Monge-Ampère Equations: Geometry, Invariants and Applications in 3D Meteorological Models

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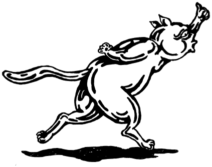
Plan

Cat Eurika:



Basic object

Figure: Monge and Ampère

$$\boxed{A\frac{\partial^2 f}{\partial q_1^2} + 2B\frac{\partial^2 f}{\partial q_1 \partial q_2} + C\frac{\partial^2 f}{\partial q_2^2} + D\left(\frac{\partial^2 f}{\partial q_1^2} \cdot \frac{\partial^2 f}{\partial q_2^2} - \left(\frac{\partial f}{\partial q_1 \partial q_2}\right)^2\right) + E = 0}$$

Global Solutions: Monge

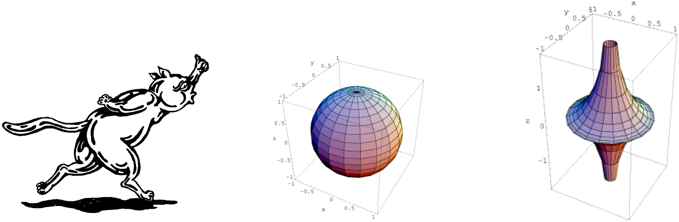


Figure: sphere and pseudosphere

An example: curvature of a surface in $\mathbb{R}^3$

$$\frac{u_{q_1 q_1} \cdot u_{q_2 q_2} - u_{q_1 q_2}^2}{(1 + u_{q_1}^2 + u_{q_2}^2)^2} = \mathcal{K}(u)$$

Monge-Ampère structure

Definition

A Monge-Ampère structure on a $2n$*-dimensional manifold* X *is a pair of differential form* $(\Omega, \omega) \in \Omega^2(X) \times \Omega^n(X)$ *such that* Ω *is symplectic and* ω *is* Ω*-effective i.e.* $\Omega \wedge \omega = 0$*.*

Main idea

- Let $F : \mathbb{R}^n \to (i)\mathbb{R}^n$ be a vector-function and its graph is a subspace in $T^*(\mathbb{R}^n) = \mathbb{R}^n \oplus (i)\mathbb{R}^n$.

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- This graph is a Lagrangian subspace in $T^*(\mathbb{R}^n)$ iff $(dF)_x$ is a symmetric endomorphism . The matrix $|| \frac{\partial F_i}{\partial x_j} ||$ is symmetric $\forall x$ iff the differential form $\sum_i F_i dx_i \in \Lambda^1(\mathbb{R}^n)$ is closed or, equivalently, exact:

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- The projection of the graph of ∇f on $(\mathbb{R}^n)_x$ is given in coordinates by $\nabla^2(f) = \det || \frac{\partial^2 f_i}{\partial x_j^2} ||$.

Correspondence: Forms -Symplectic MAO

Let M be a smooth $n-$dimensional manifold and ω is a differential n-form on T^*M. A (symplectic) Monge-Ampère operator $\Delta_\omega : C^\infty(M) \to \Omega^n(M)$ is the differential operator defined by

$$\Delta_\omega(f) = (df)^*(\omega),$$

where $df : M \to T^*M$ is the natural section associated to f.

Examples

ω	$\boldsymbol{\Delta}_\omega = \mathbf{0}$
$dq_1 \wedge dp_2 - dq_2 \wedge dp_1$	$\Delta f = 0$
$dq_1 \wedge dp_2 + dq_2 \wedge dp_1$	$\Box f = 0$
$dp_1 \wedge dp_2 \wedge dp_3 - dq_1 \wedge dq_2 \wedge dq_3$	$Hess(f) = 1$
$dp_1 \wedge dq_2 \wedge dq_3 - dp_2 \wedge dq_1 \wedge dq_3$ $+ dp_3 \wedge dq_1 \wedge dq_2 - dp_1 \wedge dp_2 \wedge dp_3$	$\Delta f - Hess(f) = 0$

Hodge-Lepage-Lychagin theorem

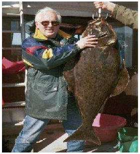


Figure: Hodge and Lychagin

The next theorem plays the fundamental role played by the effective forms in the theory of Monge-Ampère operators :

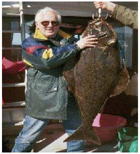
Theorem (Hodge-Lepage-Lychagin)

- *Every form* $\omega \in \Lambda^k(V^*)$ *can be uniquely decomposed into the finite sum*

$$\omega = \omega_0 + \top\omega_1 + \top^2\omega_2 + \ldots ,$$

where all ω_i *are effective forms.*

Hodge-Lepage-Lychagin theorem



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$$\omega = \omega_0 + \top\omega_1 + \top^2\omega_2 + \ldots,$$
where all ω_i are effective forms.
- *If two effective k-forms vanish on the same k-dimensional isotropic vector subspaces in (V, Ω), they are proportional.*

Symplectic Monge-Ampère Equations: Solutions

- A generalised solution of a MAE $\Delta_\omega = 0$ is a lagrangian submanifold of (T^*M, Ω) which is an integral manifold for the MA differential form ω:

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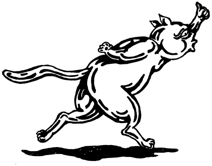
- A generalised solution (generically) locally is the graph of an 1-form df for a regular solution f.

Generalized solution



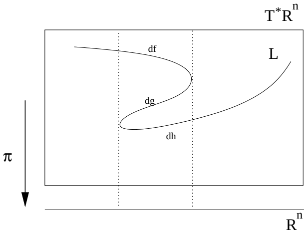
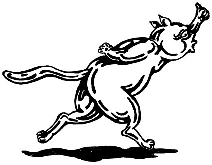


Figure: Generalised solution of a MAE

Generic types of singularities for Generalized solutions of MAE

Specific property of the graph-like Lagrangian submanifolds: their projection on the "configuration space" $\mathbb{R}^n$ is a diffeomorphism. Our generalised solutions are general Lagrangian immersions and they have Arnold's lagrangian singularities.



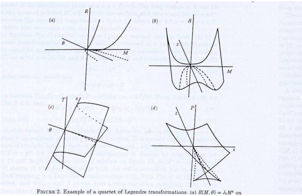


Figure: Lagrangian singularities (Wave fronts, foldings etc.)

This singularities describe the formation of fronts (Chynoweth, Porter, Sewell 1988)

Symplectic Equivalence-1

- Two SMAE $\Delta_{\omega_1} = 0$ and $\Delta_{\omega_2} = 0$ are locally equivalent iff there is exist a local symplectomorphism $F : (T^*M, \Omega) \to (T^*M, \Omega)$ such that

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- L is a generalised solution of $\Delta_{F^*\omega_1} = 0$ iff $F(L)$ is a generalised solution of $\Delta_\omega = 0$.

Legendre partial transformation

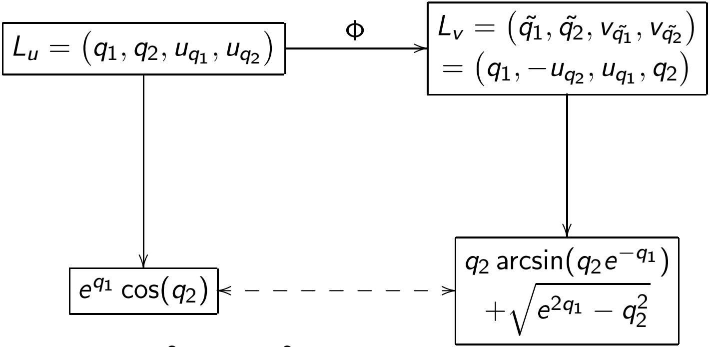
Figure: Legendre

$$u_{q_1 q_1} + u_{q_2 q_2} = 0 \quad \longleftarrow - - - \longrightarrow \quad v_{q_1 q_1} v_{q_2 q_2} - v_{q_1 q_2}^2 = 1$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

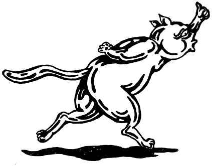
$$\omega = dq_1 \wedge dp_2 - dq_2 \wedge dp_1 \quad \xleftarrow{\Phi^*} \quad \tilde{\omega} = dp_1 \wedge dp_2 - dq_1 \wedge dq_2$$

Legendre partial transformation-2



with $\Phi : T^*\mathbb{R}^2 \to T^*\mathbb{R}^2$, $(q_1, q_2, p_1, p_2) \mapsto (q_1, -p_2, p_1, q_2)$.

Sewell-Chynoweth SG- equation



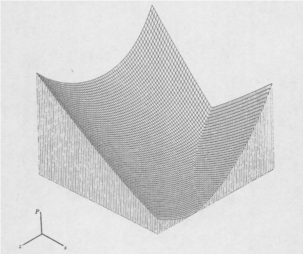


Figure: Numerical Solution of the semi-geostrophic $3D$ equation (Cullen, Sewell-Chynoweth...)

$$hess_{x,y}(u) + \frac{\partial^2 u}{\partial z^2} = hess(u) \tag{1}$$

Sewell-Chynoweth MAO form and its equivalence

- The effective form of (1):

$$\omega = dp \wedge dq \wedge dz + dx \wedge dy \wedge dr - \gamma dx \wedge dy \wedge dz,$$

$(x, y, z, p, q, r)-$ canonical coordinates system of $T^*\mathbb{R}^3$.

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- This form is a sum of two decomposable 3-forms:

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- $\phi^*(\omega) = dp \wedge dq \wedge dr - dx \wedge dy \wedge dz$ where ϕ is the symplectomorphism

$$\phi(x, y, z, p, q, r) = (x, y, r, p, q, \gamma r - z).$$

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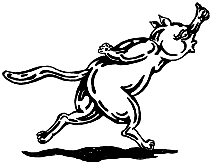
- The equation (1) is symplectically equivalent to the equation

$$hess(u) = 1. \tag{2}$$

An exact solution of the SG 3D equation



$$f(x,y,z) = \int_a^{\sqrt{xy+yz+zx}} (b + 4\xi^3)^{1/3} d\xi$$

is a regular solution of (2). Therefore,

$$L = \Big\{(x, y, (x+y)\alpha, (y+z)\alpha, (z+x)\alpha, \gamma(x+y)\alpha - z)\Big\}$$

is a generalised solution of (1) with

$$\alpha = \frac{1}{2}\Big(\frac{b}{(xy+yz+zx)^{\frac{3}{2}}} + 4\Big)^{\frac{1}{3}}.$$

Hoskins geostrophic coordinate transformation

- The SG equations are used like a good approximation to the Boussinesq primitive equations when the rate of the flow momentum is smaller than the Coriolis force, or in other words, when the Rossby number $\mathrm{Ro} << 1$.

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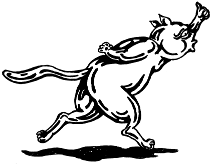
- B. Hoskins (1975) had proposed a remarkable coordinate transformation (a passage to geostrophic coordinates in $x - y$ directions such that the geostrophic velocity and potential temperature may be represented in terms of one function both in the transformed coordinates as in physical ones



$$\begin{cases} X := x + \frac{v_g}{f} = x + \frac{1}{f^2}\frac{\partial \phi}{\partial x} \\ Y := y - \frac{u_g}{f} = y + \frac{1}{f^2}\frac{\partial \phi}{\partial y} \\ Z := z; \quad T := t. \end{cases}$$

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$$\frac{1}{f^2}(\Phi_{XX} + \Phi_{YY}) - \frac{1}{f^4}(\Phi_{XX}\Phi_{YY} - \Phi_{XY}^2) + \frac{1}{N^2}\Phi_{ZZ} = 1. \quad (3)$$

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- Here (and in what follows) f is the *Coriolis parameter* taking as a constant and N is the *Brunt - Väisälä* frequency:

$$N = \sqrt{\frac{q_g g}{f\theta_0}},$$

for the uniform potential vorticity q_g and the constant potential temperature θ_0.

Hoskins geostrophic MA effective form

- This is a 3D Monge-Ampére equation with the effective form

$$\omega = \frac{1}{f^2}(dp \wedge dy \wedge dz + dx \wedge dq \wedge dz) + \frac{1}{N^2} dx \wedge dy \wedge dr -$$

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- This form is the sum of two decomposable forms:

$$\omega = \frac{1}{N^2} dx \wedge dy \wedge dr - (dx - \frac{1}{f^2} dp) \wedge (dy - \frac{1}{f^2} dq) \wedge dz.$$

Hoskins geostrophic MA effective form : equivalence

- Consider the symplectomorphism

$$F(x, y, z, p, q, r) = (p, q, z, -x + f^2 p, -y + f^2 q, r). \qquad (4)$$

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- The new canonical coordinate system $(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{p}, \tilde{q}, \tilde{r})$

$$\begin{cases} \tilde{p} := -x + f^2 p; \quad \tilde{x} := p; \\ \tilde{q} := -y + f^2 q; \quad \tilde{y} := q; \\ \tilde{r} := r; \quad \tilde{z} := z \end{cases}$$

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- The Hoskins SG (3) is equivalent to the (1):

$$\text{hess}(u) = \frac{N^2}{f^4} = \frac{(q_g g)^2}{f^6(\theta_0)^2} \tag{5}$$

  by the symplectomorphism (4).

Table 1. Effective forms with constant coefficients in $2D$

$\mathbf{\Delta_\omega = 0}$	ω	$pf(\omega)$
$\Delta f = 0$	$dq_1 \wedge dp_2 - dq_2 \wedge dp_1$	1
$\Box f = 0$	$dq_1 \wedge dp_2 + dq_2 \wedge dp_1$	-1
$\frac{\partial^2 f}{\partial q_1^2} = 0$	$dq_1 \wedge dp_2$	0

Invariants for effective 3-forms

- To each effective 3-form $\omega \in \Omega^3_\varepsilon(\mathbb{R}^6)$, we assign the following geometric invariants:

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- The Hitchin pfaffian defined by

$$pf(\omega) = \frac{1}{6} tr A_\omega^2.$$

	$\Delta_\omega = 0$	ω	$\varepsilon(q_\omega)$	$pf(\omega)$
1	$\nu\,\mathrm{hess}(f) = 1$	$-dq_1 \wedge dq_2 \wedge dq_3 + \nu \cdot dp_1 \wedge dp_2 \wedge dp_3$	$(3,3)$	ν^2
2	$\Delta f - \nu\,\mathrm{hess}(f) = 0$	$dp_1 \wedge dq_2 \wedge dq_3 - dp_2 \wedge dq_1 \wedge dq_3 + dp_3 \wedge dq_1 \wedge dq_2 - \nu \cdot dp_1 \wedge dp_2 \wedge dp_3$	$(0,6)$	$-\nu^2$
3	$\Box f + \nu\,\mathrm{hess}(f) = 0$	$dp_1 \wedge dq_2 \wedge dq_3 + dp_2 \wedge dq_1 \wedge dq_3 + dp_3 \wedge dq_1 \wedge dq_2 + \nu \cdot dp_1 \wedge dp_2 \wedge dp_3$	$(4,2)$	$-\nu^2$
4	$\Delta f = 0$	$dp_1 \wedge dq_2 \wedge dq_3 - dp_2 \wedge dq_1 \wedge dq_3 + dp_3 \wedge dq_1 \wedge dq_2$	$(0,3)$	0
5	$\Box f = 0$	$dp_1 \wedge dq_2 \wedge dq_3 + dp_2 \wedge dq_1 \wedge dq_3 + dp_3 \wedge dq_1 \wedge dq_2$	$(2,1)$	0
6	$\Delta_{q_2,q_3} f = 0$	$dp_3 \wedge dq_1 \wedge dq_2 - dp_2 \wedge dq_1 \wedge dq_3$	$(0,1)$	0
7	$\Box_{q_2,q_3} f = 0$	$dp_3 \wedge dq_1 \wedge dq_2 + dp_2 \wedge dq_1 \wedge dq_3$	$(1,0)$	0
8	$\frac{\partial^2 f}{\partial q_1^2} = 0$	$dp_1 \wedge dq_2 \wedge dq_3$	$(0,0)$	0
9		0	$(0,0)$	0

Table: Classification of effective 3-formes in dimension 6

HyperKäler triple of MAE

The conservation law (the Ertel's theorem) of the potential vorticity obtains (using the Hamiltonian representation of the system):

$$\frac{d}{dt}\left(\frac{\partial(q_1, q_2)}{\partial(a, b)}\right) =$$

$$\frac{d}{dt}(1 + \phi_{q_1 q_1} + \phi_{q_2 q_2} + \det \mathrm{Hess}\, \phi\ ) = 0,$$

This equation is a part of the HyperKähler triple of MAEs (R. and Roulstone 1997, 2001):

$$\begin{cases} \omega_I = \left[1 + a(p_{11} + p_{22}) + (a^2 - c^2)(p_{11}p_{22} - p_{12}^2)dq_1\right] \wedge dq_2 & , \\ \omega_J = \left[2cp_{12} + ac(p_{11}p_{22} - p_{12}^2)\right] dq_1 \wedge dq_2 & , \\ \omega_K = -c\Omega \end{cases}$$

$2D$ balanced model MAE

- The general family of (elliptic) MAE with constant coefficients carries all flat balanced models:

$$1 + \phi_{q_1 q_1} + a\phi_{q_2 q_2} + (a^2 - c^2)\,\det \mathrm{Hess}\,\phi = \zeta^{\mathbf{C}}/f\,, \qquad (6)$$

  Among them are:

$2D$ balanced model MAE

- The general family of (elliptic) MAE with constant coefficients carries all flat balanced models:

$$1 + \phi_{q_1 q_1} + a\phi_{q_2 q_2} + (a^2 - c^2)\det \mathrm{Hess}\,\phi = \zeta^{\mathbf{C}}/f, \qquad (6)$$

  Among them are:
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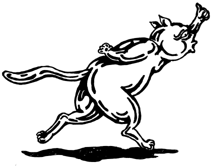
  Among them are:
- *The semi-geostrophic model*($a = 1,\ c = 0$ with $\zeta^{\mathbf{C}}/f$ positive);
- *The L_1 Salmon dynamics* with $a = c = 1$;
- *The $\sqrt{3}$ dynamics of McIntyre - Roulstone* for $a = 1, c = \sqrt{3}$ and $\zeta^{\mathbf{C}}/f < 3/2$;
  Our classification theorem in $2D$ gives a classification of all "almost-balanced"($0 < c < \sqrt{3}$) models with a uniform potential vorticity.

The subjects which I had no time to describe:

- Symmetries, conservation laws and Noether theorem for MAO and MAE
- Self-similar solutions, shock waves and Hugoniot-Rankin conditions
- Variational MAE, divergent MAE and Euler-Lagrange operators
- Jacobi 2D non-linear 1st order systems and Genralized Complex Geometry of Hitchin
- Generalized Calabi-Yau 3D structures
- Lineraisation of Dritchell-Viudez coupled MAE in 2D and 3D
- Many-many other interesting things...

Bibliography:



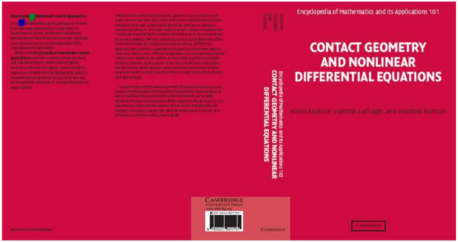


Figure: Cambridge University Press, 2007

Thank you for your attention!