Navier–Stokes Equations on a Rotating Sphere

PDE+GFD workshop, INI 2013-12-04

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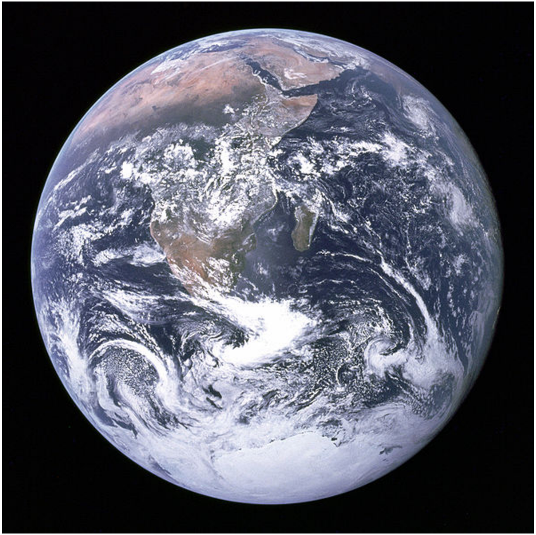
Outline:

- Physical Motivation
- Classical Results
- Zonal Flows
- Attractor and Dimension

Motivation: Observation



[NASA image]

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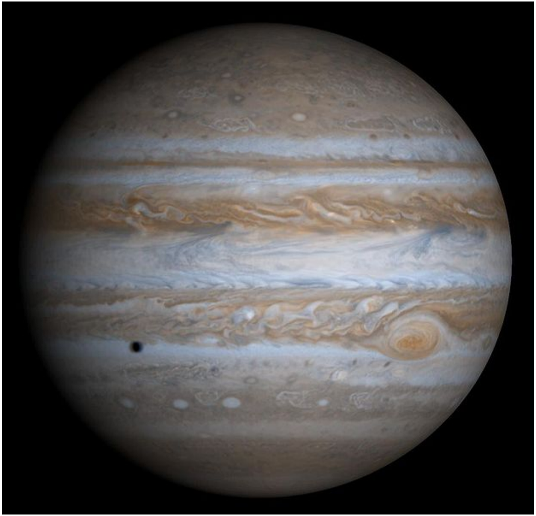

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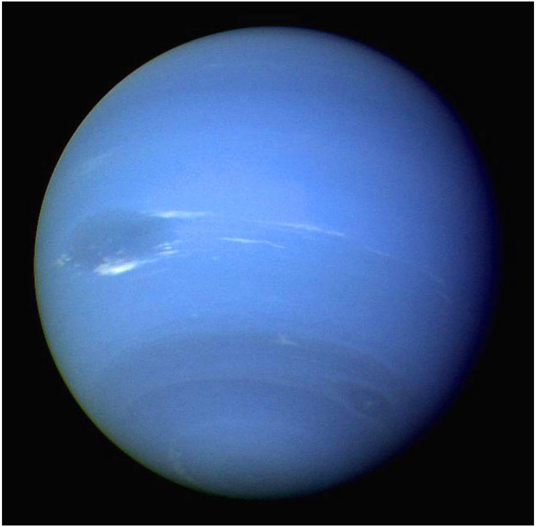

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Navier–Stokes: Rotating Sphere

On a unit sphere $S^2 \ni (\theta, \phi)$ rotating with angular velocity $1/\varepsilon$,

$$\partial_t \boldsymbol{v} + \nabla_{\boldsymbol{v}} \boldsymbol{v} + \frac{2\cos\theta}{\varepsilon} \boldsymbol{v}^\perp + \nabla p = \mu \Delta \boldsymbol{v} + \mathbf{f}$$

where

- $\boldsymbol{v}$ is 2d velocity with $\operatorname{div} \boldsymbol{v} = 0$
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Vorticity Form: with $\omega := \mathrm{curl}\, \boldsymbol{v}$ and $\partial(f, g) := (\partial_\theta f \, \partial_\phi g - \partial_\phi f \, \partial_\theta g)/\sin\theta$,

$$\partial_t \omega + \partial(\psi, \omega) + \frac{2}{\varepsilon} \partial_\phi \psi = \mu \Delta \omega + f \qquad \text{with } \psi = \Delta^{-1} \omega$$

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Write this in symbolic form as

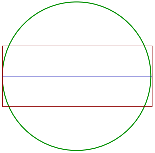
$$\partial_t \omega + B(\omega, \omega) + \frac{1}{\varepsilon} L\omega + \mu A\omega = f$$

where

- "conservation" of enstrophy: $(B(\omega^\sharp, \omega), \omega^\sharp)_{L^2} = 0$
- antisymmetric stiff term: $(L\omega, \omega)_{L^2} = 0$
- dissipative term: $(A\omega, \omega)_{L^2} =: |\nabla \omega|_{L^2}^2 \geq \frac{1}{c_p} |\omega|_{L^2}^2$

β-Plane Approximation

Linearising the curvature terms near the equator $\theta = \pi/2$,



$$\partial_t \boldsymbol{v} + \boldsymbol{v} \cdot \nabla \boldsymbol{v} + \frac{Y}{\varepsilon} \boldsymbol{v}^\perp + \nabla p = \mu \Delta \boldsymbol{v} + \mathbf{f}$$

$$\nabla \cdot \boldsymbol{v} = 0$$

where

- $\boldsymbol{v} = (u, v)$ is 2d velocity with zero integral; $\boldsymbol{v}^\perp := (-v, u)$
- $\nabla \cdot \boldsymbol{v} = 0$ gives $\Delta p = -\nabla \cdot (\boldsymbol{v} \cdot \nabla \boldsymbol{v}) + \varepsilon^{-1} \nabla^\perp \cdot \boldsymbol{v}$
- $Y(y) = y$ for $y \in (-L_2/2, L_2/2]$

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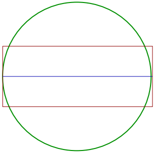

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Domain: $\boldsymbol{x} = (x, y) \in \mathcal{D} := [0, 2\pi] \times [-L_2/2, L_2/2]$, xy-periodic

Impose Symmetry: $u(x, -y, t) = u(x, y, t)$ and $v(x, -y, t) = -v(x, y, t)$

Effective domain: $y \in [0, L_2/2]$ with free-slip BC $(v = 0)$

Energy and Enstrophy

Since $(B(\omega^\sharp, \omega), \Delta^{-1}\omega) = 0$, we have ineq for the energy $\frac{1}{2}|\boldsymbol{v}|_{L^2}^2$

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- For large enough t, energy and enstrophy are bdd only by f (absorbing balls in L^2 and H^1)

Higher Derivatives and Gevrey

In higher Sobolev spaces, one uses standard ineqs

$$\frac{d}{dt}|\nabla^s \omega|^2 + \mu|\nabla^{s+1}\omega|^2 \le \frac{c(s)}{\mu}|\omega|^2|\nabla^s\omega|^2 + \frac{c'(s)}{\mu}|\nabla^{s-1}f|^2$$

to get, for large enough t,

$$|\nabla^s\omega(t)|^2 \le \frac{\|\nabla^{s-1}f\|^2}{\mu^2}\left(c(s) + \frac{c'(s)}{\mu^4}\|\nabla^{-1}f\|^2\right)^s$$

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Similarly in Gevrey space [Foias–Temam'89, Cao–Rammaha–Titi'99],

$$|e^{\sigma A^{1/2}}\omega(t)|_{L^2} \le M(\cdots)$$

for large enough time t and nice f.

Zonal Flows

Let $\bar{\omega}$ denotes the zonal component of ω,

$$\bar{\omega}(\theta, t) := \frac{1}{2\pi} \int_0^{2\pi} \omega(\theta, \phi, t)\, d\phi,$$

and $\tilde{\omega} := \omega - \bar{\omega}$ the non-zonal remainder.

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Previous analytical works:

- Gallagher–St-Raymond'07: β-plane shallow-water model
- extension of Babin–Mahalov–Nicolaenko'9x
- Cheng–Mahalov'12: Euler eqs

Motivation': β-Plane Numerics

In Fourier space:

$$\omega(\mathbf{x},t) = \sum_{\mathbf{k}} \omega_{\mathbf{k}}(t)\, e^{i\mathbf{k}\cdot\mathbf{x}}$$

$$\Rightarrow \qquad \omega_{\mathbf{k}} = \frac{1}{|\mathcal{D}|}\int_{\mathcal{D}} \omega(\mathbf{x},t)\, e^{-i\mathbf{k}\cdot\mathbf{x}}\, d\mathbf{x}$$

Plotting $|\omega_{\mathbf{k}}(t)|$ against $\mathbf{k} = (k_1, k_2)$:

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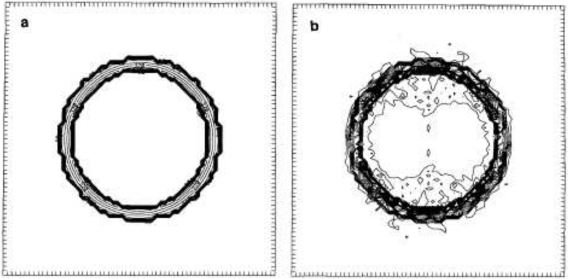

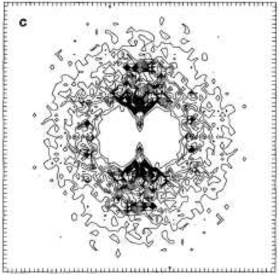

FIG. 5. Similar to Fig. 4 but now calculated using a small ensemble of direct simulations (six) of the equations of motion.

[Vallis–Maltrud 93]

Zonal Flows

Recall $\omega(\theta, \phi, t) = \bar{\omega}(\theta, t) + \tilde{\omega}(\theta, \phi, t)$ and let

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Thm 1. Suppose $\omega(0) \in L^2$ and $K_0(f) < \infty$.
Then there exist $M_0(K_0)$ and $T_0(|\omega(0)|, K_0)$ s.t. $\forall t \geq T_0$,

$$|\tilde{\omega}(\cdot,t)|_{L^2} \leq \sqrt{\varepsilon}\, M_0(K_0).$$

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Moreover, if $K_s(f) < \infty$, we also have, for all $t \geq T_s$,

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Remarks:

- bd is indep of initial data
- for meaningful result, must have $\partial_t f \sim O(1)$
- rhs is "reasonable": $M_0 \sim |\Delta\omega| |\omega| + |\Delta f| + \text{l.o.t}$
- Rhines scale: not as simple as $|\boldsymbol{v}|_{L^2}/\beta$?

Zonal Discussion

More Remarks:

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- effect of (QG) topography and structures in forcing f:

$$|\omega(t) - \Phi(f;\varepsilon)| \leq \varepsilon^{n/2} N(\cdots) \qquad \text{for } t > T_*$$

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- exponential bound [à la Matthies'01, Temam–W'08]
- shallow water [à la Gallagher–St Raymond'07] and primitive eqs (harder)

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Kaplan–Yorke dimension. Let $\{\lambda_j\}_{j=1}^{\infty}$ be the Lyapunov exponents on $\mathcal{A}$, and let J be s.t. $\lambda_1 + \cdots + \lambda_J \geq 0$ and $\lambda_1 + \cdots + \lambda_{J+1} < 0$ ...

$$\begin{aligned} \dim_{\mathrm{KY}} \mathcal{A} &:= J + \frac{\lambda_1 + \cdots + \lambda_J}{-\lambda_{J+1}} \\ &\geq \dim_{\mathrm{H}} \mathcal{A} \end{aligned}$$

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$$\geq \dim_H \mathcal{A}$$

Thm. For 2d NSE (nice domain, smooth f, etc)

$$\dim_H \mathcal{A} < \infty.$$

Dimension and Degrees of Freedom in Turbulence

Define the Grashof number by

$$G := |\mathcal{D}|\,|\mathbf{f}|_{L^2}^2/\mu^2.$$

For our case (no bdry), one has [Constantin-Foias-Temam'88; Il'yin'94]:

$$\dim_H \mathcal{A} \le c\, G^{2/3}(1+\log G)^{1/3} \qquad (\text{for } G \ge e^{-1}).$$

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Fixing the Kraichnan length scale

$$l_\eta = \mu^{1/3} L^{1/3} \langle |\nabla \omega|^2 \rangle^{-1/6},$$

$G^{2/3}$ scales extensively with the domain size.

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To do:

- Estimate $\dim_{\mathrm{H}} \mathcal{A}$ numerically

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Making use of the rotation, we have

Thm 2. There exists an $\varepsilon_*(|\nabla^2 f|; \mu)$ s.t.

$$\dim_H \mathcal{A} = 0 \qquad \text{for } \varepsilon \le \varepsilon_*.$$

Remarks: in this case

- $\mathcal{A}$ consists of a steady state determined completely by f
- this steady state is nearly, but not exactly, zonal
- $O(\sqrt{\varepsilon})$ non-zonal oscillations cannot alter it

To do:

- Estimate $\dim_H \mathcal{A}$ numerically
- Get more useful bounds

$$C_1(G, \varepsilon) \le \dim_H \mathcal{A} \le C_2(G, \varepsilon).$$

Linearised Problem

$$d_t \omega_k + B_k(\omega, \omega) + \frac{i\Omega_k}{\varepsilon} \omega_k + \mu |\mathbf{k}|^2 \omega_k = f_k$$

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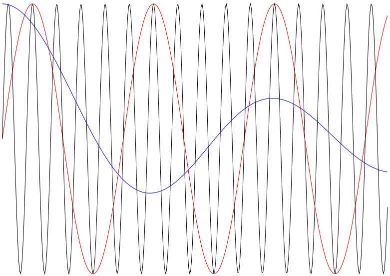
The force f_k does not act effectively for $|\Omega_k|/\varepsilon$ large.



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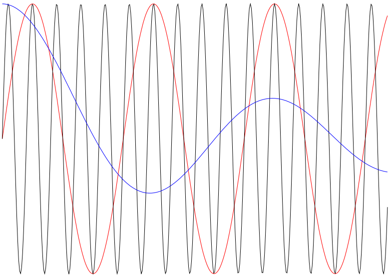

Recall that the zonal modes $\bar{\omega}_k$ have zero frequencies:

Linearised Problem

$$d_t \omega_{\mathbf{k}} + B_{\mathbf{k}}(\omega, \omega) + \frac{i\Omega_{\mathbf{k}}}{\varepsilon}\,\omega_{\mathbf{k}} + \mu\,|\mathbf{k}|^2\,\omega_{\mathbf{k}} = f_{\mathbf{k}}$$

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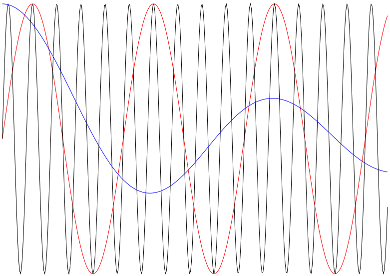

Recall that the zonal modes $\bar{\omega}_{\mathbf{k}}$ have zero frequencies:

modes with large wavenumbers $|\mathbf{k}| \gg 1$ and freqs $|\Omega_{\mathbf{k}}|/\varepsilon$ decay.

Nonlinear Term

$$\partial_t \omega + B(\omega, \omega) + \frac{1}{\varepsilon} L\omega + \mu A\omega = f$$

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Multiply this by $\tilde{\omega}$ and use $(\tilde{\omega}, \bar{\omega}) = 0$ to get

$$\tfrac{1}{2}\, d_t |\tilde{\omega}|^2 + (B(\bar{\omega} + \tilde{\omega}, \bar{\omega}), \tilde{\omega}) + \mu\, |\nabla \tilde{\omega}|^2 = (f, \tilde{\omega})$$

Now

$$(B(\bar{\omega} + \tilde{\omega}, \bar{\omega}), \tilde{\omega}) = (B(\bar{\omega}, \bar{\omega}), \tilde{\omega}) + (B(\tilde{\omega}, \bar{\omega}), \tilde{\omega})$$

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The first term vanishes:

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For the second term,

$$(B(\tilde{\omega}, \bar{\omega}), \tilde{\omega}) = -(B(\tilde{\omega}, \tilde{\omega}), \bar{\omega})$$

Non-Resonance Lemma (β-Plane)

$$\begin{aligned}(B(\tilde{\omega},\tilde{\omega}),\bar{\omega}) &= \sum_{jkl} B_{jkl}\,\tilde{\omega}_j\tilde{\omega}_k\overline{\bar{\omega}_l}\\ &= \frac{1}{2}\sum_{jkl}\left(B_{jkl}+B_{kjl}\right)\tilde{\omega}_j\tilde{\omega}_k\overline{\bar{\omega}_l}\end{aligned}$$

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This implies

- no fast–fast–slow resonance: $\Omega_j+\Omega_k = 0 \;\Rightarrow\; B_{jkl}+B_{kjl} = 0$

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- no fast–fast–slow near resonance:

$$|B_{jkl}+B_{kjl}| = |\mathbf{l}|\,|\Omega_j+\Omega_k|$$

Non-Resonance Lemma (Sphere)

Expanding in sph harmonics, $\omega(\theta,\phi,t) = \sum_{\mathbf{k}} \omega_{\mathbf{k}}(t) Y_{\mathbf{k}}(\theta,\phi)$ with $\mathbf{k} = (k, \hat{k})$, gives the nlt

$$\begin{aligned} B_{\mathbf{jkl}} &= (\partial(\Delta^{-1} Y_{\mathbf{j}}, Y_{\mathbf{k}}), Y_{\mathbf{l}})_{L^2} \\ &= \frac{\sqrt{(l+k-j)(l+1+j-k)}}{4i\sqrt{\pi}\, j(j+1)} \begin{pmatrix} l & j & k-1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} l & k & j \\ 0 & \hat{k} & \hat{j} \end{pmatrix} S_{jkl} \end{aligned}$$

where $4\sqrt{\pi}\, S_{jkl} = [(2j+1)(2k+1)(2l+1)(j+k+l+1)(j+k-l)]^{1/2}$.

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After some computation, one finds when $\hat{l} = 0$

$$\frac{i}{2}(B_{\mathbf{j}\mathbf{k}\mathbf{l}} + B_{\mathbf{k}\mathbf{j}\mathbf{l}}) = \frac{1}{4i\hat{j}} \mathcal{J}_{\mathbf{j}\mathbf{k}\mathbf{l}}(\Omega_{\mathbf{j}} + \Omega_{\mathbf{k}}) \quad \text{with} \quad \mathcal{J}_{\mathbf{j}\mathbf{k}\mathbf{l}} = (\partial(Y_{\mathbf{j}}, Y_{\mathbf{k}}), Y_{\mathbf{l}})_{L^2} .$$

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$\Rightarrow$ control on fast–fast–slow interactions.

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Note: computation was essentially brute force – possible group-theoretic structure as in Zeitlin'04?