figure: Basic state for the Eady problem, from Volkert & Bishop, *Tellus*, 42A (1990). 3D flow: $\partial\theta/\partial y = const$, while $\hat{y}\cdot u_S = 0$, $\partial(\hat{x}\cdot u_S)/\partial z = const$ in a vertical x-z slice. This basic flow is unstable to y-independent perturbations in all three components of velocity and temperature, which rapidly produce fronts with large x-gradients that are translation-invariant in the y-direction. This y-independence suggests a slice model.

The perturbed evolution in Eady's problem is unstable

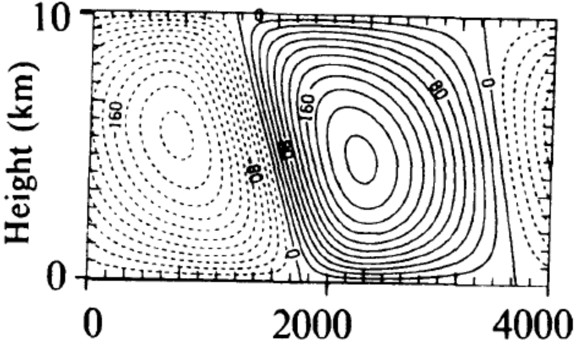


Figure: The Eady problem is interesting, because y-independent perturbations create sharp fronts. From Volkert & Bishop, *Tellus*, 42A (1990). We'll build a slice model which has similar Eady behaviour

We'll derive a new variational Eady slice model

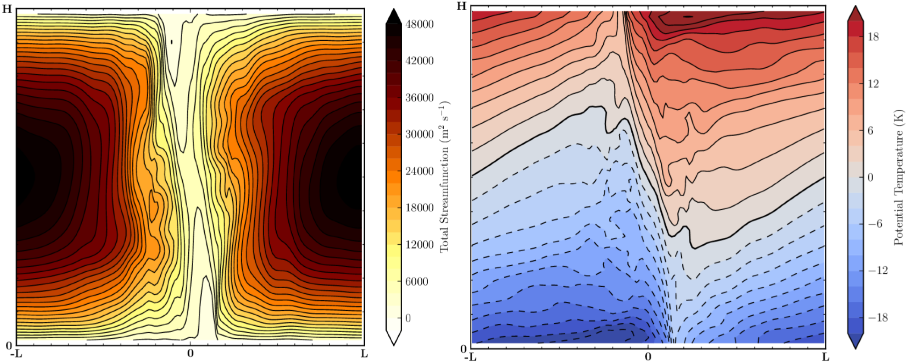
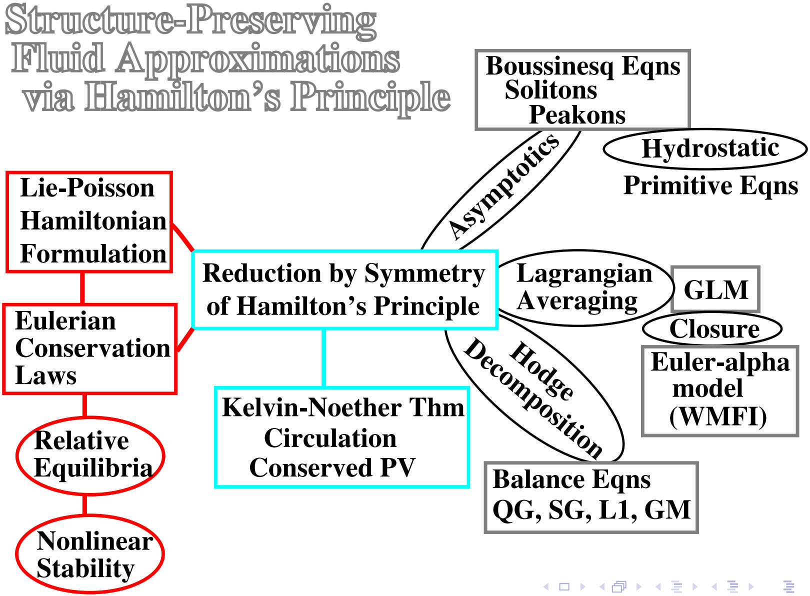


Figure: We'll build a slice model which (1) Has proper Eady behaviour in the incompressible case and (2) Generalises easily to the compressible case. The figure shows stream function (left) and temperature (right) in the slice. (Figure courtesy of A Visram).

We'll make approximations in Hamilton's principle



Building a variational vertical slice model.1

(1) Assume that the forward Lagrangian map takes the form

$$\phi(X, Y, Z, t) = (x(X, Z, t), y(X, Z, t) + Y, z(X, Z, t)) \,, \qquad (1)$$

where (X, Y, Z) are Lagrangian labels, (x, y, z) are particle locations and t is time, *i.e.*

$$\frac{\partial \phi}{\partial Y} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} .$$

Such maps form a subgroup of the diffeomorphisms[1] $\mathrm{Diff}(\Omega \times \mathbb{R})$
– $\Omega \in \mathbb{R}^2$ is the domain in the x-z plane, and
– $\mathbb{R}$ represents an infinite line in the transverse y-direction

[1] Diffeomorphisms are smooth invertible maps with smooth inverses.

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Building a variational vertical slice model.2

(2) Let $\phi \in \text{Diff}(\Omega)$ and $f \in \mathcal{F}(\Omega)$ specify, respectively, the displacement of Lagrangian particles in Ω and along $y \in \mathbb{R}$ at each point in Ω.

Composition of these smooth functions obeys a standard formula

$$(\phi_1, f_1) \cdot (\phi_2, f_2) = (\phi_1 \circ \phi_2, \phi_1 \circ f_2 + f_1). \quad (2)$$

In the tangent space at the identity $T_e \text{Diff}(\Omega \times \mathbb{R})$, the vector field $u_S \in \mathfrak{X}(\Omega)$ represents the two components of the velocity in the x-z plane, and the smooth function $u_T \in \mathcal{F}(\Omega)$ represents the y-component of the velocity.

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Thus, in $T_e \text{Diff}(\Omega \times \mathbb{R})$, a tangent vector is represented by (u_S, u_T).

Building a variational vertical slice model.3

(3) The tangent space $T_e \,\mathrm{Diff}(\Omega \times \mathbb{R})$ has a *Lie bracket* derived by taking the tangent to the composition (2) and evaluating at the identity.

This Lie bracket is

$$[(u_S, u_T), (w_S, w_T)] = ([u_S, w_S],\, u_S \cdot \nabla w_T - w_S \cdot \nabla u_T)\,. \tag{3}$$

Here $[u_S, w_S] = u_S \cdot \nabla w_S - w_S \cdot \nabla u_S$ is the standard Lie bracket for time-dependent vector fields $(u_S, w_S) \in \mathfrak{X}(\Omega)$, and ∇ denotes the gradient in the x-z plane.

Remark

Note that the variations of the Lagrangian paths in Hamilton's variational principle for the slice model will also lie in $T_e \,\mathrm{Diff}(\Omega \times \mathbb{R})$.

Introduce two types of advected quantities

(i) The mass density $D(x, y, z, t)$ satisfies

$$\partial_t D + \nabla \cdot (u_S D) + \partial_y (u_T D) = 0\,,$$

with partial time derivative $\partial_t = \partial/\partial t$ and partial space derivative $\partial_y = \partial/\partial y$ in the y-direction normal to the x-z plane.

For the slice motion u_T and D are taken to be y-independent.

Then conservation of mass reduces to advection of an *areal density* $D\,\mathrm{d}S \in \Lambda^2(\Omega)$, in which $D(x, z, t)$ satisfies the continuity equation,

$$\partial_t D + \nabla \cdot (u_S D) = 0\,. \tag{4}$$

Introduce two types of advected quantities

(ii) Assume potential temperature has a constant gradient in the y-direction, $\partial\theta/\partial y = s = const$. Then advected scalars $\theta(x, y, z, t)$ may be decomposed into dynamic and static parts, as

$$\theta(x, y, z, t) = \theta_S(x, z, t) + (y - y_0)s. \tag{5}$$

Consequently, the scalar tracer equation in 3D

$$\partial_t\theta_S + u_S \cdot \nabla\theta_S + u_T\partial_y\theta = 0\,,$$

becomes a dynamic equation for $\theta_S(x, z, t) \in \mathcal{F}(\Omega)$, which satisfies,

$$\partial_t\theta_S + u_S \cdot \nabla\theta_S + u_T s = 0\,. \tag{6}$$

The variational principle for the Eady model.1

The Euler–Boussinesq Eady model in a periodic domain $(x, z) \in \Omega$ of width L and height b, has Lagrangian

$$
\begin{aligned}
l[u_S, u_T, D, \theta, p] = \int_\Omega \frac{D}{2}\left(|u_S|^2 + u_T^2\right) + D\, f\, u_T x \\
+ \frac{g}{\theta_0} D \left(z - \frac{b}{2}\right)\theta_S + p(1 - D)\, \mathrm{d}S,
\end{aligned}
\tag{7}
$$

where
– f is the Coriolis parameter,
– g is the acceleration due to gravity,
– θ_0 is the reference temperature, and
– the Lagrange multiplier p enforces constant areal density, D.

The variational principle for the Eady model.2

After taking variations and using the Lie bracket on $T_e \,\mathrm{Diff}(\Omega \times \mathbb{R})$, we derive the *standard* Eady model equations,

$$\begin{aligned} \partial_t u_S + u_S \cdot \nabla u_S - f u_T \hat{x} &= -\nabla p + \frac{g}{\theta_0} \theta_S \hat{z}, \\ \partial_t u_T + u_S \cdot \nabla u_T + f u_S \cdot \hat{x} &= -\gamma_S s, \\ \nabla \cdot u_S &= 0, \\ \partial_t \theta_S + u_S \cdot \nabla \theta_S + u_T s &= 0, \end{aligned} \tag{8}$$

where $\hat{x} = \nabla x$ is the unit vector in the x-direction and we've defined

$$\gamma_S := \frac{g}{\theta_0}\left(z - \frac{b}{2}\right).$$

Conserved Kelvin circulation & PV for the Eady model

Define the circulation velocity $v_S := u_S - s^{-1}(u_T + f\,x)\nabla\theta_S$.

Theorem

The Eady model conserves the circulation of v_s on loops carried by u_S

$$\frac{d}{dt}\oint_{c(u_s)} v_S \cdot \mathrm{d}x = \oint_{c(u_s)} \mathrm{d}\left(\frac{1}{2}|u_S|^2 - p + \gamma_S\theta_S\right) = 0 \,. \qquad (9)$$

By Stokes theorem, this becomes a conserved ***area integral***,

$$\frac{d}{dt}\iint_{S(u_s)} \mathrm{curl}\, v_s \cdot \mathrm{d}S = 0\,, \quad \text{where} \quad \partial S(u_s) = c(u_s) \subset \Omega.$$

Corollary

The PV $q := \hat{y} \cdot \mathrm{curl} v_s$ is conserved on fluid parcels,

$$\frac{Dq}{Dt} = \partial_t q + u_S \cdot \nabla q = 0\,.$$

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Energy and Enstrophy conservation for Eady model

Theorem

The Eady model (8) *conserves energy and generalised enstrophy.*

$$h = \int_\Omega \frac{1}{2}|u_S|^2 + \frac{1}{2}u_T^2 - \gamma_S\theta_S \,\mathrm{d}z\,\mathrm{d}x \quad \textit{(Energy)} \quad . \tag{10}$$

$$C_\Phi = \int_\Omega \Phi(q)\,\mathrm{d}z\,\mathrm{d}x \quad \textit{(Generalised enstrophy)} \quad , \tag{11}$$

for any differentiable function Φ *of the PV,* $\quad q := \hat{y}\cdot \mathrm{curl}\, v_S$.

Proof.

The easiest proofs of these conservation laws will follow later from the Hamiltonian formulation of the Eady model equations (8). □

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Variational criterion for Eady equilibrium solutions

Theorem

Critical points of $H_\Phi = h + C_\Phi$ are Eady equilibrium solutions of (8).

Proof.

After a brief calculation one finds

$$\begin{aligned}\delta H_\Phi = \int_\Omega \Big[& \delta u_S \cdot \big(u_S + \nabla\Phi'(q) \times \hat{y}\big) \\ & + \delta u_T \big(u_T - s^{-1}\nabla\Phi'(q) \times \hat{y} \cdot \nabla\theta_S\big) \\ & + \delta\theta_S \big(\nabla\Phi'(q) \times \hat{y} \cdot \nabla(u_t + fx)/s - \gamma_S\big)\Big] \,\mathrm{d}z\,\mathrm{d}x\,. \end{aligned} \tag{12}$$

This calculation demonstrates that vanishing of all of the coefficients of the variational quantities at a critical point $\delta H_\Phi = 0$ provides sufficient conditions for Eady equilibria. (The Hamiltonian proof is easier.) □

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Stability of Eady equilibrium solutions

The linearised equations in the neighbourhood of an Eady equilibrium are Hamiltonian with conserved energy $\delta^2 H_\Phi$, (Hamiltonian proof later.)

$$\delta^2 H_\Phi = \int_\Omega |\delta u_S|^2 + (\delta u_T)^2 + \Phi''(q_e)(\delta q)^2 \, \mathrm{d}z \, \mathrm{d}x \,,$$

where q_e is PV of the equilibrium solution and we have used $\delta\gamma_S = 0$.

Remark

When $\delta^2 H_\Phi$ provides a norm, then the Eady equilibrium solution is linearly Lyapunov stable, since $\delta^2 H_\Phi$ is the conserved Hamiltonian for the linearised flow dynamics. There are two cases.

(1) The case of $\delta^2 H_\Phi > 0$ for $\Phi''(q_e) \geq 0$ corresponds to the Rayleigh Arnold inflection point criterion for stability of 2D Euler equilibria.

(2) Arnold's second criterion, for stability of 2D Euler equilibria when $\delta^2 H_\Phi < 0$ for $\Phi''(q_e)$ sufficiently negative, is not available here.

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Hamiltonian formulation of the Eady model equations

The Legendre transformation yields the variational relations

$$m_S = \frac{\delta l}{\delta u_S}\,, \quad u_S = \frac{\delta h}{\delta m_S}\,, \quad m_T = \frac{\delta l}{\delta u_T}\,, \quad u_T = \frac{\delta h}{\delta m_T}\,, \quad \frac{\delta h}{\delta \theta_S} = -\frac{\delta l}{\delta \theta_S}\,,$$

The corresponding Hamiltonian matrix for the Eady model equations is

$$\frac{\partial}{\partial t}\begin{bmatrix} m_S \\ m_T \\ \theta_S \\ D \end{bmatrix} = -\begin{bmatrix} \mathrm{ad}^*_{\Box} m_S & \Box \diamond m_T & \Box \diamond \theta_S & \Box \diamond D \\ \mathcal{L}_{\Box} m_T & 0 & -s & 0 \\ \mathcal{L}_{\Box}\theta_S & s & 0 & 0 \\ \mathcal{L}_{\Box} D & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \delta h/\delta m_S \\ \delta h/\delta m_T \\ \delta h/\delta \theta_S \\ \delta h/\delta D \end{bmatrix}, \qquad (13)$$

in which the box $\Box$ indicates the appropriate substitutions, $\mathcal{L}$ is Lie derivative and $\diamond$ is minus its dual with respect to the natural pairing.

This equation may be abbreviated as

$$\dot{X} = \mathbb{J}(X) Dh(X)\,, \quad \text{with} \quad X \in \{m_S, m_T, \theta_S, D\}$$

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Energy and enstrophy in the Hamiltonian formulation

In the Hamiltonian formulation, a functional F of the dynamical variables obeys

$$\frac{dF}{dt} = \{F\,,\,h\} = DF\mathbb{J}(X)Dh$$

For the Hamiltonian matrix $\mathbb{J}(X)$ in (19), antisymmetry of the Poisson bracket yields

$$\frac{dh}{dt} = \{h\,,\,h\} = 0\,, \quad \text{since} \quad \{F\,,\,h\} = -\,\{h\,,\,F\}\,.$$

For the Casimir C_Φ, one has

$$\{F\,,\,C_\Phi\} = 0 \text{ for all } F \quad \text{since} \quad \mathbb{J}(X)DC_\Phi = 0\,.$$

Thus, critical points of $H_\Phi = h + C_\Phi$ are equilibrium solutions, since

$$\frac{dF}{dt} = \{F\,,\,H_\Phi\} = DF\mathbb{J}(X)DH_\Phi = 0 \quad \text{when} \quad DH_\Phi = 0\,.$$

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$\delta^2 H_\Phi$ is the Hamiltonian for the linearised motion

We rewrite the Hamiltonian equations as

$$\frac{dX}{dt} = \mathbb{J}(X) DH_\Phi(X)$$

and take variations around the equilibrium solution X_e to find the linearised equations

$$\frac{d(\delta X)}{dt} = D\mathbb{J}(\delta X) DH_\Phi(X_e) + \mathbb{J}(X_e) D^2 H_\Phi(\delta X, \delta X)$$

If the equilibrium is a critical point satisfying $DH_\Phi(\delta X) = 0$, then

$$\frac{d(\delta X)}{dt} = \mathbb{J}(X_e) D^2 H_\Phi(\delta X, \delta X)$$

Remark

Thus, the Hamiltonian for linearised motion is $\delta^2 H_\Phi = D^2 H_\Phi(\delta X, \delta X)$ and the Poisson bracket is given by $\mathbb{J}(X_e)$.

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Compressible slice models

The Lagrangian for the Compressible Slice Model (CSM) in Eulerian (x, y, z) coordinates is

$$l[u_S, u_T, D, \theta_S] = \int_\Omega \frac{D}{2}\left(|u_S|^2 + u_T^2\right) + f\,Du_Tx + gDz - Dc_v\theta_S\Pi\,\mathrm{d}S,$$

where Π is the Exner function given by

$$\Pi = \left(\frac{p}{p_0}\right)^{R/c_p}$$

Here, p_0 is a reference pressure level and c_p and R are gas constants.

Compressible slice models

The equation of state for an ideal gas is

$$p_0 \Pi^{c_p/R} = DR\theta_S \Pi \,.$$

Differentiating with respect to θ_S and D gives

$$\frac{\partial \Pi}{\partial \theta_S} = \frac{R}{c_v \theta_S} \Pi \quad \text{and} \quad \frac{\partial \Pi}{\partial D} = \frac{R}{c_v D} \Pi \,.$$

Remark

Note that θ_S appears in both the internal energy term in the Lagrangian, and in the equation of state. This removes all y-dependence from the Lagrangian, making a slice model possible.

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Note that θ_S appears in both the internal energy term in the Lagrangian, and in the equation of state. This removes all y-dependence from the Lagrangian, making a slice model possible.

Compressible slice models

The variational derivatives of the CSM Lagrangian are

$$\begin{aligned} \frac{1}{D}\frac{\delta l}{\delta u_S} &= u_S\,, \\ \frac{1}{D}\frac{\delta l}{\delta u_T} &= u_T + fx\,, \\ \frac{\delta l}{\delta D} &= \frac{1}{2}\left(|u_S|^2 + u_T^2\right) + fu_T x + gz - c_p\Pi\theta_S\,, \\ \gamma_S := \frac{1}{D}\frac{\delta l}{\delta \theta_S} &= -\,c_p\Pi\,, \end{aligned} \tag{14}$$

in which γ_S is now a dynamical variable.

Compressible slice models

After taking variations, we have the CSM equations,

$$\begin{aligned} \partial_t u_S + u_S \cdot \nabla u_S - f u_T \hat{x} &= -c_p \theta \nabla \Pi - g\hat{z}, \\ \partial_t u_T + u_S \cdot \nabla u_T + f u_S \cdot \hat{x} &= s\, c_p \Pi, \\ \partial_t \theta_S + u_S \cdot \nabla \theta_S &= -s\, u_T\,, \\ \frac{\partial D}{\partial t} + \nabla \cdot (D u_S) &= 0\,. \end{aligned} \tag{15}$$

Compare these with the *standard* Eady model equations,

$$\begin{aligned} \partial_t u_S + u_S \cdot \nabla u_S - f u_T \hat{x} &= -\nabla p + \frac{g}{\theta_0}\theta_S \hat{z}, \\ \partial_t u_T + u_S \cdot \nabla u_T + f u_S \cdot \hat{x} &= -s\gamma_S, \\ \partial_t \theta_S + u_S \cdot \nabla \theta_S &= -s\, u_T, \\ \nabla \cdot u_S &= 0. \end{aligned} \tag{16}$$

where $\gamma_S := \frac{g}{\theta_0}\left(z - \frac{b}{2}\right)$ in the Boussinesq-Euler approximation.

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Compressible slice models

Define the circulation velocity $v_S := u_S - s^{-1}(u_t + f\,x)\nabla\theta_S$.

Theorem

The CSM model conserves the circulation of v_s on loops carried by u_S.

$$\frac{d}{dt}\oint_{c(u_S)} v_S \cdot \mathrm{d}x = \frac{d}{dt}\oint_{c(u_S)} \Big(u_S - s^{-1}(u_T + fx)\nabla\theta_S \Big) \cdot \mathrm{d}x = 0\,. \qquad (17)$$

Corollary

Potential vorticity q is conserved along flow lines of velocity u_S,

$$\partial_t q + u_S \cdot \nabla q = 0 \quad \textit{with potential vorticity} \quad q := \frac{1}{D}\operatorname{curl} v_S\,. \qquad (18)$$

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Theorem

The CSM equations are Hamiltonian, with conserved energy

$$h = \int_\Omega \frac{D}{2}\left(|u_S|^2 + u_T^2\right) - gDz + c_v D\Pi\theta_S \,\mathrm{d}z\,\mathrm{d}x.$$

Compare with the Eady model Hamiltonian in the Boussinesq-Euler approximation

$$h_{Eady} = \int_\Omega \frac{1}{2}\left(|u_S|^2 + u_T^2\right) - \gamma_S\theta_S \,\mathrm{d}z\,\mathrm{d}x\,.$$

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Compressible slice models

Theorem

The Casimir function for the CSM equations is the generalised enstrophy, weighted by the density, D:

$$C_\Phi = \int_\Omega D\Phi(q)\,\mathrm{d}z\,\mathrm{d}x\,,$$

for any differentiable function Φ of

$$q := \frac{1}{D}\,\hat{y}\cdot\mathrm{curl}\,v_S\,.$$

Hamiltonian formulation of CSM

Although the Hamiltonian is different, the Hamiltonian matrix for the CSM equations is the same as for the Eady model

$$\frac{\partial}{\partial t}\begin{bmatrix} m_S \\ m_T \\ \theta_S \\ D \end{bmatrix} = -\begin{bmatrix} \mathrm{ad}^*_{\square} m_S & \square \diamond m_T & \square \diamond \theta_S & \square \diamond D \\ \mathcal{L}_{\square} m_T & 0 & -s & 0 \\ \mathcal{L}_{\square}\theta_S & s & 0 & 0 \\ \mathcal{L}_{\square} D & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \delta h/\delta m_S \\ \delta h/\delta m_T \\ \delta h/\delta \theta_S \\ \delta h/\delta D \end{bmatrix}, \qquad (19)$$

in which the box $\square$ indicates the appropriate substitutions, $\mathcal{L}$ is Lie derivative and $\diamond$ is minus its dual (or adjoint) wrt the natural pairing.

Thus, the compressible flow equations may still be abbreviated as

$$\dot{X} = \mathbb{J}(X)Dh(X)$$

with the same $\mathbb{J}(X)$ but for a different Hamiltonian.

For CSM, critical points of $H_\Phi = h + C_\Phi$ are still equilibrium solutions,

$$\frac{dF}{dt} = \{F\,,\,H_\Phi\} = DF\mathbb{J}(X)DH_\Phi = 0 \quad \text{for} \quad DH_\Phi(X_e) = 0\,.$$

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Arnold stability for CSM equilibria

The linearised equations for equilibria that are critical points, with $DH_\Phi(\delta X) = 0$, are still obtained from

$$\frac{d(\delta X)}{dt} = \mathbb{J}(X_e) D^2 H_\Phi(\delta X, \delta X).$$

Thus, the Hamiltonian for linearised motion is $\delta^2 H_\Phi = D^2 H_\Phi(\delta X, \delta X)$, and the Poisson bracket is given by $\mathbb{J}(X_e)$.

Remark

However, the sound waves that accompany compressibility hinder the Arnold approach for determining stability conditions, although the Hamiltonian formulation still gives the critical point equilibria satisfying $\delta H_\Phi = 0$ *for* $H_\Phi = h + C_\Phi$.

Thanks for listening!

Here are the main points of my lecture:

1. These slice models assume that transverse velocity u_T is swept by the slice velocity $\mathbf{u}_S$.
2. A general theory exists for creating such models, based on the actions of Lie groups on each other and on vector spaces for advected quantities. (Used implicitly in this talk.)
3. The theory is based on the Euler-Poincaré variational principle, in which Eulerian velocities and advected quantities are varied, not Lagrangian particle paths.
4. The Poisson bracket that emerges from the Legendre transformation in this case is not canonical.
5. Fluid equilibria and their linearised stability may be determined by a method based on variations of the sum of the energy and the Casimirs. The first variation determines a class of equilibria. The second variation determines their stability conditions.