Lagrangian solutions for semigeostrophic system in physical space

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Derivation of Semigeostrophic Model

Boussinesq system with Coriolis force in 3D:

$$\frac{D}{Dt}(u_1, u_2) + (\frac{\partial p}{\partial x_1}, \frac{\partial p}{\partial x_2}) = (u_2, \, -u_1),$$

$$\frac{D}{Dt}\rho = 0,$$

$$\text{div} u = 0,$$

$$\frac{\partial p}{\partial x_3} + \rho = 0,$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u \cdot \nabla,$$

$$u = (u_1, u_2, u_3), \qquad \nabla = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})$$

Derivation of Semigeostrophic Model

Strong Coriolis forcing:

$$\varepsilon \frac{D}{Dt}(u_1, u_2) + (\frac{\partial p}{\partial x_1}, \frac{\partial p}{\partial x_2}) = (u_2, -u_1),$$

$$\frac{D}{Dt}\rho = 0,$$

$$\text{div} u = 0,$$

$$\frac{\partial p}{\partial x_3} + \rho = 0,$$

where

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u \cdot \nabla,$$

$$u = (u_1, u_2, u_3), \qquad \nabla = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3})$$

Derivation of Semigeostrophic Model

Rotation dominated motion: set $\varepsilon = \mathbf{0}$, thus drop inertial term $\frac{D}{Dt}(u_1, u_2)$ to obtain Geostrophic Balance. This defines Geostrophic Velocities (horizontal)

$$(v_1^g,\ v_2^g) = (\frac{\partial p}{\partial x_2}, -\frac{\partial p}{\partial x_1}),$$

Substitute geostrophic velocities into inertial term $\frac{D}{Dt}(v_1^g,\ v_2^g)$ (where still $\frac{D}{Dt} = \frac{\partial}{\partial t} + u \cdot \nabla$), get Semigeostrophic System

Semigeostrophic System in 3D

Model with rigid boundaries i.e. in domain $\Omega \subset \mathbb{R}^3$.

$$
\begin{aligned}
&\frac{D}{Dt}(v_1^g,\, v_2^g) + (\frac{\partial p}{\partial x_1}, \frac{\partial p}{\partial x_2}) = (u_2,\, -u_1),\\
&\frac{D}{Dt}\rho = 0,\\
&\text{div} u = 0,\\
&\frac{\partial p}{\partial x_3} + \rho = 0,\\
&\frac{D}{Dt} = \frac{\partial}{\partial t} + u \cdot \nabla, \quad (v_1^g,\, v_2^g) = (\frac{\partial p}{\partial x_2}, -\frac{\partial p}{\partial x_1}),
\end{aligned}
$$

in $(0, T) \times \Omega$, with initial and boundary conditions:

$$
\begin{aligned}
&u \cdot \nu = 0 \;\text{ on }\; (0, T) \times \partial\Omega,\\
&p(0, x) = p_0(x) \;\text{ in }\; \{t = 0\} \times \Omega.
\end{aligned}
$$

Semigeostrophic System in 3D

Model was introduced by Eliassen(1948), Hoskins(1975).
Rewrite system:
Use $(u_1, u_2, u_3) = \frac{D}{Dt}(x_1, x_2, x_3)$.
Introduce

$$P(t,x) := p(t,x) + \frac{1}{2}(x_1^2 + x_2^2),$$
$$P_0(x) = p_0(x) + \frac{1}{2}(x_1^2 + x_2^2),$$

and $\frac{\pi}{2}$-rotation in horizontal plane matrix:

$$J = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Semigeostrophic System in 3D

SG system takes form:

$$\frac{DX}{Dt} = J(X - x)$$
$$\mathrm{div} u = 0,$$
$$X = \nabla P, \qquad \frac{D}{Dt} = \frac{\partial}{\partial t} + u \cdot \nabla,$$

in $(0, T) \times \Omega$, with initial and boundary conditions:

$$u \cdot \nu = 0 \ \text{ on } \ (0, T) \times \partial\Omega,$$
$$P(0, x) = P_0(x) \ \text{ in } \ \{t = 0\} \times \Omega.$$

Cullen-Purser stability condition: $P(t, \cdot)$ is convex

Dual Space

Dual space: change of variables:

$$(t, x) \to (t, X), \qquad \text{where} \;\; X = \nabla P_t(x),$$

where we use notation $P_t(\cdot) = P(t, \cdot)$.

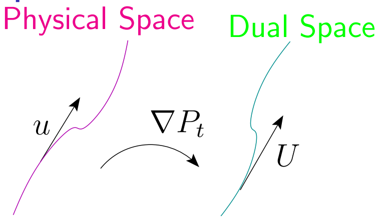
If $P_t(\cdot)$ is strictly convex, then inverse transform is given by

$$x = \nabla P_t^*(X),$$

where $P_t^*(\cdot)$ is the convex dual (Legendre transform) of $P_t(\cdot)$:

$$P_t^*(X) = \sup_{x \in \Omega} [x \cdot X - P(x, t)].$$

Velocity in Dual Space



Let $x(t)$ be a particle path in physical space: $\dot{x}(t) = u(t, x(t))$.
Then $X(t) = \nabla P_t(x(t))$ is a particle path in physical space.
Then velocity in dual space is (using SG system):

$$\begin{aligned} U(t, X(t)) = \dot{X}(t) &= \frac{d}{dt}\left(\nabla P_t(x(t))\right) \\ &= \frac{\partial}{\partial t}\nabla P_t(x) + (\dot{x}(t)\cdot\nabla)\nabla P_t(x) \\ &= \frac{\partial}{\partial t}X + (u\cdot\nabla)X = J(X - x) = J(X - \nabla P_t^*(X)). \end{aligned}$$

Density in Dual Space

Recall: if μ, ν are measures on metric spaces X, Y, and $r : X \to Y$ is Borel, then r pushes forward μ to ν, denoted $r_{\#}\mu = \nu$ if

$$\mu(r^{-1}(A)) = \nu(A) \quad \text{for each Borel } \; A \subset Y.$$

For $t \geq 0$, denote $\alpha_t = \nabla P_{t\#}\chi_\Omega$. Then α_t is density in dual space.
Equation for $\alpha(t, X) = \alpha_t(X)$ (heuristic argument):
Equation $\operatorname{div} u = 0$ in Ω together with condition $u \cdot \nu = 0$ on $\partial\Omega$ imply

$$\partial_t \chi_\Omega + \operatorname{div}(u\chi_\Omega) = 0 \quad \text{in } \; \mathbb{R}^3.$$

Then changing variables $X = \nabla P_t(x)$, and using that velocity in dual space is $U(t, X) = J(X - x) = J(X - \nabla P_t^*(X))$, yields

$$\partial_t \alpha + \operatorname{div}(U\alpha) = 0 \quad \text{in } \; (0, \infty) \times \mathbb{R}^3.$$

Semigeostrophic system in Dual Space

$$\partial_t \alpha + \mathrm{div}(U\alpha) = 0 \quad \text{in} \quad (0,\infty)\times\mathbb{R}^3,$$
$$U(t,X) = J(X - \nabla P_t^*(X)),$$
$$\nabla P_{t\#}\chi_\Omega = \alpha_t,$$
$$\alpha_{|t=0} = \alpha_0.$$

Existence in dual space:

Benamou, Brenier (1998) for 3D rigid boundaries model, case $\alpha_0 \in L^q, \quad q \geq 3$.

Cullen, Gangbo (2001) for 2D shallow water SG model, case $\alpha_0 \in L^q, \quad q \geq 1$.

Lopes-Filho, Nussenzveig-Lopes (2002) extended to $q = 1$.

Loeper (2006) case α_0 a measure. Then ∇P^* is replaced by barycentric projection

Ambrosio, Gangbo (2008) case α_0 a measure: SG in dual space is a Hamiltonian ODE in the Wasserstein spaces.

Relation to Monge-Kantorovich mass transport

Semigeostrophic system in Dual Space

$$\partial_t \alpha + \mathrm{div}(U\alpha) = 0 \quad \text{in} \quad (0,\infty)\times\mathbb{R}^3,$$
$$U(t,X) = J(X - \nabla P_t^*(X)),$$
$$\nabla P_{t\#}\chi_\Omega = \alpha_t,$$
$$\alpha_{|t=0} = \alpha_0.$$

Recall: $P_t(\cdot)$ is convex. Thus $\nabla P_t(\cdot)$ is the optimal map for Monge's problem between measures χ_Ω and α_t with cost $=$ distance^2:

$$I[\nabla P_t] = \min_{s_\# \chi_\Omega = \alpha_t} I[s], \quad I[s] = \int_\Omega |s(x) - x|^2 dx.$$

Monge-Kantorovich problem

Kantorovich problem: Weak formulation of Monge problem:
Define projections $\pi_1, \pi_2 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ by $\pi_1(x, y) = x$, $\pi_2(x, y) = y$. If γ is a measure on $\mathbb{R}^n \times \mathbb{R}^n$, then measures $\mu = \pi_{1\#}\gamma$, $\nu = \pi_{2\#}\gamma$ satisfy

$$\mu(A) = \gamma(A \times \mathbb{R}^n), \quad \nu(A) = \gamma(\mathbb{R}^n \times A) \quad \text{for each Borel } \ A \subset \mathbb{R}^n.$$

μ, ν are called marginals of γ.

Kantorovich problem: Given measures μ, ν on $\mathbb{R}^n$ with $\mu(\mathbb{R}^n) = \nu(\mathbb{R}^n)$, find a measure on $\mathbb{R}^n \times \mathbb{R}^n$ which minimizes

$$J[\gamma] = \int_{\mathbb{R}^n \times \mathbb{R}^n} c(x, y)\, d\gamma(x, y)$$

among all measures γ on $\mathbb{R}^n \times \mathbb{R}^n$ with marginals μ, ν.
Measures γ are called transport plans from between μ and ν.

Monge-Kantorovich problem

Note: if map $s : \mathbb{R}^n \to \mathbb{R}^n$ is a solution of Monge problem for μ, ν, then transport plan

$$\gamma = (Id \times s)_{\#}\mu$$

is a solution of the Kantorovich problem.

Monge-Kantorovich problem

Dual Kantorovich problem: Maximize

$$K[u,v] = \int_{\mathbb{R}^n} u(x)d\mu(x) + \int_{\mathbb{R}^n} v(y)d\nu(y)$$

among admissible pairs of functions (u, v), i.e. satisfying

$$u(x) + v(y) \leq c(x,y) \quad \text{for all } \; x \in \text{supp } \mu, \; y \in \text{supp } \nu.$$

Lemma (Duality): Suppose $s_{\#}\mu = \nu$ and (u, v) is an admissible pair. Then $I[s] \geq K[u, v]$. If

$$u(x) + v(s(x)) = c(x, s(x)) \quad \text{for } \mu\text{-a.e. } \; x,$$

then s is an optimal map, (u, v) is an optimal pair, and $I[s] = K[u, v]$.

Then for optimal pair:

$$u(x) = \min{}_y (c(x,y) - v(y)), \quad v(y) = \min{}_x (c(x,y) - u(x))$$

Monge-Kantorovich problem: Quadratic cost.

$$c(x,y) = \frac{1}{2}|x-y|^2.$$

Measures with finite second moments $\int |x|^2 d\mu(x) < \infty$.
Changing $u(x)$ to $\frac{1}{2}|x|^2 - u(x)$ and $v(y)$ to $\frac{1}{2}|y|^2 - v(y)$, get that admissible pair condition is

$$u(x) + v(y) \geq x \cdot y \quad \text{for all} \;\; x \in \text{supp}\ \mu, \;\; y \in \text{supp}\ \nu,$$

and optimal pair satisfies

$$u(x) = \text{sup}\ _y(x \cdot y - v(y)), \quad v(y) = \text{sup}\ _x(x \cdot y - u(x)).$$

Thus optimal (u, v) are both convex, and mutually convex dual. If equality holds at (x, y) in the admissibility condition, then equality for first derivatives holds at (x, y), thus

$$y = \nabla u(x).$$

Monge-Kantorovich problem: Quadratic cost.

Lemma. If $u(x)$ is convex, and μ is a measure absolutely continuous w.r.t $\mathcal{L}^n$, then the map $s(x) = \nabla u(x)$ is optimal between μ and $\nu = (\nabla u)_\# \mu$.

Theorem (Brenier, Rachev-Ruchendorff, ...). If measures μ and ν satisfy $\mu(\mathbb{R}^n) = \nu(\mathbb{R}^n)$ and have finite second moments, and $\mu << \mathcal{L}^n$, then there exists an optimal map s for Monge problem with quadratic cost. Moreover, $s = \nabla u$ where u is convex.

Connection with Monge-Ampere equations: If $\mu = f(x)dx$, $\nu = g(y)dy$, and convex u satisfies $\nu = (\nabla u)_\# \mu$, that is

$$\int h(\nabla u(x)) f(x) dx = \int h(y) g(y) dy \quad \text{for all} \quad h \in C_c(\mathbb{R}^n),$$

change of variables implies Monge-Ampere equation for u:

$$g(\nabla u(x)) \det D^2 u(x) = f(x)$$

Solving Semigeostrophic system in Dual Space

$$\partial_t \alpha + \mathrm{div}(U\alpha) = 0 \quad \text{in} \quad (0,\infty) \times \mathbb{R}^3,$$
$$U(t,X) = J(X - \nabla P_t^*(X)),$$
$$\nabla P_{t\#}\chi_\Omega = \alpha_t,$$
$$\alpha_{|t=0} = \alpha_0.$$

Existence in dual space: time stepping (Benamou-Brenier, Cullen-Gangbo): let $\Delta t = h$.
Suppose, at time at $t_k = kh$, the $P_k(x)$ convex, and measure $\alpha_k(x)$ are given with $\int_{\mathbb{R}^3} \alpha_k dx = |\Omega|$. Determine velocity

$$U_k(X) = J(X - \nabla P_k^*(X))$$

(plus some regularization...). Solve transport equation

$$\partial_t \alpha + \mathrm{div}(U_k \alpha) = 0 \quad \text{in} \quad (kh, (k+1)h) \times \mathbb{R}^3,$$
$$\alpha_{|t=kh} = \alpha_k.$$

Solving Semigeostrophic system in Dual Space

Then define $\alpha_{k+1} = \alpha((k+1)h)$.

From $\operatorname{div} U_k = 0$ get $\int \alpha_k dx = \int \alpha_{k+1} dx$.

Determine P_{k+1} by solving Monge-Kantorovich problem: P_{k+1} is convex and ∇P_{k+1} is the optimal map between χ_Ω and α_{k+1}.

Then send h to $0+$. Using convexity, can pass to the limit in equations.

Semigeostrophic system in Physical Space

Let (P, α) be a solution in dual space.
Obtain solution (P, u) in physical space, i.e. define physical velocity u. Formally, use relation $x(t) = \nabla P_t(X(t))$ for particle paths. Differentiate:

$$\begin{aligned} u(t,x) =& \partial_t \nabla P_t^*(X) + U \cdot \nabla(\nabla P_t^*(X)) \\ =& \partial_t \nabla P^*(t, \nabla P_t(x)) + D^2 P_t^*(\nabla P_t(x))[J(\nabla P(t,x) - x)], \end{aligned}$$

Here P_t, P_t^* are convex, i.e $D^2 P_t^*$ is a measure, and $\nabla P_t \in L^\infty$. Their product is not well-defined.

Semigeostrophic system in Physical Space: Eulerian solutions

Also, (P, u) is a weak (Eulerian) solution of SG if $\text{div}\, u = 0$ and

$$\int_{(0,T)\times\Omega} \{\nabla P(t,x) \cdot [\partial_t \phi(t,x) + (u(t,x)\cdot\nabla)\phi(t,x)]$$

$$+ J[\nabla P(t,x) - x] \cdot \phi(t,x)\} dtdx + \int_\Omega \nabla P_0(x) \cdot \phi(0,x) dx = 0.$$

for any $\phi \in C_c^1([0,T) \times \Omega; \mathbb{R}^3)$.
Since $\nabla P_t \in L^\infty$, need $u \in L^1_{loc}$.
Existence of $u \in L^1_{loc}$ is not known.
Apriori estimates of u as a measure: Loeper (2005).

Semigeostrophic system in Physical Space: Eulerian solutions

Recent works: De Philippis, Figalli (2011) regularity for Monge-Ampere: if $\Lambda \geq f(x) \geq \lambda > 0$ in Ω and

$$\det D^2 u = f \text{ in } \Omega,$$

then $u \in W^{2,1}(\Omega)$ (and slightly better). Boundary regularity if $\partial\Omega$ is convex and smooth.

Ambrosio, Colombo, De Philippis, Figalli (2011, 2012): existence of Eulerian solutions of SG in 2D-periodic and 3D cases if the density in dual space α_0 is strictly positive $+$...

In the case when the boundary of the support of α_0 is nonempty, say when supp(α_0) is compact, existence of Eulerian solutions is not known, not clear whether can be expected.

The case when the support of α_0 is compact is physically interesting: related to modeling of front formation in atmospheric flows.

Weak Lagrangian Solutions in Physical Space

Cullen-F. 2006.

If (P, u) is smooth, then define flow map of u:
$F : [0, T] \times \Omega \to \Omega$ satisfying

$$\partial_t F(t, x) = u(t, F(t, x))$$
$$F_{|t=0} = Id.$$

Since $u \cdot \nu = 0$ on $\partial\Omega$, it follows for each $t \geq 0$ that $F_t : \Omega \to \Omega$ is diffeomorphism. Then F determines u.

SG system in terms of (P, F):

$$F_{t\#}\chi_\Omega = \chi_\Omega \quad \text{for all} \quad t > 0,$$
$$F_0 = Id,$$

and $Z(t, x) = \nabla P(t, F_t(x))$ is a solution of the ODE

$$\partial_t Z(t, x) = J[Z(t, x) - F(t, x)] \qquad \text{in} \quad [0, T) \times \Omega,$$
$$Z(0, x) = \nabla P_0(x).$$

Weak Lagrangian Solutions in Physical Space

Let $\Omega \subset \mathbb{R}^3$ be an open bounded set, and $T > 0$. Let $P_0(x) \in W^{1,\infty}(\Omega)$ be convex. Let $r \in [1, \infty)$. Let

$$P \in L^\infty([0,T); W^{1,\infty}(\Omega)) \cap C([0,T); W^{1,r}(\Omega)),$$
$$P_t(\cdot) \quad \text{is convex in} \quad \Omega \quad \text{for each} \quad t \in [0,T).$$

Let $F : [0,T) \times \Omega \to \Omega$ satisfy $F \in C([0,T); L^r(\Omega; \mathbb{R}^3))$. (P, F) is a weak Lagrangian solution of SG system if

$$F_{t\#}\chi_\Omega = \chi_\Omega \quad \text{for all} \quad t > 0,$$
$$F_0 = Id,$$

and $Z(t,x) = \nabla P(t, F_t(x))$ is a weak solution of the ODE

$$\partial_t Z(t,x) = J[Z(t,x) - F(t,x)] \qquad \text{in} \quad [0,T) \times \Omega,$$
$$Z(0,x) = \nabla P_0(x).$$

Existence of Weak Lagrangian Solutions in Physical Space

Cullen, Feldman (2006): if $\alpha_0 := \nabla P_0 \# \chi_\Omega \in L^q$, $q > 1$, for 3D rigid boundaries and 2D shallow water SG models.

Outline of proof: Combining Cullen-Gangbo time-stepping procedure, and Ambrosio theory of Hamilton-Jacobi equations and ODE with BV vector fields, obtain Lagrangian flow $\Phi(t, X)$ in dual space, and

$$\alpha_t = \Phi_{t\#}\alpha_0.$$

Here we use that $U(t, X) = J(X - \nabla P_t^*(X))$ is BV (as a gradient of convex function) and divergence-free (by $J\nabla$-structure).

Then the flow in physical space is

$$F_t = \nabla P_t^* \circ \Phi_t \circ \nabla P_0.$$

Faria, Lopes-Filho, Nussenzveig-Lopes (2009): $q = 1$ case.

Remark on condition $\alpha_0 := \nabla P_0 \# \chi_\Omega \in L^q$

This condition is a form of strict convexity of P_0. For example, if P_0 is uniformly strictly convex, i.e. $P_0(x) - \varepsilon x^2$ is convex, then $\alpha_0 \in L^\infty$.
If P_0 is affine on a set of positive measure, then α_0 has a delta-function (i.e. a point of nonzero measure).

Case of $\alpha_0 := \nabla P_0 \# \chi_\Omega$ is a measure

This case is physically relevant.

Solutions in dual space Loeper (2006), Ambrosio, Gangbo (2008): P_t is convex and α_t is a measure, and satisfy:

$$
\begin{aligned}
&\partial_t \alpha + \mathrm{div}(U\alpha) = 0 \quad \text{in} \quad (0,\infty) \times \mathbb{R}^3, \\
&U(t,X) = J(X - \bar{\gamma}_t(X)), \\
&\nabla P_{t\#}\chi_\Omega = \alpha_t, \\
&\alpha_{|t=0} = \alpha_0,
\end{aligned}
$$

where $\bar{\gamma}_t(X)$ is the barycentric projection of the optimal Kantorovich plan $\gamma_t := \big(\nabla P_t \times \mathrm{Id}\big)_{\#}\chi$ having α_t and χ as first and second marginals, respectively. It is defined by

$$
\int_{\mathbb{R}^3} \xi(X) \cdot \bar{\gamma}_t(X) d\alpha_t(X) = \iint_{\mathbb{R}^3 \times \Omega} \xi(X) \cdot y d\gamma_t(X,y)
$$

for all continuous $\xi : \mathbb{R}^3 \to \mathbb{R}^3$ of at most quadratic growth.

Case $\alpha_0 := \nabla P_0 \# \chi_\Omega$ is a measure: Flow map in physical space

(F.-Tudorascu 2012)
Define Lagrangian solutions in physical space when α_0 is singular: Since $\bar{\gamma}_t$ replaces ∇P_t^*, try

$$F_t = \bar{\gamma}_t \circ \Phi_t \circ \nabla P_0.$$

Example: $\Omega = B_1$, $P_0(x) = 0$.
Then $P_t(x) = 0$ on B_1 for all t, and $\alpha_t = \delta_0$. Also $P_t^*(X) = |X|$.
Thus $\bar{\gamma}_t(0) = 0$ which defined $\bar{\gamma}_t(X)$ for α_t-a.e. $X \in \mathbb{R}^3$. Can set $\bar{\gamma}_t(X) = \nabla P_t^*$ for $X \neq 0$.
Also, $U(t, X) = J(X - \bar{\gamma}_t(X))$, thus $\Phi(t, 0) = 0$ is a solution of ODE $\frac{d}{dt}\Phi(t, 0) = U(t, U(t, \Phi(t, 0))$. In fact, this is a continuous extension to $X = 0$ of the regular flow $\Phi(t, X)$ for vector field $U(t, X) = J(X - \nabla P_t^*)$.

Case $\alpha_0 := \nabla P_0 \# \chi_\Omega$ is a measure: Flow map in physical space

We get $F_t(x) = \bar{\gamma}_t \circ \Phi_t \circ \nabla P_0(x) = 0$ for any $x \in B_1$, $t > 0$.
In particular, $F_{t\#}\chi_\Omega = \delta_0 \neq \chi_\Omega$: Incompressibility of $F_t(\cdot)$ is violated.
Also can show: for regularizations $P_0^\varepsilon = \varepsilon \|x|^2$ get $F^\varepsilon \rightharpoonup F$ weakly-* in $L^\infty([0,T] \times B_1)$, but not in $L^p(B_1)$ for each t.

$F_t = \bar{\gamma}_t \circ \Phi_t \circ \nabla P_0$. Issues to address:
(i) If Φ_t is a Lagrangian flow ∇P_t^* (or, equivalently, for $\bar{\gamma}_t$), then it is not clear if $\alpha_t = \Phi_{t\#}\alpha_0$ holds (or even well-defined);
(ii) If $\alpha_t = \nabla P_{t\#}\chi_\Omega$ is a singular measure, then $(\bar{\gamma}_t \circ \nabla P_t)_{\#}\chi_\Omega \neq \chi_\Omega$. Thus $F_0 \# \chi_\Omega \neq \chi_\Omega$. Then probably $F_{t\#}\chi_\Omega \neq \chi_\Omega$ for $t > 0$. Instead, define "reduced domain" measures $\mu_t = (\bar{\gamma}_t \circ \nabla P_t)_{\#}\chi_\Omega$. Require: $F_{t\#}\chi_\Omega = \mu_t$, and $F_{t\#}\mu_0 = \mu_t$. Note: if $\alpha_t \in L^1(\mathbb{R}^3)$, then $\mu_t = \chi_\Omega$.

Case $\alpha_0 := \nabla P_0 \# \chi_\Omega$ is a measure: Existence of Lagrangian Solutions in Physical Space

Proposition Given solution (P_t, α_t) in dual space: If there exists Lagrangian flow in dual space $\Phi(t, X)$ satisfying

$$\partial_t \Phi(t, X) = J(\Phi(t, X) - \bar{\gamma}_t(\Phi(t, X))), \quad \Phi_{|t=0} = Id,$$
$$\alpha_t = \Phi_{t\#}\alpha_0,$$

then (P, F) with $F_t = \bar{\gamma}_t \circ \Phi_t \circ \nabla P_0$, is a Lagrangian solution in physical space.

Theorem If $P_0 = \max_{k=1,\dots,n} L_k(X)$, where each L_k is an affine function, then there exists a Lagrangian solution (P, F) in physical space.

Remark In the conditions of theorem, α_0 is a convex combination of delta-functions.

Properties of Weak Lagrangian Solutions in Physical Space

Geostrophic energy:

$$E(t) = \int_{\Omega} |\nabla P_t(x) - x|^2 dx.$$

Formally $E(t) = const$ on solutions of SG system.

Theorem, If (P, F) is a weak Lagrangian solution. Then

- Let $\alpha_t := \nabla P_{t\#}\chi_\Omega$. Then (P, α) is a solution of SG in dual space:

$$\partial_t \alpha + \text{div}(U\alpha) = 0 \quad \text{in} \quad (0, \infty) \times \mathbb{R}^3,$$
$$U(t, X) = J(X - \bar{\gamma}_t(X)).$$

- $E(t) = const$.

Remarks: (i) $\alpha_t = \nabla P_{t\#}\chi_\Omega$ may be a singular measure;
(ii) Once we know (P, α) is a solution, then $E(t) = const$ follows from work of Ambrosio-Gangbo.

Relaxed Lagrangian solutions in physical space

(F.-Tudorascu, 2013)
Existence for arbitrary (possibly non-strictly) convex initial P_0:
replace flow map $F_t : \Omega \to \Omega$ by transport plan σ_t on $\Omega \times \Omega$.

Let $\alpha_0 = \nabla P_{0\#}\chi_\Omega \in L^q$ and (P_t, F_t) is a Lagrangian solution.
Define measure $\sigma_t = (Id \times F_t)_{\#}\chi_\Omega$ on $\Omega \times \Omega$. Then:
(i) $\sigma_0 = (Id \times Id)_{\#}\chi_\Omega$;
(ii) $\pi_{1\#}\sigma_t = \chi_\Omega$, $\pi_{2\#}\sigma_t = \chi_\Omega$, where $\pi_k(\mathbf{x}) = \mathbf{x}_k$ for $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2) \in \Omega \times \Omega$, $k = 1, 2$.
(iii) for any $\varphi \in C_c^1([0,T) \times \Omega; \mathbb{R}^3)$

$$\int_0^T \int_{\Omega\times\Omega} \left[\nabla P_t(y) \cdot \partial_t \varphi(t,x) + J(\nabla P_t(y) - y) \cdot \varphi(t,x)\right] d\sigma_t(x,y)\, dt + \int_\Omega \nabla P_0(x) \cdot \varphi(0,x)\, dx = 0.$$

Such definition of relaxed solutions is too weak: depends only on global averages. Modify to get local averages.

Renormalized Lagrangian solutions

If $\alpha_0 = \nabla P_{0\#}\chi_\Omega \in L^q$, $q > 1$ and $P_0 \in W^{1,\infty}(\Omega)$, then Lagrangian solutions (P, F) satisfy: $Z, \partial_t Z \in L^\infty([0,T) \times \Omega)$ where recall that $Z_t(x) = \nabla P_t(F_t(X))$, and

$$\partial_t Z(t,x) = J[Z(t,x) - F(t,x)] \text{ a.e. in } [0,T) \times \Omega.$$

Thus if $\xi \in C^1(\mathbb{R}^3)$

$$\partial_t(\xi(Z(t,x)) = \nabla\xi(Z(t,x)) \cdot J[Z(t,x) - F(t,x)] \quad \text{a.e. in } [0,T) \times \Omega.$$

Thus for any $\xi \in C_b^1(\mathbb{R}^3)$, $\varphi \in C_c^1([0,T) \times \Omega)$

$$\int_0^T \int_\Omega \Big[\xi(Z(t,x))\partial_t\varphi(t,x) + \nabla\xi(Z(t,x)) \cdot J\Big(Z(t,x) - F(t,x)\Big)\varphi(t,x)\Big]\,dx\,dt + \int_\Omega \xi(\nabla P_0(x)\varphi(0,x))\,dx = 0.$$

Renormalized Relaxed Lagrangian solutions

Definition. Let P_0 be convex on Ω. Let $P : [0,T) \times \Omega \to \mathbb{R}^1$, and let $\sigma = \int_0^T \sigma_t dt$ be a Borel measure on $[0,T) \times \Omega \times \Omega$.

Then (P, σ) is a renormalized relaxed Lagrangian solution of SG with initial data P_0 if

(i) P_t is convex for each $t \in [0,T)$,

(ii) $\pi_{1\#}\sigma_t = \chi_\Omega$, $\pi_{2\#}\sigma_t = \chi_\Omega$, where $\pi_k(\mathbf{x}) = \mathbf{x}_k$ for $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2) \in \Omega \times \Omega$, $k = 1, 2$.

(iii) for any $\xi \in C_b^1(\mathbb{R}^3)$, $\varphi \in C_c^1([0,T) \times \Omega)$

$$\int_0^T \int_{\Omega\times\Omega} \Big[\xi(\nabla P_t(y))\partial_t\varphi(t,x) + \nabla\xi(\nabla P_t(y)) \cdot J\Big(\nabla P_t(y) - y\Big)\varphi(t,x)\Big] d\sigma_t(x,y)\, dt + \int_\Omega \xi(\nabla P_0(x))\varphi(0,x)\, dx = 0.$$

Equation (iii) well-defined for $\nabla P \in L^\infty([0,T); L^1(\Omega))$ by (ii).

Renormalized Relaxed Lagrangian solutions

These solutions are somewhat underdeterminate: heuristically, $\sigma_t(x,y)$ can be modified on flat parts of P_0 and P_t resp.

Example: $\Omega = B_1$, $P_0(x) = 0$. Then $P_t(x) = 0$ for all t. Equation is: for any $\xi \in C_b^1(\mathbb{R}^3)$, $\varphi \in C_c^1([0,T) \times \Omega)$

$$\int_0^T \int_{\Omega\times\Omega} \Big[\xi(0)\partial_t\varphi(t,x) - \varphi(t,x)\nabla\xi(0)\cdot Jy\Big] d\sigma_t(x,y)\,dt + \int_\Omega \xi(0)\varphi(0,x)\,dx = 0,$$

and using $\pi_{1\#}\sigma_t = \chi_\Omega$,

$$\int_0^T \int_{\Omega\times\Omega} \varphi(t,x)\nabla\xi(0)\cdot Jy\,d\sigma_t(x,y)\,dt = 0.$$

Thus any Borel family $\sigma_t(x,y)$ with marginals χ_Ω and y-barycenter $\bar{\sigma}_t(x) = 0$ works.

Such underdeterminacy is physically natural/expected.

Renormalized Relaxed Lagrangian solutions: Results

(F.-Tudorascu, 2013)

- Existence for any convex $P_0 \in H^1(\Omega)$.
- Stability: for a bounded in $H^1(\Omega)$ sequence of initial data P_0^k such that $P_0^k \to P_0$ in $L^2(\Omega)$, let (P^k, σ^k) be a solution for P_0^k. Then a subsequence of (P^k, σ^k) converges in some weak sense to a solution (P, σ) for P_0.
- Time continuity: $t \to (P_t, \sigma_t)$ is continuous in an appropriate weak sense (recall underdeterminacy!!). Initial condition $\sigma_0 = \delta_{x=y}$ holds in that sense.
- Return to dual space, conservation of geostrophic energy: Let $\alpha_t = \nabla P_{t\#}\chi_\Omega$. Then (P, α) is a distribution solution in dual space. In particular geostrophic energy is conserved.

Open problems

- Uniqueness of weak (renormalized) solutions. Possibly weak-strong uniqueness.
- Existence of solutions for the case of variable Coriolis parameter: dual space is not defined.