From simple particle models to PDE dynamics

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GENERIC

General nonequilibrium equations

Nonequilibrium framework (GENERIC)

General Equation for Non-Equilibrium Reversible-Irreversible Coupling
(H.-Ch. Öttinger, Beyond Equilibrium Thermodynamics, Wiley, 2005)

$$\partial_t z = \mathsf{L}(z)\,\mathrm{dE}(z) + \mathsf{K}(z)\,\mathrm{dS}(z), \tag{1}$$

- $\mathsf{E}, \mathsf{S} \colon Z \to \mathbb{R}$ are the energy and entropy functionals,
- dE, dS are appropriate derivatives (Fréchet derivative or a gradient with respect to some inner product);
- $\mathsf{L} = \mathsf{L}(z)$ is an antisymmetric operator satisfying the Jacobi identity

$$\{\{\mathsf{F}_1, \mathsf{F}_2\}_\mathsf{L}, \mathsf{F}_3\}_\mathsf{L} + \{\{\mathsf{F}_2, \mathsf{F}_3\}_\mathsf{L}, \mathsf{F}_1\}_\mathsf{L} + \{\{\mathsf{F}_3, \mathsf{F}_1\}_\mathsf{L}, \mathsf{F}_2\}_\mathsf{L} = 0$$

  for all functions $\mathsf{F}_i \colon Z \colon Z \to \mathbb{R},$, with Poisson bracket $\{\cdot, \cdot\}_\mathsf{L}$

$$\{\mathsf{F}, \mathsf{G}\}_\mathsf{L} := \mathrm{dF} \cdot \mathsf{L}\,\mathrm{dG}$$

- $\mathsf{K} = \mathsf{K}(z)$ is symmetric and positive semidefinite.

General nonequilibrium equations

Nonequilibrium framework (GENERIC)

The building blocks $\{\mathsf{L}, \mathsf{K}, \mathsf{E}, \mathsf{S}\}$ are required to fulfil the *degeneracy conditions*: for all $z \in Z$,

$$\mathsf{L}\,\mathrm{d}\mathsf{S} = 0, \quad \mathsf{K}\,\mathrm{d}\mathsf{E} = 0. \tag{2}$$

As a consequence, energy is conserved along a solution, and entropy is non-decreasing:

$$\frac{d\mathsf{E}(z(t))}{dt} = \mathrm{d}\mathsf{E} \cdot \frac{dz}{dt} = \mathrm{d}\mathsf{E} \cdot (\mathsf{L}\,\mathrm{d}\mathsf{E} + \mathsf{K}\,\mathrm{d}\mathsf{S}) = 0,$$
$$\frac{d\mathsf{S}(z(t))}{dt} = \mathrm{d}\mathsf{S} \cdot \frac{dz}{dt} = \mathrm{d}\mathsf{S} \cdot (\mathsf{L}\,\mathrm{d}\mathsf{E} + \mathsf{K}\,\mathrm{d}\mathsf{S}) = \mathrm{d}\mathsf{S} \cdot \mathsf{K}\,\mathrm{d}\mathsf{S} \geq 0.$$

A GENERIC system is then fully characterised by $\{Z, \mathsf{E}, \mathsf{S}, \mathsf{L}, \mathsf{K}\}$.

GENERIC

Use so far

- Öttinger and group: formal scale-bridging from microscopic Hamiltonian models to GENERIC as continuum description
- Thermodynamic consistent modelling (fluids, quantum systems coupled to macroscopic dissipative ones, elastoplasticity, ...: e.g., Mielke, *Contin. Mech. Thermodyn.*, **23** (2011), 233–256)

Current questions

- Can we derive simple GENERIC systems rigorously from mesoscopic models?
- Analysis for GENERIC systems (existence / time splitting methods, ...).

Example of a GENERIC system

Thermoviscoelasticity

$$u_{tt} = k u_{xx} + \alpha\theta_x + \mu u_{xxt} - \gamma u_{xxxx}$$
$$\theta_t = \kappa\theta_{xx} + \alpha\theta u_{xt} + \mu u_{xt}^2$$

(+ initial and boundary conditions). Then with $p = \dot{u}$

$$\mathsf{E} = \int_\Omega \left[\frac{1}{2}p^2 + \theta + \frac{k}{2}u_x^2 + \frac{\gamma}{2}u_{xx}^2\right] \mathrm{d}x,$$
$$\mathsf{S} = \int_\Omega \left[\log\theta - \alpha u_x\right] \mathrm{d}x$$

Mielke, *Contin. Mech. Thermodyn.*, **23** (2011), 233–256: write $\mathsf{L} = \mathsf{Q_S L_0 Q_S^*}$, $\mathsf{K} = \mathsf{Q_E K_0 Q_E^*}$, with given k_{visc} and k_{heat},

$$\mathsf{L}_0 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathsf{K}_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & k_{\mathrm{visc}} & 0 \\ 0 & 0 & k_{\mathrm{heat}} \end{pmatrix}.$$

Example of a GENERIC system

Thermoviscoelasticity

$$\mathsf{Q_S} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{D_\theta S} D_u S[\cdot] & -\frac{1}{D_\theta S} D_p S[\cdot] & \frac{1}{D_\theta S} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\theta D_u S[\cdot] & 0 & \theta \end{pmatrix},$$

which yields

$$\begin{aligned} \mathsf{Q_S^* dE} &= \begin{pmatrix} 1 & 0 & -\theta \star \partial_u S \\ 0 & 1 & 0 \\ 0 & 0 & \theta \end{pmatrix} \begin{pmatrix} \partial_u \mathsf{E} \\ \partial_p \mathsf{E} \\ \partial_\theta \mathsf{E} \end{pmatrix} \\ &= \begin{pmatrix} k u_{xx} - \gamma u_{xxxx} - \theta \star \partial_u S \\ p \\ \theta \end{pmatrix} = \begin{pmatrix} k u_{xx} - \gamma u_{xxxx} - \alpha \theta u_x \\ p \\ \theta \end{pmatrix}, \end{aligned}$$

which is the *driving force*.

Purely entropic systems

Particles and deterministic gradient flows

From particles to diffusion

Brownian motion (Pollen grains in fluid, R. Brown 1827)

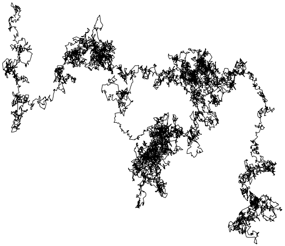


Figure: Brownian motion, taken from Mörters & Peres, *Brownian Motion*

Macroscopic (many-particle) limit:
Let $\rho = \rho(x,t)$ be the particle density (number of particles per unit volume) at position x and time t. Then ρ satisfies the diffusion equation with diffusion constant κ,

$$\frac{\partial}{\partial t}\rho(x,t) = \kappa\Delta\rho(x,t) = \kappa \operatorname{div}\left(\rho(x,t)\nabla\frac{\delta \mathsf{S}}{\delta\rho}(\rho)\right) \tag{3}$$

with entropy
$\mathsf{S}(\rho) := \int_{\mathbb{R}^n}\rho(x)\log\rho(x)\,\mathrm{d}x$.

Question: How can we *derive (3) and stochastic "corrections" directly from particles?*

Diffusion as entropic gradient flow

Jordan, Kinderlehrer & Otto, 1998

$$\Delta\rho = \mathrm{div}\left(\rho\nabla\frac{\delta\,\mathsf{S}}{\delta\rho}\right) = -\,\mathrm{grad}_W\,\mathsf{S}(\rho) = \text{-}\textit{Wasserstein gradient of}\ \mathsf{S}\,.$$

Variational formulation: for linear diffusion

$$\frac{\partial}{\partial t}\rho(x,t) = \Delta\rho(x,t), \tag{4}$$

via time-stepping, define at time k the function ρ^k as minimiser of

$$K_h(\rho\,;\rho^{k-1}) := \frac{1}{2h}d(\rho,\rho^{k-1})^2 + \mathsf{S}(\rho),$$

where d is the *Wasserstein metric*

$$d(\rho_0,\rho_1)^2 := \inf_{\gamma\,\big|\,\pi_0\gamma=\rho_0,\pi_1\gamma=\rho_1}\int_{\mathbb{R}^n\times\mathbb{R}^n}(x-y)^2\,\gamma(dxdy).$$

This approximation converges to the solution ρ of (4) as $h\to 0$.

What is a large deviation principle? An example

Sanov's theorem

In *equilibrium* (static situation):
Let X_i ($i = 1, 2, \dots$) be independent and identically distributed stochastic variables with distribution μ on a state space X (positions of particles)
Their concentration is given by the *empirical measure* $\rho_n := \frac{1}{n}\sum_{i=1}^n \delta_{X_i}$.
Sanov's theorem states: the random measure ρ_n satisfies the *large-deviation principle*

$$\mathrm{Prob}(\rho_n \approx \rho) \sim \exp[-nJ(\rho)], \qquad \text{as } n \to \infty, \tag{5}$$

where the *rate function* $I \geq 0$ is the *relative entropy* of ρ with respect to μ,

$$J(\rho) = H(\rho|\mu) := \begin{cases} \int f \log f \, d\mu & \text{if } \rho \ll \mu \text{ and } \rho = f\mu, \\ +\infty & \text{otherwise.} \end{cases}$$

Is there a 'dynamic Sanov' (away from equilibrium, time step $h > 0$)?

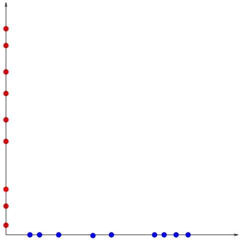
Dynamics: The microscopic picture

Particle model

Brownian motion: the probability that a particle jumps in time $h > 0$ from $x \in \mathbb{R}$ to $y \in \mathbb{R}$ is

$$p_h(x, y) := \frac{1}{2\sqrt{\pi h}} e^{-(y-x)^2/4h}.$$

Possible particle jumps:



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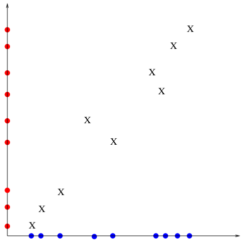

Every blue particle has to jump to a red one (be identified with a red one)

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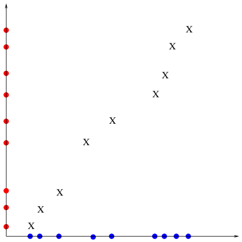


Jumps maximise $\Pi e^{(x_i - y_i)^2/(4h)} = e^{\sum (x_i - y_i)^2/(4h)}$, yields cost functional $(x-y)^2$ of Wasserstein metric $d(\rho_0, \rho_1)^2 :=$

$$\inf_{\gamma \in \Gamma(\rho_0, \rho_1)} \int_{\mathbb{R}^n \times \mathbb{R}^n} (x-y)^2 \, \gamma(dxdy),$$

$\Gamma(\rho_0, \rho) = \big\{ \gamma \in \mathcal{M}_1(\mathbb{R}^n \times \mathbb{R}^n) \colon \pi_0 \gamma = \rho_0, \pi_1 \gamma = \rho \big\}.$

Dynamics: From particles to the gradient flow

Large deviation principle

Let $L_n^t = n^{-1} \sum_{i=1}^n \delta_{X^{(i)}(t)}$ be the empirical measure of n Brownian particles.

Let $L_n^0 \approx \rho_0$ and consider the *large deviation principle* describing the probability to approximate at time h a measure ρ by L_n^h:

$$\mathbb{P}\big(L_n^h \approx \rho \,|\, L_n^0 \approx \rho_0\big) \approx \exp\big[-nJ_h(\rho\,;\rho_0)\big] \qquad \text{as } n \to \infty.$$

Here the *rate function* J_h is (Léonard; Adams, Peletier, Dirr & Z.)

$$J_h(\rho\,;\,\rho_0) := \inf_{q:\ \pi_0 q = \rho_0, \pi_1 q = \rho} H(q\,|\,q_0)$$

with

$$H(q\,|\,p) := \begin{cases} \int_{\mathbb{R}\times\mathbb{R}} f(x,y) \log f(x,y)\; p(d(x,y)) & \text{if } q \ll p, f = \frac{dq}{dp} \\ +\infty & \text{else} \end{cases}.$$

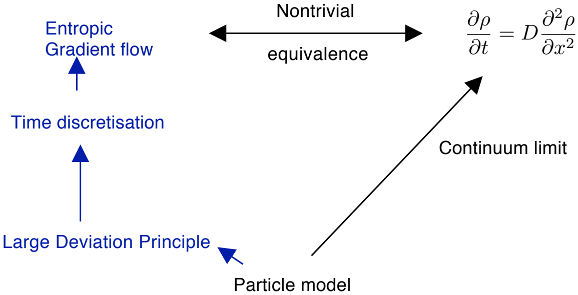
What has the rate functional J_h to do with the time discretisation $K_h(\rho\,;\rho^{k-1}) := \frac{1}{2h} d(\rho, \rho^{k-1})^2 + \mathrm{S}(\rho)$ of Otto's entropic flow?

Dynamics: From particles to the gradient flow

What has the rate functional J_h to do with the time discretisation K_h of Otto's entropic flow?

Theorem (Adams, Dirr, Peletier, Z. / Dirr, Laschos, Z.)

J_h and K_h agree asymptotically in the limit $h \to 0$, in the sense of Γ-convergence [ADPZ: Measures on a flat torus with Lebesgue density in $\left\{\rho \in L^\infty(0,L) : \int_0^L \rho \,\mathrm{d}x = 1 \text{ and } \|\rho - L^{-1}\|_\infty < \delta\right\}$, for suitable $0 < \delta < 1$; DLZ: Gaussian measures].



Large deviations: from particles to PDEs

Dawson-Gärtner theorem

Now let us consider the *dynamic* situation:
Given n Brownian particles and a fixed terminal time $T > 0$, consider the path of *empirical measures* $[0, T] \ni t \mapsto \rho_n(t) = \frac{1}{n}\sum_{j=1}^{n} \delta_{X(j)}$.
Then (Dawson, Gärtner, *Mem. Amer. Math. Soc.*, **78** (1989); Kipnis, Olla, *Stochastics Rep.*, **33** (1990), 17–25):

$$\mathrm{Prob}(\rho_n \approx \rho) \sim \exp[-nJ(\rho)], \tag{6}$$

with the *rate functional*

$$J(\rho) := \frac{1}{2}\int_0^T \left\| \frac{\partial \rho}{\partial t} - \Delta\rho \right\|^2_{\rho(t),*} \mathrm{d}t \tag{7}$$

(the norm will be defined later).

- Limit particle number $n \to \infty$: (6) gives vanishing probability to all states ρ except those for which $J(\rho) = 0$, the solution of the heat equation.

Pathwise approach

Large deviation principle for paths

Re-write the functional from the previous slide by expanding the square:

$$
\begin{aligned}
&\frac{1}{2}\int_0^T \left\| \frac{\partial \rho}{\partial t} - \Delta\rho \right\|_{\rho,*}^2 \mathrm{d}t \\
&= \frac{1}{2}\int_0^T \left[\left\| \frac{\partial \rho}{\partial t} \right\|_{\rho,*}^2 - 2\left(\frac{\partial \rho}{\partial t}, \Delta\rho \right)_{\rho,*} + \|\Delta\rho\|_{\rho,*}^2 \right] \mathrm{d}t \\
&= \mathsf{S}(\rho(T)) - \mathsf{S}(\rho(0)) + \frac{1}{2}\int_0^T \left[\left\| \frac{\partial \rho}{\partial t} \right\|_{\rho,*}^2 + \|\Delta\rho\|_{\rho,*}^2 \right] \mathrm{d}t \\
&= \mathsf{S}(\rho(T)) - \mathsf{S}(\rho(0)) + \frac{1}{2}\int_0^T \left[\left\| \frac{\partial \rho}{\partial t} \right\|_{\rho,*}^2 + \left\| -\frac{\delta\,\mathsf{S}}{\delta\rho} \right\|_{\rho}^2 \right] \mathrm{d}t;
\end{aligned}
$$

here we used
$\Delta\rho = \operatorname{div}\rho\nabla\left(\frac{\delta\,\mathsf{S}}{\delta\rho}\right) = -\operatorname{grad}_W \mathsf{S}(\rho)$= -*Wasserstein gradient of* S.
Insight: *the mixed term gives the entropy* (but we need to recognise it).
Nonlinear diffusion, boundary conditions: preprint by Bodineau, Lebowitz, Mouhot and Villani.

Simplest energy- and entropy-driven system

(More) general nonequilibrium equations

Vlasov-Fokker-Planck equation

Duong, Peletier, Z.: Revisit Vlasov-Fokker-Planck equation in light of GENERIC equation (1), $\rho = \rho(p, q)$.

$$\partial_t \rho = -\mathrm{div}_q \Big(\rho \frac{p}{m}\Big) + \mathrm{div}_p \rho \Big(\nabla_q V + \nabla_q \psi * \rho + \gamma \frac{p}{m}\Big) + \gamma\theta \Delta_p \rho. \qquad (8)$$

This is many particle limit of *interacting Brownian particles with inertia*,

$$\begin{aligned} dQ_i(t) &= \frac{P_i(t)}{m}\, dt, \\ dP_i(t) &= -\nabla V(Q_i(t))\, dt - \sum_{j=1}^{n} \nabla\psi(Q_i(t) - Q_j(t)) \\ &\quad - \frac{\gamma}{m} P_i(t)\, dt + \sqrt{2\gamma\theta}\, dW_i(t) \end{aligned}$$

(Q_i and P_i position and momentum of particle $i = 1, \ldots, n$, mass m, potential V, interaction potential ψ, drift term $-\gamma P_i dt/m$, stochastic forcing by n independent Wiener measures W_i).

Vlasov-Fokker-Planck equation as GENERIC equation

Microscopic energy balance

The VFP equation (8) is *not* of GENERIC form: there is no conserved functional E. Physical reason: Particle model from last slide has heat bath interaction, which affects natural Hamiltonian

$$H_n(Q_1, \dots, Q_n, P_1, \dots, P_n) := \frac{1}{n} \sum_{i=1}^{n} \left[\frac{P_i^2}{2m} + V(Q_i) \right] + \frac{1}{2n^2} \sum_{i,j=1}^{n} \psi(Q_i - Q_j);$$

a calculation shows $dH_n = -\frac{1}{n} \sum_{i=1}^{n} \left[\frac{\gamma}{m^2} P_i^2 \, dt - \frac{\gamma \theta d}{m} \, dt + \frac{\sqrt{2\gamma\theta}}{m} P_i \, dW_i \right]$.

Remedy: add scalar e_n, define evolution such that $H_n + e_n = \mathrm{const}$:

$$dQ_i = \frac{P_i}{m} \, dt, \tag{9a}$$

$$dP_i = -\nabla V(Q_i) \, dt - \sum_{j=1}^{n} \nabla \psi(Q_i - Q_j) - \frac{\gamma}{m} P_i \, dt + \sqrt{2\gamma\theta} \, dW_i, \tag{9b}$$

$$de_n = \frac{1}{n} \sum_{i=1}^{n} \left[\frac{\gamma}{m^2} P_i^2 \, dt - \frac{\gamma \theta d}{m} \, dt + \frac{\sqrt{2\gamma\theta}}{m} P_i \, dW_i \right]. \tag{9c}$$

Vlasov-Fokker-Planck equation as GENERIC equation

GENERIC setting

Proposition

The system (9) from the previous slide is a GENERIC evolution in $Z = \mathcal{P}_2(\mathbb{R}^{2d}) \times \mathbb{R}$*, with the following building blocks for* $z = (\rho, e)$*:*

$$\begin{aligned} \mathsf{E}(\rho, e) &= \mathcal{H}(\rho) + e, & \mathsf{L} &= \mathsf{L}(\rho, e) = \begin{pmatrix} \mathsf{L}_{\rho\rho} & 0 \\ 0 & 0 \end{pmatrix}, \\ \mathsf{S}(\rho, e) &= \mathcal{S}(\rho) + e, & \mathsf{K} &= \mathsf{K}(\rho, e) = \gamma \begin{pmatrix} \mathsf{K}_{\rho\rho} & \mathsf{K}_{\rho e} \\ \mathsf{K}_{e\rho} & \mathsf{K}_{ee} \end{pmatrix}, \end{aligned} \tag{10}$$

with (ξ, r) *for* (ρ, e)*,* $\mathsf{L}_{\rho\rho}\xi = \mathrm{div}\rho\mathbf{J}\nabla\xi$ *and*

$$\begin{aligned} \mathsf{K}_{\rho\rho}\xi &= -\mathrm{div}_p\rho\nabla_p\xi, & \mathsf{K}_{\rho e}r &= r\mathrm{div}_p\Big(\rho\frac{p}{m}\Big), \\ \mathsf{K}_{e\rho}\xi &= -\int_{\mathbb{R}^{2d}} \frac{p}{m}\cdot\nabla_p\xi\,\rho(dqdp) & \mathsf{K}_{ee}r &= r\int_{\mathbb{R}^{2d}} \frac{p^2}{m^2}\,\rho(dqdp); \end{aligned}$$

$\mathcal{S}(\rho) := -\theta\int_{\mathbb{R}^{2d}} f(x)\log f(x)\,\mathrm{d}x$ *if* ρ *has Lebesgue density* f*;* dE, dS *are* L^2 *gradients.*

(More) general nonequilibrium equations

GENERIC as variational principle

Suggested result (formal for now):
An GENERIC evolution $\{Z, \mathsf{E}, \mathsf{S}, \mathsf{L}, \mathsf{K}\}$,

$$\partial_t z = \mathsf{L}\,\mathrm{d}\mathsf{E} + \mathsf{K}\,\mathrm{d}\mathsf{S}, \tag{11}$$

can be associated with a variational principle. Namely, define

$$2\theta J(z) = \mathsf{S}(z(T)) - \mathsf{S}(z(0)) + \frac{1}{2}\int_0^T \left[\|\partial_t z - \mathsf{L}\,\mathrm{grad}\,\mathsf{E}\|^2_{\mathsf{K}^{-1}} + \|\mathrm{grad}\,\mathsf{S}\|^2_{\mathsf{K}} \right] \mathrm{d}t.$$

Then a function $z\colon [0, T] \to Z$ is a solution of the GENERIC equation (11) iff $J(z) = 0$.

Proposition

The formal statement is rigorous for the Vlasov-Fokker-Planck equation.

(All the quantities appearing in the statement are here well-defined.)

Vlasov-Fokker-Planck and GENERIC

Vlasov-Fokker-Planck and large deviations

Proposition

For deterministic initial data $(Q_i(0), P_i(0))$*,* $i = 1, \ldots, n$ *with* $\rho_n(0) \rightharpoonup \rho^0$ *for some* $\rho^0 \in P(\mathbb{R}^{2d})$*, the empirical process* $\{\rho_n\}$ *satisfies a large-deviation principle in the space* $C([0,T], P(\mathbb{R}^{2d}))$*, with good rate function*

$$I(\rho) = \begin{cases} \dfrac{1}{4\gamma\theta} \displaystyle\int_0^T \left\| \partial_t \rho_t - A^\tau_{\rho_t} \rho_t \right\|^2_{-1,\rho_t} \mathrm{d}t & \rho \in AC([0,T]; P(\mathbb{R}^{2d})), \rho|_{t=0} = \rho^0, \\ +\infty & \textit{otherwise}, \end{cases}$$

where $A_\nu f := \frac{p}{m} \cdot \nabla_p f - \left[\nabla_q V + \nabla_q \psi * \nu + \gamma \frac{p}{m} \right] \cdot \nabla_p f + \gamma\theta \Delta_p f$*.*

Interpretation: Vlasov-Fokker-Planck equation appears as minimiser of rate functional I (this follows from Dawson, Gärtner, *Stochastics*, **20** (1987), 247–308; Budhiraja, Dupuis, Fisher, *Ann. Prob.*, **40** (2012), 74–102)). Up to constant factor, I and J are *identical*.

Vlasov-Fokker-Planck and GENERIC

Summary

- Vlasov-Fokker-Planck equation appears in rate functional for particle process
- One can show (Duong, Peletier, Z.) that this functional can be re-written in GENERIC form, namely for $z = (\rho, e) \in AC([0, T]; Z)$ with $\rho_{t=0} = \rho^0$ as

  $$J(z) = \int_0^T \frac{1}{4\theta} \left\| \partial_t z_t - \mathsf{L}(z_t)\,\mathrm{grad}\,\mathsf{E}(z_t) - \mathsf{K}(z_t)\,\mathrm{grad}\,\mathsf{S}(z_t) \right\|^2_{\mathsf{K}(z_t)^{-1}} \mathrm{d}t,$$

  and $J(z) = +\infty$ otherwise.

So in this case the macroscopic GENERIC evolution can be obtained as a large deviation principle (cf. Öttinger, Grmela, *Phys. Rev. E*, **56** (1997), 6633–6655).

Noise

Extracting more information

The key argument revisited

We consider two objects:

- Continuum description (e.g., K_h, variational time-discrete formulation of PDE),
- Rate functional J_h.

The theorem said: these are *asymptotically* equivalent (in the sense of Γ-convergence); but $J_h \neq K_h$!
In time-continuous setting:

$$J(\rho) := \frac{1}{2} \int_0^T \left\| \frac{\partial \rho}{\partial t} - \frac{\partial^2 \rho}{\partial x^2} \right\|_{\rho,*}^2 \mathrm{d}t$$

Rate functionals J and J_h describes fluctuations and contains *more* information than the PDE.
Question: how can we use this additional information?

Extracting more information: Noise

Rate functional contains "more" information than time-step from PDE. How to extract this information? Preliminary answer (with Rob Jack, Physics, Bath, v. Renesse): stochastic equation.

Dean's equation (Dean, *J. Phys. A: Math. Gen.*, **29** (1996), L613–L617)

$$\frac{\partial \rho}{\partial t} = \Delta \rho + \frac{1}{n} \operatorname{div}(\sqrt{\rho} \dot{W}) \tag{12}$$

with space-time white noise $\dot{W}$. Saddle point argument: (12) gives for $n \to \infty$ the Wasserstein formulation of the diffusion equation (time discretisation, limiting action functional has integrand $\left\| \frac{\partial \rho}{\partial t} - \Delta \rho \right\|_{\rho,*}^2$).

In other words:

1. The noise $\operatorname{div}(\sqrt{\rho}\dot{W})$ is the "right" noise from the perspective of the Wasserstein geometry (Sturm, von Renesse, *Ann. Probab.*, **37** (2009), 1114–1191)
2. It can formally be derived by a large deviation principle.

Open questions:

1. Other noises (e.g., from different particle model with the same hydrodynamic limit)
2. Procedure applicable for other equations?

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Thank you!