# Generalized Large Scale Semigeostrophic Equations: geometric structure and global well-posedness

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# 1. Introduction

**GLSG equations:**

- Family of balance models for rotating shallow water in semigeostrophic scaling
- Derived by M. Oliver 06, for Coriolis par-r $f = \mathrm{const}$
- Hamiltonian models
- Have PV advection-inversion formulation
- Family contains Salmon's $L_1$ and LSG models
- Global existence result. Now extends to $L_1$ model for $f = \mathrm{const}$.
- Extension to spatially varying Coriolis parameter and bottom topography.

**Talk outline:**

- Derivation of the models
- Global well-posedness for constant Coriolis parameter
- Classes of GLSG models for variable Coriolis parameter and their well-posedness

# 2. GLSG derivation

## Scaled Rotating Shallow Water equations

$$\underbrace{\varepsilon(\partial_t \boldsymbol{u} + \boldsymbol{u}\cdot\nabla\boldsymbol{u})}_{\text{Inertial term}} + \underbrace{+f\boldsymbol{u}^{\perp}}_{\text{Coriolis force}} = \underbrace{= -\tfrac{B}{\varepsilon}\nabla(h-b),}_{\text{Force of gravity}}$$

$$\partial_t h + \boldsymbol{\nabla}\cdot(h\boldsymbol{u}) = 0$$

$\boldsymbol{u} = (u_1(\boldsymbol{x}), u_2(\boldsymbol{x}))$ is horizontal velocity, $\boldsymbol{x} = (x_1, x_2)$
$f = f(\boldsymbol{x})$ is the scaled $O(1)$ Coriolis parameter
$\boldsymbol{u}^{\perp} = (-u_2, u_1)$
$h$ is a fluid layer depth
$b = b(\boldsymbol{x})$ is the bottom topography
$\varepsilon$ is a Rossby number, $B$ is a Burger number

## SG scaling

$$\varepsilon = \frac{U}{FL} \ll 1, \qquad B = \frac{gH_0}{F^2L^2} = O(\varepsilon)$$

## Balance models

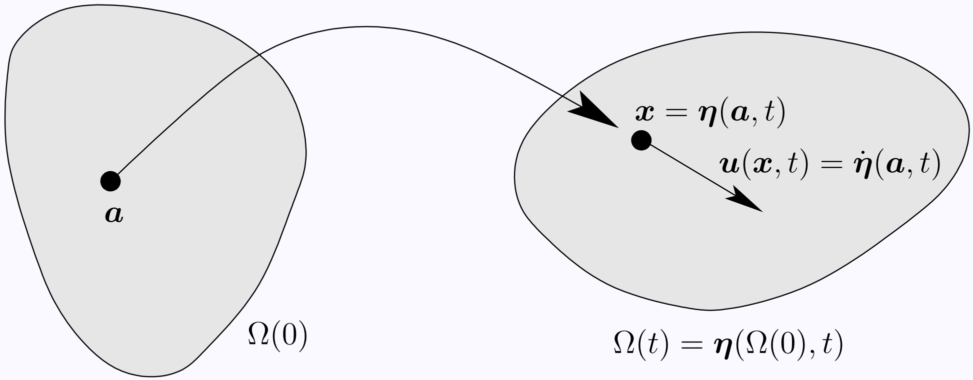
- RSW equations support high frequency waves generated by the balance between inertia and gravity terms
- Balance models filter out gravity waves by approximating momentum eq-n with a balance relationship.

$$\boldsymbol{u} = F(\varepsilon, h)$$

$$\lim_{\varepsilon \to 0} F(\varepsilon, h) = \boldsymbol{u}_G \equiv \frac{\nabla^{\perp}(h - b)}{f}$$

- Goal: Derive approximate balance models with PV and energy conservation

## 2.1. Variational structure of RSW




**Lagrangian particle trajectories** define a curve in the group of diffeos s.t.

$$\dot{\eta} = \boldsymbol{u} \circ \eta \qquad \eta(\cdot, 0) = Id$$

i.e. $\eta$ is a flow of Eulerian velocity $\boldsymbol{u}$ and $\det \nabla\eta > 0$.

**Define layer depth** as a function of flow:

$$h(\eta) \equiv \det \nabla \eta^{-1} = \frac{1}{\det \nabla \eta} \circ \eta^{-1}$$

**RSW Lagrangian**

$$\begin{aligned} L_{RSW}(\eta, \dot{\eta}) &= \int_{\mathcal{D}} \boldsymbol{R} \circ \eta \cdot \dot{\eta} + \frac{\varepsilon}{2} |\dot{\eta}|^2 - \frac{1}{2h} \tilde{h}^2 \circ \eta \, da \\ &= \int_{\mathcal{D}} h \left( \boldsymbol{R} \cdot \boldsymbol{u} + \frac{\varepsilon}{2} |\boldsymbol{u}|^2 \right) - \frac{1}{2} \tilde{h}^2 \, dxdy \, . \end{aligned}$$

where $\boldsymbol{R}$ is Coriolis potential, $\operatorname{curl} \boldsymbol{R} = f$, $\tilde{h} = h - b$.

**Configuration space** is a group of diffeomorphisms (“flow maps”)

**RSW momentum equation** is Euler-Lagrange eq-n for $L_{RSW}$:

$$\delta \int_{t_1}^{t_2} L_{RSW}(\eta, \dot{\eta}) = 0$$

with $\delta\eta = 0$ at $t = t_1, t_2$ implies that $\boldsymbol{u} = \dot{\eta} \circ \eta^{-1}$ satisfies RSW momentum equation.

**Observation:** Lagrangian for balance model must be affine in $\dot{\eta}$:

$$u = F(h) \Leftrightarrow \frac{d}{dt}\frac{\delta L}{\delta \dot{\eta}} - \frac{\delta L}{\delta \eta} = 0$$

iff $\frac{\delta L}{\delta \dot{\eta}}$ is independent of $u, \dot{\eta}$.

**Blueprint for deriving GLSG balanced models:**

- Assume flows $\eta_\varepsilon$ in physical and $\eta$ in computational coordinates are related by near identity transformation:

  $$\eta_\varepsilon = \xi_\varepsilon \circ \eta$$

  $$\xi_\varepsilon(\boldsymbol{x}) = \boldsymbol{x} + \varepsilon \boldsymbol{v}(\boldsymbol{x}) + O(\varepsilon^2)$$

  This is near identity transformation of configuration space as well as near identity transformation of Eulerian coordinates.
- The flow in in computational coordinates comes from affine Lagrangian $L$ with particle relabelling symmetry $\Rightarrow$ balanced, with PV conservation law.
- Approximation:

  $$L_{RSW}(\eta_\varepsilon, \dot{\eta}_\varepsilon) = L(\eta, \dot{\eta}) + O(\varepsilon^2) \quad \forall \eta$$

- Choose $\boldsymbol{v}$ s.t. $L_{RSW}(\eta_\varepsilon, \dot{\eta}_\varepsilon)$ truncated at $O(\varepsilon^2)$ is affine in $\dot{\eta}$

# 3. Resulting models

**Transformation** to physical coordinates is explicit ($\tau$ is a free parameter):

$$\boldsymbol{v} = \frac{1}{2f}\,\boldsymbol{u}^{\perp} + \tau(\boldsymbol{x})\,\boldsymbol{\nabla}\tilde{h} + \cancel{g(\boldsymbol{x})\,\boldsymbol{u}} + \cancel{\mu(\boldsymbol{x})\,\boldsymbol{\nabla}^{\perp}\tilde{h}}\,.$$

$$\boldsymbol{u}_{\varepsilon} = \boldsymbol{u} + \varepsilon\,(\dot{\boldsymbol{v}} + [\boldsymbol{u},\boldsymbol{v}]), \qquad h_{\varepsilon} = h - \varepsilon\,\boldsymbol{\nabla}\cdot(h\boldsymbol{v})$$

**Computational Lagrangian** ($\alpha = f\tau + \frac{1}{2f}$):

$$L = \int h\left[\left(\boldsymbol{R} + \varepsilon\,(\alpha\boldsymbol{\nabla}^{\perp}\tilde{h})\right)\cdot\boldsymbol{u} - \varepsilon\,\tau\,|\boldsymbol{\nabla}\tilde{h}|^2 - \frac{\tilde{h}^2}{2h}\right]\mathrm{d}\boldsymbol{x}$$

**Balance equation**

$$\begin{aligned} &hq\boldsymbol{u} + \varepsilon\,\boldsymbol{\nabla}\alpha\,\boldsymbol{\nabla}\cdot(h\boldsymbol{u}) + \varepsilon\,\boldsymbol{\nabla}(h\boldsymbol{u}\cdot\boldsymbol{\nabla}\alpha) - \varepsilon\Delta(\alpha h\boldsymbol{u}) \\ &= \boldsymbol{\nabla}^{\perp}\left[\tilde{h} - \varepsilon\left(2\,\boldsymbol{\nabla}\cdot(\tau h\boldsymbol{\nabla}\tilde{h}) - \tau\,|\boldsymbol{\nabla}\tilde{h}|^2\right)\right] \end{aligned}$$

**PV advection**

$$q = \frac{f + \varepsilon\,\boldsymbol{\nabla}\cdot(\alpha\boldsymbol{\nabla}\tilde{h})}{h}\,, \qquad \partial_t q + \boldsymbol{u}\cdot\boldsymbol{\nabla}q = 0.$$

# 4. Well-posedness of GLSG

**Simplifications:** $f = 1$, $b = \text{const}$, $\alpha = \text{const} > 0$, $\tau = \text{const}$

**GLSG on $\mathbb{T}^2$ in vorticity form** :

$$\partial_t q + u \cdot \nabla q = 0 \text{ (PV advection)}$$
$$u = K(q) \text{ (PV inversion)}$$
$$q(0) = q^{in} > 0$$

where $K$ is defined implicitly:

$$L_q h \equiv (q - \sigma\,\Delta)h = 1$$
$$\Lambda_h u \equiv \big[1 - \sigma\,(h\,\Delta + 2\,\nabla h \cdot \nabla)\big]u = \nabla^\perp\big[h - \mu\,(2\,h\,\Delta h + |\nabla h|^2)\big]$$

with $\sigma = \varepsilon\alpha = \varepsilon(\tau + \frac{1}{2}) > 0$, $\mu = \varepsilon\tau$.

**Special cases:**

- $\tau = \frac{1}{2}$: Salmon's $L_1$ dynamics
- $\tau = -\frac{1}{2}$: Salmon's LSG, *ill-posed*
- $\tau = 0$: Optimal regularity!

## 4.1. Blueprint: classical solutions for 2D Euler

**Abstract vorticity formulation**

$$\partial_t q + \boldsymbol{u} \cdot \nabla q = 0$$
$$\boldsymbol{u} = Kq$$

where $K$ is a linear "Biot–Savart" operator, $K^{-1}\boldsymbol{u} = \operatorname{curl} \boldsymbol{u}$.

**$H^s$ estimate**

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|q\|_{H^s}^2 \le \left| \int \nabla \cdot \boldsymbol{u} \, |\mathrm{D}^s q|^2 \, dx dy \right| + \text{lots of other terms}$$
$$\le C \|\boldsymbol{u}\|_{W^{1,\infty}} \|q\|_{H^s}^2$$

**Observation:** need

$$\|\boldsymbol{u}\|_{W^{1,\infty}} \le c \, \|q\|_{L^\infty} \left(1 + \ln_+ \|q\|_{H^s}\right)$$
$$\|\boldsymbol{u}\|_{W^{m+1,p}} \le c \|q\|_{W^{m,p}}$$

This holds in 2D and 3D with $s = 2$, implying global classical solutions for 2D Euler and the Beale, Kato, and Majda (1984) criterion for 3D Euler (follows from estimates on Biot-Savart kernel).

## 4.2. The Yudovich trick

- Direct verification of the $W^{1,\infty}$ bound is hard when the integral kernel of $K$ is not explicitly known.
- If $K$ is of the general structure $\boldsymbol{u} = \mathrm{D}\Delta^{-1} q$, elliptic $L^p$ theory tells us that

$$\|\boldsymbol{u}\|_{W^{1,p}} \leq c \, \frac{p^2}{p-1} \, \|q\|_{L^p}$$

  for $p \in (1, \infty)$ (which easily propagates up the Sobolev scale).
- Use Gagliardo–Nirenberg interpolation

$$\|\boldsymbol{\nabla u}\|_{L^\infty} \leq c(p) \, \|\boldsymbol{\nabla u}\|_{H^{s-1}}^{\theta} \, \|\boldsymbol{\nabla u}\|_{L^p}^{1-\theta}$$

  where $c(p) \to 1$ as $p \to \infty$ and

$$\theta = \frac{1}{p \left( \frac{s-1}{d} - \frac{1}{2} \right) + 1}$$

- Now optimize in $p$ to obtain the desired estimate

$$\|\boldsymbol{u}\|_{W^{1,\infty}} \leq c \, \|q\|_{L^\infty} \left( 1 + \ln_+ \|q\|_{H^s} \right)$$

- *Problems for GLSG:* Required estimates are sub-linear, while $K(q)$ is nonlinear, $K(q)$ is not defined for all $q$, $\boldsymbol{u}$ is not divergence free.

Theorem. (Çalik, Oliver, and V., Archives, 2013)
*Suppose* $m \geq 3$ *and* $q^{in} \in H^m(\mathbb{T}^2)$ *is strictly positive. Then the GLSG equations possess a global classical solution* $q \in C^k([0,\infty); H^{m-k}(\mathbb{T}^2))$ *for every* $k = 0, \ldots, m-2$*. The associated velocity field satisfies* $\boldsymbol{u} \in C^k([0,\infty); H^{m-k+1}(\mathbb{T}^2))$ *when* $\mu \neq 0$ *and* $C^k([0,\infty); H^{m-k+3}(\mathbb{T}^2))$ *when* $\mu = 0$ *for* $k = 0, \ldots, m-2$*.*

**Remark.** Result as strong as the one available for 2D Euler for $\mu \neq 0$ and 2D Euler-$\alpha$ for $\mu = 0$.

## 4.3. $q - h$ inversion

Suppose $q^{in} > 0$.

$$L_q h \equiv qh - \sigma \Delta h = 1$$

**Observation :** $q$ is advected, hence $q > 0$ for all times and

$$q_- = \operatorname*{ess\,inf}_{x \in \mathbb{T}^2} q(x) \qquad \text{and} \qquad q_+ = \operatorname*{ess\,sup}_{x \in \mathbb{T}^2} q(x)\,.$$

are the constants of the motion.

**Global $L^\infty$ bound on $h, h^{-1}$:**

$$L_q\left(h - \tfrac{1}{q_+}\right) = 1 - \tfrac{q}{q_+} \geq 0\,, \qquad L_q\left(h - \tfrac{1}{q_-}\right) = 1 - \tfrac{q}{q_-} \leq 0$$

Since $L_q$ is uniformly elliptic, from strong maximum modulus principle

$$\frac{1}{q_+} \leq h \leq \frac{1}{q_-}\,.$$

## 4.4. $W^{m,p}$ bounds on $h$:

Since $q > 0$ the solution $h \in H^1$ exists by Lax-Milgram.

**Dimension independent Gagliardo-Nirenberg inequalities:** $(0 < \theta < 1)$

$$\|\mathrm{D}^j v\|_{L^p} \leq C \|\mathrm{D}^{j/\theta} v\|^{\theta}_{L^{p\theta}} \|v\|^{1-\theta}_{L^\infty}$$

By induction, elliptic regularity, and G-N

$$\|h\|_{W^{m+2,p}} \leq C(1 + \|q\|_{W^{m,p}})$$

**Proof** .

$$\mathrm{D}^\alpha h = (1 - \sigma\Delta)^{-1} \left( \mathrm{D}^\alpha 1 - \sum_{\beta \leq \alpha} \binom{\alpha}{\beta} \mathrm{D}^{\alpha-\beta} \tilde{q}\, \mathrm{D}^\beta h \right).$$

Bootstrap the induction hypothesis into the estimate with $\theta = \frac{|\alpha-\beta|}{|\alpha|}$

$$\begin{aligned}
\|\mathrm{D}^{\alpha-\beta} \tilde{q}\, \mathrm{D}^\beta h\|_{L^p} &\leq \|\mathrm{D}^{\alpha-\beta} \tilde{q}\|_{L^{p/\theta}} \|\mathrm{D}^\beta h\|_{L^{p/(1-\theta)}} \\
&\leq \|\mathrm{D}^{\alpha-\beta} \tilde{q}\|_{L^{p/\theta}} C_1 \left(1 + \|\tilde{q}\|_{W^{|\beta|,p/(1-\theta)}}\right) \\
&\leq C_2 \|\mathrm{D}^{|\alpha|} \tilde{q}\|^{\theta}_{L^p} \|\tilde{q}\|^{1-\theta}_{L^\infty} \left[1 + \|\tilde{q}\|^{1-\theta}_{W^{|\alpha|,p}} \|\tilde{q}\|^{\theta}_{L^\infty}\right] \\
&\leq C_3 \left(1 + \|\tilde{q}\|_{W^{|\alpha|,p}}\right).
\end{aligned}$$

## 4.5. $\boldsymbol{u} - h$ estimates

Consider

$$\Lambda_h \boldsymbol{u} \equiv (1 - \sigma\,(h\Delta + 2\nabla h\cdot\nabla))\boldsymbol{u} = g$$

**Coercitivity:**

$$\begin{aligned}
\int \Lambda_h \boldsymbol{u}\cdot\boldsymbol{u} &= \int_{\mathbb{T}^2} \left(|\boldsymbol{u}|^2 + \sigma\,h\,|\nabla\boldsymbol{u}|^2 - \tfrac{1}{2}\,\sigma\,\nabla h\cdot\nabla|\boldsymbol{u}|^2\right)\mathrm{d}x \\
&= \int_{\mathbb{T}^2} \left(|\boldsymbol{u}|^2 + \sigma\,h\,|\nabla\boldsymbol{u}|^2 + \tfrac{1}{2}\,\sigma\,\Delta h\,|\boldsymbol{u}|^2\right)\mathrm{d}x \\
&= \int_{\mathbb{T}^2} \left(\tfrac{1}{2}\,(1+qh)\,|\boldsymbol{u}|^2 + \sigma\,h\,|\nabla\boldsymbol{u}|^2\right)\mathrm{d}x \\
&\geq \min\{\tfrac{1}{2}, \sigma h_-\}\,\|u\|_{H^1}\,.
\end{aligned}$$

By Lax-Milgram

$$\|\boldsymbol{u}\|_{H^1} \leq \tfrac{2}{\sigma}\|g\|_{H^{-1}}$$

**$W^{1,p}$ estimate:** Write $h^{-1} \equiv 1 + \tilde{b}$. Then

$$(1 - \sigma\Delta)\boldsymbol{u} = \frac{g}{h} - \tilde{b}\,\boldsymbol{u} + \frac{2\sigma}{h}\nabla h \cdot \nabla \boldsymbol{u}\,,$$

therefore, for $p \geq 2$,

$$\|\boldsymbol{u}\|_{W^{1,p}} \leq \frac{c\,p}{\sigma^2}\|g\|_{W^{-1,p}} \leq \frac{Cp}{\sigma^2}\,.$$

**$W^{m,p}$ estimate:**

$$\|\boldsymbol{u}\|_{W^{m+1,p}} \leq C\,\|q\|_{W^{m,p}}\,.$$

In the special case when $\mu = 0$, $\boldsymbol{u} \in W^{m+3,p}$ with bound

$$\|\boldsymbol{u}\|_{W^{m+3,p}} \leq C\,\|q\|_{W^{m,p}}\,.$$

**Proof of $W^{m,p}$ estimate** by induction. Write

$$(1-\sigma\Delta)\mathrm{D}^\alpha \boldsymbol{u} = \mathrm{D}^\alpha(gh^{-1}) - \mathrm{D}^\alpha(\tilde{b}\,\boldsymbol{u}) + 2\sigma\,\mathrm{D}^\alpha(h^{-1}\nabla h\cdot\nabla\boldsymbol{u})\,,$$

**Worst term estimate:** by G-N

$$\begin{aligned}\|\mathrm{D}^\alpha(h^{-1}\,\nabla h\cdot\nabla\boldsymbol{u})\|_{L^p} &\le c\sum_{\alpha^1+\alpha^2+\alpha^3=\alpha}\|\mathrm{D}^{\alpha^1}h^{-1}\|_{L^{p/s1}}\,\|\mathrm{D}^{\alpha^2}\nabla h\|_{L^{p/s_2}}\,\|\mathrm{D}^{\alpha^3}\nabla\boldsymbol{u}\|_{L^{p/s_3}}\\ &\le C_4\sum_{\alpha^1+\alpha^2+\alpha^3=\alpha}\|h\|_{W^{|\alpha^1|,p/s_1}}\,\|h\|_{W^{|\alpha^2|+1,p/s_2}}\,\|h\|_{W^{|\alpha^3|+2,p/s_3}}\\ &\le C_5\,\|h\|_{W^{|\alpha|+3,p}}\,\|h\|_{L^\infty}\\ &\le C_6\|q\|_{W^{|\alpha|+1,p}}\,,\end{aligned}$$

$$s_1=\frac{|\alpha^1|}{|\alpha|+3}\,,\qquad s_2=\frac{|\alpha^2|+1}{|\alpha|+3}\,,\qquad \text{and}\qquad s_3=\frac{|\alpha^3|+2}{|\alpha|+3}\,,$$

$$s_1+s_2+s_3=1$$

# 5. Back to variable Coriolis parameter

**Balance equation**

$$\Lambda_h \equiv hq\boldsymbol{u} + \varepsilon\,\boldsymbol{\nabla}\alpha\,\boldsymbol{\nabla}\cdot(h\boldsymbol{u}) + \varepsilon\,\boldsymbol{\nabla}(h\boldsymbol{u}\cdot\boldsymbol{\nabla}\alpha) - \varepsilon\Delta(\alpha h\boldsymbol{u})$$
$$= \boldsymbol{\nabla}^{\perp}\left[\tilde{h} - \varepsilon\left(2\,\boldsymbol{\nabla}\cdot(\tau h\boldsymbol{\nabla}\tilde{h}) - \tau\,|\boldsymbol{\nabla}\tilde{h}|^2\right)\right]$$

**PV inversion.** Denote $\tilde{f} = f - \varepsilon\boldsymbol{\nabla}\cdot(\alpha\boldsymbol{\nabla}b)$

$$L_q h \equiv qh - \varepsilon\boldsymbol{\nabla}\cdot(\alpha\boldsymbol{\nabla}h) = \tilde{f}$$

**PV advection**

$$\partial_t q + \boldsymbol{u}\cdot\boldsymbol{\nabla}q = 0.$$

**Continuity equation**

$$\partial_t h + \boldsymbol{\nabla}\cdot(h\boldsymbol{u}) = 0$$

Complete set of equations are BE+PVI+PVA or BE+CE.

**Note:** two sets are equivalent iff $L_q$ is invertible with $h > 0$.

$$\partial_t q + \boldsymbol{u}\cdot\boldsymbol{\nabla}q = -h^{-1}L_q(\partial_t h + \boldsymbol{\nabla}\cdot(h\boldsymbol{u}))$$

**PV inversion.** Suppose:

$$\alpha(\boldsymbol{x}) \geq \alpha_0 > 0$$
$$q^{in} \geq q_0 > 0$$
$$\tilde{f} = f - \varepsilon\, \boldsymbol{\nabla} \cdot (\alpha \boldsymbol{\nabla} b) \geq \tilde{f}_0 > 0$$

**$q - h$ inversion:** $L_q h = \tilde{f}$

$$L_q \left( h - \inf_{\boldsymbol{x}} \tilde{f}/q \right) = \tilde{f} - q \inf_{\boldsymbol{x}} \tilde{f}/q \geq 0\,, \quad L_q \left( h - \sup_{\boldsymbol{x}} \tilde{f}/q \right) = \tilde{f} - q \sup_{\boldsymbol{x}} \tilde{f}/q \leq 0$$

Maximum principle for elliptic PDEs $\Rightarrow$

$$0 < \inf_{\boldsymbol{x}} \frac{\tilde{f}}{q} \leq h \leq \sup_{\boldsymbol{x}} \frac{\tilde{f}}{q}$$

**Sufficient h-u invertibility condition** is positivity: for all $\boldsymbol{w} \neq 0$

$$\int \tfrac{1}{2} \left[ \tilde{f} + qh - \varepsilon\, \boldsymbol{\nabla} \cdot (h \boldsymbol{\nabla} \alpha) \right] |\boldsymbol{w}|^2 \mathrm{d}\boldsymbol{x} + \varepsilon \int \left[ \alpha h\, |\boldsymbol{\nabla} \boldsymbol{w}|^2 + (\boldsymbol{\nabla} \alpha \cdot \boldsymbol{w})\, (\boldsymbol{\nabla} h \cdot \boldsymbol{w}) \right] \mathrm{d}\boldsymbol{x} > 0$$

Condition holds if either:

- $\alpha = \mathrm{const}$
- $3\,\varepsilon\, |\boldsymbol{\nabla} h|\, |\boldsymbol{\nabla} \alpha| \leq qh + \tilde{f} - \varepsilon\, h\, \Delta\alpha$

**Distinguished choices of parameters:**

**Salmon's $L_1$ model:**

$$\tau = \tfrac{1}{2f^2} \Leftrightarrow \alpha = \frac{1}{f}$$

+ Physical coordinates coincide with computational
- Solvability condition for PV inversion

**Salmon's LSG model:**

$$\tau = -1/(2f^2) \Leftrightarrow \alpha = 0$$

+ Simplest equations of motion
- Ill posed

**New models:** $\alpha = \text{const} > 0 \Leftrightarrow \tau = \frac{\alpha}{f} - \frac{1}{2f^2}$
+ Simple equations of motion
= PV inversion gains 1 derivative
+ Globally well posed
- One needs to transform back to physical coordinates

**$\tau = 0$ model:** $\Leftrightarrow \alpha = \frac{1}{2f}$

+ PV inversion gains 3 derivatives

- Condition for PV inversion

# 6. Conclusions

- We have a good framework for producing Hamiltonian balance models with PV conservation
- Produced models have the same formal accuracy as SG, but easier analytic treatment (elliptic PDEs with non-constant parameters).
- Some of the previously known models are distinguished members of the family.
- Variable Coriolis parameter can easily be treated in the framework.
- Some models in the family have robust physical solvability conditions. For $\alpha = const > 0$, the global well-posedness is evident.
- The downside currently is inability to treat boundary conditions.