Renormalized relaxed Lagrangian solutions for the Semi-Geostrophic system in physical space

A. Tudorascu, West Virginia U.
joint work with M. Feldman (UW-Madison)

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Standard formulation (3D with rigid boundary)

Let $\Omega \subset \mathbb{R}^3$ be an open bounded set and $T > 0$. The semigeostrophic (SG) system (Eliassen 1948, Hoskins 1974) in $(0,T) \times \Omega$:

$$
\begin{aligned}
&D_t \mathbf{v}_g + \mathbf{e}_3 \times \mathbf{v} = -\bar{\nabla} p, \\
&\nabla p + \mathbf{e}_3 \times \mathbf{v}_g = \theta \mathbf{e}_3, \\
&\nabla \cdot \mathbf{u} = 0, \\
&D_t \theta = 0.
\end{aligned}
$$

The initial and boundary data:

$$
\begin{aligned}
&\mathbf{u} \cdot \nu = 0, \quad \text{on } [0,T) \times \partial\Omega, \\
&p(0,x) = p_0(x) \quad \text{in } \Omega.
\end{aligned}
$$

Notation: $\mathbf{v}_g = \mathbf{e}_3 \times \bar{\nabla} p$ is the *geostrophic wind*, $\mathbf{u} = \mathbf{v}_g + \mathbf{v}_a = \mathbf{v} + u_3 \mathbf{e}_3 \in \mathbb{R}^3$ is the total wind velocity ($\mathbf{v}_a$ is the *ageostrophic wind*), p is the pressure, θ is the potential temperature; $\bar{\nabla} = (\partial_{x_1}, \partial_{x_2}, 0)$, $D_t = \partial_t + \mathbf{u} \cdot \nabla$.

SG in terms of the modified pressure

- Introducing the function

$$P(t,x) = p(t,x) + (x_1^2 + x_2^2)/2,$$

we rewrite SG as the following system of equations for $(P, \mathbf{u})$:

$$\begin{aligned}
&D_t X = J(X - x),\\
&\nabla \cdot \mathbf{u} = 0,\\
&X = \nabla P,\\
&\mathbf{u} \cdot \nu = 0, \quad \text{on } [0,T) \times \partial\Omega,\\
&P(0,x) = P_0(x) \quad \text{in } \Omega,
\end{aligned}$$

- Here $J = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

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Cullen-Purser stability

- Let $(P, \mathbf{u})$ be a solution and set $\mathcal{X}(t) := \{Y : \Omega \to \mathbb{R}^3 \text{ Borel } : Y_{\#}\chi = \nabla P(t,\cdot)_{\#}\chi\}$.
- (Cullen & Shutts) Then $X(t,\cdot) = \nabla P(t,\cdot)$ is a critical point for

$$I(Y) = \int_\Omega |Y(x) - x|^2 dx$$

among all maps $Y \in \mathcal{X}(t)$, for all $t \in (0,T)$.
- (Cullen & Purser) We are looking for minimizers.
- In the language of Optimal Transport, this means $X(t,\cdot)$ must be the gradient of a convex function, i.e. $P(t,\cdot)$ must be convex for all $t \in (0,T)$.

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Do we have solutions?

- In the general case, $\mathbf{u}$ may be a singular measure. We need to understand what we mean by "solution".

Theorem (Feldman & T.)

Let $(P, \mathbf{u})$ be a distributional solution for SG in the physical space such that $\nabla P \in H^1(0,T; L^2(\Omega;\mathbb{R}^3))$. Then $\alpha_t := \nabla P_{t\#}\chi$ is atom-free for $\mathcal{L}^1$–a.e. $t \in (0,T)$.

- We would like the model to accommodate solutions for which ∇P_t could be locally constant.

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A change of variable and a new system

- Let $\nabla P(t,\cdot)_{\#}\chi =: \alpha(t,\cdot)$. If $\mathcal{L}^3(\Omega) = 1$, then α_t is a Borel probability on $\mathbb{R}^3$.
- The path of measures $t \mapsto \alpha_t$ solves:

$$\partial_t \alpha + \nabla \cdot (U\alpha) = 0 \quad \text{in } [0,T) \times \mathbb{R}^3, \tag{1}$$

$$\nabla P(t,\cdot)_{\#}\chi_\Omega = \alpha(t,\cdot) \quad \text{for any } t \in [0,T);$$

$$U(t,X) = J[X - \bar{\gamma}(t,X)], \tag{2}$$

$$\alpha(0,X) = \alpha_0(X) \quad \text{for a.e. } X \in \mathbb{R}^3,$$

where $\bar{\gamma}(t,\cdot)$ is the barycentric projection of $\nabla P^*(t,\cdot)$.

- Note that equation (1) represents the fact that the dual flow $t \mapsto \alpha_t$ is weakly (in the sense of distributions) transported by the dual velocity U defined by (2).

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Existence for SG

- Solutions in "dual variables" Benamou & Brenier, Cullen & Gangbo, Cullen & Maroofi.
- Hamiltonian system treatment Gangbo & Pacini, Ambrosio & Gangbo.
- Dual-space solutions are obtained either through variational techniques, or by iterative methods.
- Lagrangian solutions in physical space in the case of nonsingular dual-space solutions Cullen & Feldman.
- Eulerian solutions in physical space in special cases ("nice" dual-space solutions) Ambrosio, Colombo, De Philippis & Figalli.
- Case of singular measures in dual space: existence of solutions for discrete measures Feldman & T.

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Formal Lagrangian flow

- Assume one has a classical solution to SG in physical space.
- If $\mathbf{u}$ has a flow $F : [0,T] \times \Omega \to \Omega$ defined by

$$\partial_t F(t,x) = \mathbf{u}(t, F(t,x)), \quad F(0,x) = x,$$

  then F can replace $\mathbf{u}$ as an unknown. Let $Z(t,x) := \nabla P(t, F(t,x))$.
- The system for (P,F) becomes

$$\begin{aligned} &\partial_t Z(t,x) = J\left[Z(t,x) - F(t,x)\right] && \text{in } [0,T) \times \Omega, \\ &Z(0,x) = \nabla P_0(x) && \text{in } \Omega. \end{aligned} \tag{LE}$$

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Definition in the general case

Let $P_0 \in W^{1,\infty}(\Omega)$, $F : [0,T) \times \Omega \to \Omega$ and $P : [0,T) \times \Omega \to \mathbb{R}$ be as before. Set $\alpha_t := \nabla P_{t\#}\chi$, $\mu_t := \bar{\gamma}_{t\#}\alpha_t$. Then the pair (P,F) is called a weak Lagrangian solution for SG in $[0,T) \times \Omega$ if

- $F(0,x) = x$ for μ_0-a.e. $x \in \Omega$, $P(0,x) = P_0(x)$ for a.e. $x \in \Omega$,
- for any $t > 0$ the mapping $F_t = F(t,\cdot) : \Omega \to \Omega$ satisfies $F_{t\#}\mu_0 = \mu_t$;
- there exists a Borel mapping $F^* : [0,T] \times \Omega \to \Omega$ satisfying $F^*_{t\#}\mu_t = \mu_0$ and $F^*_t \circ F_t = \mathrm{Id}$ μ_0-a.e., $F_t \circ F^*_t = \mathrm{Id}$ μ_t-a.e. ;
- The function $Z : (0,T) \times \Omega \to \mathbb{R}^3$ defined by $Z_t = \nabla P_t \circ F_t$ lies, along with F, in $L^1\big((0,T) \times \Omega; \mathbb{R}^3\big)$ and is a distributional solution of (LE).

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Let $P_0 \in W^{1,\infty}(\Omega)$, $F : [0,T) \times \Omega \to \Omega$ and $P : [0,T) \times \Omega \to \mathbb{R}$ be as before. Set $\alpha_t := \nabla P_{t\#}\chi$, $\mu_t := \bar{\gamma}_{t\#}\alpha_t$. Then the pair (P,F) is called a weak Lagrangian solution for SG in $[0,T) \times \Omega$ if

- $F(0,x) = x$ for μ_0-a.e. $x \in \Omega$, $P(0,x) = P_0(x)$ for a.e. $x \in \Omega$,
- for any $t > 0$ the mapping $F_t = F(t,\cdot) : \Omega \to \Omega$ satisfies $F_{t\#}\mu_0 = \mu_t$;
- there exists a Borel mapping $F^* : [0,T] \times \Omega \to \Omega$ satisfying $F^*_{t\#}\mu_t = \mu_0$ and $F^*_t \circ F_t = \mathrm{Id}$ μ_0-a.e., $F_t \circ F^*_t = \mathrm{Id}$ μ_t-a.e. ;
- The function $Z : (0,T) \times \Omega \to \mathbb{R}^3$ defined by $Z_t = \nabla P_t \circ F_t$ lies, along with F, in $L^1\big((0,T) \times \Omega; \mathbb{R}^3\big)$ and is a distributional solution of (LE).

Existence and consistency

- Cullen & Feldman used Ambrosio's theory of regular Lagrangian flows to construct weak Lagrangian solutions in physical space in the case $\alpha_0 \in L^p(\mathbb{R}^3)$ for some $p > 1$.
- Recall $U_t = J[\mathrm{Id} - \nabla P_t^*]$. There exists (Ambrosio) a unique Borel bounded, $\mathcal{L}^3$–measure-preserving (for a.e. t) vector-map $\Phi(t, X)$ satisfying $\dot{\Phi} = U(\cdot, \Phi)$, $\mathcal{L}^4 -$ a.e. in $(0,T) \times \nabla P_0(\Omega)$ and $\Phi(0, \cdot) = \mathrm{Id}$, $\alpha_0 -$ a.e. in $\mathbb{R}^3$.
- Set $F_t := \nabla P_t^* \circ \Phi_t \circ \nabla P_0$. Then (P, F) is a weak Lagrangian solution of SG in physical space (Cullen & Feldman).
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When α_0 is singular; an example

- If $\alpha_0 = \delta_{z_0}$ for some $z_0 \in \mathbb{R}^3$, one can readily check that if $\dot{z}(t) = Jz(t),\ z(0) = z_0$, then $\alpha_t := \delta_{z(t)}$ solves the SG in dual space.
- There are no $\mathcal{L}^3$–measure preserving maps F_t as in the definition of weak Lagrangian solutions.
- In general, it is natural to ask if one can keep (LE) valid by putting $F_t := \bar{\gamma}_t \circ \Phi_t \circ \nabla P_0$.
- This F_t pushes $\mu_0 := \bar{\gamma}_{0\#}\alpha_0 = \delta_0$ forward to the "reduced domain"–measure $\mu_t := \bar{\gamma}_{t\#}\alpha_t = \delta_0$.
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Dual flow gives weak Lagrangian solutions in physical space

Theorem (Feldman & T.)

Assume there exists a Borel flow-map $\Phi : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^3$ for U such that $\Phi_{t\#}\alpha_0 = \alpha_t$ for all $t \in [0,T)$. Then the pair (P,F) is a weak Lagrangian solution in physical space satisfying all properties but the existence of F^* from the previous definition. Furthermore, if there also exists a Borel map $\Phi^* : [0,T] \times \mathbb{R}^3 \to \mathbb{R}^3$ such that $\Phi^*_{\#}\alpha_t = \alpha_0$ for all $t \in [0,T)$, then (P,F) is a weak Lagrangian solution in the physical space.

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- To obtain Φ in the general case.
- Stability: if one approximates α_0 by averages of point masses, can one get convergence of Φ^n (to Φ) in a reasonable sense?
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An observation (1)

- If $\alpha_0 \in L^p(\mathbb{R}^3)$ for some $1 \leq p < \infty$, we have (P, F) a weak Lagrangian solution with F_t volume preserving for all $t \in [0, T)$ and $F_0 = \mathrm{Id}_\Omega$.
- Set $\sigma_t := (\mathrm{Id} \times F_t)_{\#}\chi$. This is a probability measure on $\Omega \times \Omega$ with both marginals χ. Also, $\sigma_0 = (\mathrm{Id} \times \mathrm{Id})_{\#}\chi$.
- Set $\sigma(dt, dx, dy) := \sigma_t(dx, dy)dt$.
- Let $\xi \in C_b^1(\mathbb{R}^3)$. Then $t \mapsto \xi(Z(t, x))$ is absolutely continuous for χ-a.e. $x \in \Omega$ and

$$\frac{\mathrm{d}}{\mathrm{d}t}\xi(Z(t,x)) = -J\nabla\xi(Z(t,x)) \cdot \left[Z(t,x) - F(t,x)\right] \text{ for a.e. } t \in [0,T).$$

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- So, for any $\xi \in C_b^1(\mathbb{R}^3)$, $\zeta \in C_c^1([0,T)\times\Omega)$, we have

$$\int_0^T \int_\Omega \Big\{ \xi(Z(t,x))\partial_t\zeta(t,x) + J\nabla\xi(Z(t,x))\cdot\big[F(t,x) - Z(t,x)\big]\zeta(t,x)\Big\} dx\,dt + \int_\Omega \xi(\nabla P_0(x))\zeta(0,x)\,dx = 0.$$

- In terms of σ, this reads

$$\int_0^T \iint_{\Omega\times\Omega} \Big\{ \xi(\nabla P_t(y))\partial_t\zeta(t,x) + J\nabla\xi(\nabla P_t(y))\cdot\big[y - \nabla P_t(y)\big]\zeta(t,x)\Big\}\,\sigma(dt,dx,dy) + \int_\Omega \xi(\nabla P_0(x))\zeta(0,x)\,dx = 0.$$

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Renormalized Relaxed Lagrangian solutions

Definition. Let P_0 be convex on Ω. Let $P : [0,T) \times \Omega \to \mathbb{R}$, and let $\sigma = \int_0^T \sigma_t dt$ be a Borel measure on $[0,T) \times \Omega \times \Omega$. Then (P, σ) is a renormalized relaxed Lagrangian solution of SG with initial data P_0 if

(i) P_t is convex for each $t \in [0,T)$;

(ii) $\pi_{k\#}\sigma_t = \chi$ for each $t \in [0,T)$, where $\pi_k(\mathbf{x}) = x_k$ for $\mathbf{x} = (x_1, x_2) \in \Omega \times \Omega$, $k = 1, 2$;

(iii) for any $\xi \in C_b^1(\mathbb{R}^3)$, $\zeta \in C_c^1([0,T) \times \Omega)$

$$\int_0^T \iint_{\Omega\times\Omega} \Big\{ \xi(\nabla P_t(y)) \partial_t \zeta(t,x) + J\nabla\xi(\nabla P_t(y)) \cdot [y - \nabla P_t(y)] \zeta(t,x) \Big\} \sigma(dt,dx,dy) + \int_\Omega \xi(\nabla P_0(x)) \zeta(0,x)\, dx = 0. \qquad (\sigma\text{-eq})$$

- Equation (iii) well-defined for $\nabla P \in L^\infty([0,T); L^1(\Omega))$ by (ii).

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(ii) $\pi_{k\#}\sigma_t = \chi$ for each $t \in [0,T)$, where $\pi_k(\mathbf{x}) = x_k$ for $\mathbf{x} = (x_1,x_2) \in \Omega \times \Omega$, $k = 1,2$;
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- Equation (iii) well-defined for $\nabla P \in L^\infty([0,T); L^1(\Omega))$ by (ii).

Renormalized Relaxed Lagrangian solutions

Definition. Let P_0 be convex on Ω. Let $P : [0,T) \times \Omega \to \mathbb{R}$, and let $\sigma = \int_0^T \sigma_t dt$ be a Borel measure on $[0,T) \times \Omega \times \Omega$. Then (P, σ) is a renormalized relaxed Lagrangian solution of SG with initial data P_0 if
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Global Lipschitz solutions in dual space. Weak stability

Theorem (Ambrosio & Gangbo, Feldman & T.)

Let $\alpha_0 \in \mathcal{P}_2(\mathbb{R}^3)$ be given. Then there exists a distributional solution $\alpha \in AC^\infty(0,\infty;\mathcal{P}_2(\mathbb{R}^3))$ for SG in dual space with $\alpha(0) = \alpha_0$, where $\bar{\gamma}_t$ is the barycentric projection of the optimal plan between α_t and χ. In other words, the curve α is globally Lipschitz and satisfies SG in dual space in the sense of distributions in $(0,\infty) \times \mathbb{R}^3$. Furthermore, the Hamiltonian energy $t \to H(\alpha_t) := -W_2^2(\alpha_t,\chi)/2$ is conserved.

- Since $[0,\infty) \ni t \to \alpha_t \in \mathcal{P}_2(\mathbb{R}^3)$ is continuous, we conclude that there is a family $P \in C([0,\infty); H^1(\Omega))$ of convex functions $P(t,\cdot)$ such that $\nabla_x P(t,\cdot) =: \nabla P_t$ pushes χ forward to α_t optimally for all $t \geq 0$.

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Weak stability in dual space

Theorem (Feldman & T.)

Let $\{\alpha_0\} \cup \{\alpha_0^n\}_{n\geq 1} \subset \mathcal{P}_2(\mathbb{R}^3)$ be such that

$$\alpha_0^n \rightharpoonup \alpha_0 \text{ weakly } \star \text{ as measures, and } \sup_{n\geq 1} W_2(\alpha_0^n, \alpha_0) < \infty.$$

Let α^n be solutions as in previous theorem corresponding to the initial data α_0^n. Then, possibly up to a subsequence, α^n converges to a solution $\alpha \in AC^\infty(0,\infty;\mathcal{P}_2(\mathbb{R}^3))$ with initial data α_0. More precisely, we have
(i) $\alpha_t^n \to \alpha_t$ in $\mathcal{P}_p(\mathbb{R}^3)$ for all $t \in [0,\infty)$ and all $p \in [1,2)$.
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Stability of optimal maps

Theorem (Feldman & T.)

Let $\Omega \in \mathbb{R}^d$ be open and bounded and consider the Borel measures $\{\mu_n\}_{n\geq 1}$, μ in $\mathbb{R}^d$ with total mass equal to $\mathcal{L}^d(\Omega)$ and with finite second moments. Denote by M_n, M the Brenier optimal maps for pushing $\mathcal{L} := \mathcal{L}^d|_\Omega$ forward to μ_n, μ for the quadratic cost (i.e., gradients of convex functions such that $M_{n\#}\mathcal{L} = \mu_n$, $M_\#\mathcal{L} = \mu$). Then, for any $1 < p \leq 2$, we have

$$\lim_{n\to\infty} W_p(\mu_n, \mu) = 0 \text{ is equivalent to } \lim_{n\to\infty} \left\| M_n - M \right\|_{L^p(\Omega;\mathbb{R}^d)} = 0.$$

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Weak stability for renormalized relaxed solutions

Corollary

Let $P_0 \in H^1(\Omega)$, $\{P_0^n\}_n \subset H^1(\Omega)$ be convex and such that $\nabla P_0^n \to \nabla P_0$ in $L^2(\Omega;\mathbb{R}^3)$. Assume (P^n, σ^n) are renormalized relaxed solutions for SG in physical space corresponding to the initial data P_0^n. Then, possibly up to a subsequence, (P^n, σ^n) converges to a renormalized relaxed solution (P, σ) corresponding to the initial datum P_0. This convergence is in the following sense:
(i) $\nabla P_t^n \to \nabla P_t$ in $L^p(\Omega;\mathbb{R}^3)$ for all $t \in [0, T)$ and all $1 \le p < 2$;
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Furthermore, the dual space solutions satisfy $W_p(\alpha_t^n, \alpha_t) \to 0$ for all $t \in [0, T)$ and all $1 \le p < 2$.

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Existence of renormalized relaxed solutions

Corollary

Let $P_0 \in H^1(\Omega)$. Then there exists a renormalized relaxed solution (P, σ) corresponding to the initial datum P_0.

- Idea of proof: approximate P_0 in $H^1(\Omega)$ by "nice" convex functions P_0^n such that $\nabla P^n_{0\#}\chi =: \alpha_0^n \in L^1(\mathbb{R}^3)$, then use Cullen & Feldman's solutions (P^n, F^n) to construct $\sigma_t^n := (\mathrm{Id} \times F_t^n)_{\#}\chi$. Then use the *a priori* stability result (previous corollary) to conclude.

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Time-continuity

- Define $G_t : \Omega \times \Omega \to \mathbb{R}^3 \times \mathbb{R}^3$ by $G_t(x, y) = (x, \nabla P_t(y))$, then $t \to G_{t\#}\sigma_t$ is continuous with respect to narrow convergence on $\Omega \times \mathbb{R}^3$: i.e., from the equation, for any $\xi,\ \zeta \in C_c^1(\mathbb{R}^3)$

$$t \to \int_{\Omega\times\Omega} \xi(\nabla P_t(x))\zeta(x)\,\sigma_t(dx, dy) = \int_{\Omega\times\mathbb{R}^3} \zeta(x)\,\xi(Y)\,G_{t\#}\sigma_t(dx, dY)$$

  is continuous.
- Also, for any $s \in [0, T]$: $t \to G_{s\#}\sigma_t$ is continuous at $t = s$ with respect to narrow convergence on $\Omega \times \mathbb{R}^3$.
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Under-determinacy in the singular case; an example

- Let $\Omega = B(0,1)$, $P_0 \equiv 0$ and $T = 2\pi$.
- The solution obtained by the described procedure is $d\tilde{\sigma}dt$, where

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  where $\mathcal{S}$ is the hypersurface in $\Omega \times \Omega$ of equation $x^2 = y^2$ for $\mathbf{x} := (x, y) \in \Omega \times \Omega$.
- Any measure $\tilde{\sigma}$ on $\Omega \times \Omega$ with marginals χ and barycenter at the origin works, e.g. $\tilde{\sigma} = \mathcal{L}^6|_{\Omega\times\Omega}$.
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Renormalization property enables return to dual space (1)

- Let (P, σ) be a renormalized relaxed Lagrangian solution and $\alpha_t = \nabla P_{t\#}\chi$. Then (P, α) is a distribution solution in dual space. In particular, the geostrophic energy is conserved.
- Renormalization property is used: for test functions $\varphi(t)\xi(X)$ in dual space with $\varphi(0) = 0$ (for simplicity), we have

$$\int_0^T \int_{\mathbb{R}^3} \dot{\varphi}(t)\xi(X)\alpha_t(dX)dt = \int_0^T \int_\Omega \dot{\varphi}(t)\xi(\nabla P_t(y))dydt$$
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Renormalization property enables return to dual space (2)

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- Denote $\gamma_t = (\mathrm{Id} \times \nabla P_t)_\# \chi$.

$$I_2 = -\int_0^T \int_\Omega \varphi(t)\nabla\xi(\nabla P_t(y)) \cdot Jy\, dy\, dt = -\int_0^T \int_{\Omega\times\mathbb{R}^3} \varphi(t)\nabla\xi(X) \cdot Jy\, \gamma_t(dy, dX)\, dt$$
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- We have obtained $\int_0^T \int_{\mathbb{R}^3} [\partial_t\zeta + \nabla\zeta \cdot U]\alpha_t(dX) dt = 0$, for $U(t, X) = J(X - \bar{\gamma}_t(X))$, $\zeta(t, x) = \varphi(t)\xi(X)$.

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- Denote $\gamma_t = (\mathrm{Id} \times \nabla P_t)_{\#}\chi$.

$$I_2 = -\int_0^T \int_\Omega \varphi(t)\nabla\xi(\nabla P_t(y)) \cdot Jy\, dy\, dt = -\int_0^T \int_{\Omega\times\mathbb{R}^3} \varphi(t)\nabla\xi(X) \cdot Jy\, \gamma_t(dy, dX)\, dt$$
$$= -\int_0^T \int_{\mathbb{R}^3} \varphi(t)\nabla\xi(X) \cdot J\bar{\gamma}_t(X)\, \alpha_t(dX)\, dt$$

- We have obtained $\int_0^T \int_{\mathbb{R}^3} [\partial_t\zeta + \nabla\zeta \cdot U]\alpha_t(dX)dt = 0$, for $U(t, X) = J(X - \bar{\gamma}_t(X))$, $\zeta(t, x) = \varphi(t)\xi(X)$.

Renormalization property enables return to dual space (2)

- $I_1 = \int_0^T \int_\Omega \varphi(t)\nabla\xi(\nabla P_t(y)) \cdot J\nabla P_t(y)dy\,dt = \int_0^T \int_{\mathbb{R}^3} \varphi(t)\nabla\xi(X) \cdot JX\,\alpha_t(dX)\,dt.$
- Denote $\gamma_t = (\mathrm{Id} \times \nabla P_t)_{\#}\chi$.

$$I_2 = -\int_0^T \int_\Omega \varphi(t)\nabla\xi(\nabla P_t(y)) \cdot Jy\,dy\,dt = -\int_0^T \int_{\Omega\times\mathbb{R}^3} \varphi(t)\nabla\xi(X) \cdot Jy\,\gamma_t(dy, dX)\,dt$$
$$= -\int_0^T \int_{\mathbb{R}^3} \varphi(t)\nabla\xi(X) \cdot J\bar{\gamma}_t(X)\,\alpha_t(dX)\,dt$$

- We have obtained $\int_0^T \int_{\mathbb{R}^3} [\partial_t\zeta + \nabla\zeta \cdot U]\alpha_t(dX)dt = 0$, for $U(t, X) = J(X - \bar{\gamma}_t(X))$, $\zeta(t, x) = \varphi(t)\xi(X)$.

Thank you!