Heteroscedasticity and Autocorrelation Robust Structural Change Detection

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Figure: Upper panel: Time series plot of the Argentina rainfall data. Lower panel: Fitted marginal standard deviation of Argentina rainfall data.

Yearly rainfall in Tucumán Province, Argentina in millimeters from 1884 to 1996.
CUSUM test of change in mean.

Observe time series \( \{X_i\}_{i=1}^n \) with \( \mathbb{E}[X_i] = \mu_i \).

\[ H_0 : \mu_1 = \mu_2 = \cdots = \mu_n = \mu, \quad \leftrightarrow \quad H_a : \mu_i \neq \mu_j \text{ for some } 1 \leq i < j \leq n. \]

CUSUM test:

\[ T_n = \max_{1 \leq i \leq n} |S_i - t_i S_n|, \text{ where } S_i = \sum_{j=1}^{i} X_j \text{ and } t_i = i/n. \] (2)
Write $X_i = \mu_i + e_i$. We say $\{e_i\}_{i=1}^n$ is piecewise locally stationary with $r$ break points (PLS($r$)) if there exist constants $0 = b_0 < b_1 < \cdots < b_r < b_{r+1} = 1$ and nonlinear filters $G_0, G_1, \cdots, G_r$, such that

$$e_i = G_j(t_i, \mathcal{F}_i), \text{ if } b_j < t_i \leq b_{j+1}, \quad (3)$$

where $t_i = i/n$, $\mathcal{F}_i = (\cdots, \varepsilon_0, \cdots, \varepsilon_{i-1}, \varepsilon_i)$ and $\varepsilon_i$'s are i.i.d. random variables.
(i) (PLS linear processes)

\[ G_k(t, \mathcal{F}_i) = \sum_{j=0}^{\infty} a_{k,j}(t) \varepsilon_{i-j} \quad b_k < t \leq b_{k+1}, \]  

(4)

where \( a_{k,j}(\cdot) \)’s are Lipschitz continuous functions.

(ii) (PLS nonlinear processes)

\[ G_k(t, \mathcal{F}_i) = R_k(t, G_k(t, \mathcal{F}_{i-1}), \varepsilon_i), \quad b_k < t \leq b_{k+1}, \]  

(5)

where \( R_k(t, \cdot, \cdot) \) is a smooth function of \( t \).
Limiting behavior of CUSUM test for PLS time series

Define the long-run variance function

\[ \sigma^2(t) = \sum_{k=-\infty}^{\infty} \text{Cov}(G_i(t, \mathcal{F}_0), G_i(t, \mathcal{F}_k)) \text{ if } t \in (b_i, b_{i+1}]. \]

Let \( \sigma^2(0) = \lim_{t \downarrow 0} \sigma^2(t) \).

**Theorem**

*Under mild conditions and the null hypothesis \( H_0 \), we have*

\[ T_n/\sqrt{n} \Rightarrow \sup_{0 \leq t \leq 1} |U(t) - tU(1)|, \quad (6) \]

*where \( \Rightarrow \) stands for convergence in distribution and \( U(t) \) is a zero-mean Gaussian process with covariance function

\[ \gamma(t, s) = \int_0^{\min(t, s)} \sigma^2(r) \, dr. \]
Behavior of traditional testing procedures under PLS

Lagged window estimate of long-run variance:

\[
\hat{\sigma}^2_{LW} = \frac{m}{n - m + 1} \sum_{j=1}^{n-m+1} \left( \frac{1}{m} \sum_{i=j}^{j+m-1} X_i - \frac{1}{n} S_n \right)^2.
\] (7)

Theorem

Under \( H_0 \), the assumption that \( m \to \infty \) with \( m/n \to 0 \) and some mild conditions, we have

\[
\hat{\sigma}^2_{LW} \to \int_0^1 \sigma^2(t) \, dt \text{ in probability.}
\]

\[
T_n/(\hat{\sigma}_{LW} \sqrt{n}) \Rightarrow \sup_{0 \leq t \leq 1} \left| U(t) - tU(1) \right|/\sqrt{\int_0^1 \sigma^2(t) \, dt}.
\] (8)
The robust bootstrap for a fixed window size $m$, define the process

$$
\Phi_{i,m} = \sum_{j=1}^{i} \frac{1}{\sqrt{m(n-m+1)}}(S_{j,m} - \frac{m}{n}S_n)R_j, \quad i = 1, \cdots, n-m+1,
$$

where $S_{j,m} = \sum_{r=j}^{j+m-1} X_r$ and $(R_i)_{i=1}^{n}$ are iid standard normal and are independent of $(X_i)_{i=1}^{n}$. On $[1/n, (n-m+1)/n]$, define the associated linear interpolation

$$
\tilde{\Phi}_{m,n}(t) = \Phi_{t^*n,m} + n(t - t^*)(\Phi_{t^*n,m} - \Phi_{t^*n,m}),
$$

where $t^* = [tn]/n$ and $t^* = t^* + 1/n$. 

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The robust bootstrap

**Theorem**

Assume that $H_0$ holds and $m \to \infty$ with $m/n \to 0$. Then conditional on $(X_i)_{i=1}^n$ and under some mild conditions, we have

$$
\tilde{\Phi}_{m,n}(t) \Rightarrow U(t) \text{ on } C(0,1) \text{ with the uniform topology.} \tag{9}
$$

And, conditional on $(X_i)_{i=1}^n$,

$$
\max_{m+1 \leq i \leq n-m+1} \left| \Phi_{i,m} - \frac{i}{n-m+1} \Phi_{n-m+1,m} \right| \Rightarrow \sup_{0 \leq t \leq 1} |U(t) - tU(1)|. \tag{10}
$$
Select the window size $m$. 
Select the window size $m$.

Generate $B$ (say 2000) conditionally i.i.d. copies $\{\Phi_{i,m}\}_{i=1}^{n-m+1}$, $r = 1, 2, \ldots, B$. Let

$$M_r = \max_{m+1 \leq i \leq n-m+1} |\Phi_{i,m}^{(r)} - \frac{i}{n-m+1} \Phi_{n-m+1,m}^{(r)}|.$$
Select the window size \( m \).

Generate \( B \) (say 2000) conditionally i.i.d. copies \( \{ \Phi_{i,m} \}_{i=1}^{n-m+1} \), \( r = 1, 2, \ldots, B \). Let

\[
M_r = \max_{m+1 \leq i \leq n-m+1} |\Phi_{i,m}^{(r)} - \frac{i}{n-m+1}\Phi_{n-m+1,m}^{(r)}|.
\]

Let \( M^{(1)} \leq M^{(2)} \leq \cdots \leq M^{(B)} \) be the ordered statistics of \( M_r \), \( r = 1, 2, \cdots, B \). Reject \( H_0 \) at level \( \alpha \) if

\[
T_n / \sqrt{n} > M^{(\lfloor B(1-\alpha) \rfloor)},
\]

where \( \lfloor x \rfloor \) denotes the largest integer smaller or equal to \( x \). Let

\[
B^* = \max\{ r : M^{(r)} \leq T_n / \sqrt{n} \}.
\]

The \( p \)-value of the test can be obtained by \( 1 - B^*/B \).
Accuracy of the robust bootstrap

Theorem

Under mild conditions, we have

$$\max_{i,j} \| \text{Cov}(\Phi_{i,m}, \Phi_{j,m} | \{X_i\}_{i=1}^n) - \text{Cov}(U(t_i), U(t_j)) \| = O\left(\sqrt{\frac{m}{n}} + \frac{1}{m}\right). \quad (11)$$

Consequently the optimal window size $m = O(n^{1/3})$ with optimal uniform MSE $= O(n^{-1/3})$. 

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Theorem

Assume that \( m \to \infty \) with \( m/n \to 0 \) and conditional on \((X_i)_{i=1}^n\). Then under the local alternative that \( \mu(\cdot) = n^{-1/2}f(\cdot) \), where \( f(\cdot) \) is a nonconstant piecewise Lipschitz continuous function, we have

\[
\max_{m+1 \leq i \leq n-m+1} \left| \Phi_{i,m} - \frac{i}{n-m+1} \Phi_{n-m+1,m} \right| \Rightarrow \sup_{0 \leq t \leq 1} |U(t) - tU(1)|. \quad (12)
\]

On the other hand, note that, under the above local alternative

\[
T_n/\sqrt{n} \Rightarrow \sup_{0 \leq t \leq 1} |U(t) - tU(1) + \int_0^t f(s) \, ds - t \int_0^1 f(s) \, ds|.
\]
Consider testing structure changes of
\[ \eta_i = \mathbb{E}\phi(X_i, X_{i-1}, \cdots, X_{i-q}), \text{ where } \{X_i\} \text{ is a } d\text{-dimensional PLS process(13)} \]
and \( \phi : \mathbb{R}^{qd} \rightarrow \mathbb{R}^p. \)
CUSUM test statistic
\[ T_n(\phi) = \max_{1+q \leq i \leq n} |S_i(\phi) - t_i S_n(\phi)|, \text{ with } S_i(\phi) = \sum_{j=q+1}^{i} \phi(X_j, X_{j-1}, \cdots, X_{j-q}) \]
Write \( Y_i = \phi(X_i, X_{i-1}, \cdots, X_{i-q}). \) Then
\[ Y_i = \eta_i + W_j[t_i, (\cdots, \varepsilon_{i-1}, \varepsilon_i)], \text{ if } b_j < t_i \leq b_{j+1}, \tag{14} \]
Write \( \Sigma^2(t) = \sum_{k=-\infty}^{\infty} \text{Cov}(W_i(t, \mathcal{F}_0), W_i(t, \mathcal{F}_k)) \text{ if } t \in (b_i, b_{i+1}]. \)
Extensions to multidimensional statistics of PLS time series

Theorem

i) : Under $H_0$, $T_n(\phi)/\sqrt{n} \Rightarrow \sup_{0 \leq t \leq 1} |U(t) - tU(1)|$, where $U(t)$ is a $p$-dimensional zero-mean Gaussian process with covariance function $\gamma(t, s) = \int_0^{\min(t, s)} \Sigma^2(r) dr$. ii) : Let

$$\Phi_{i, m}(\phi) = \sum_{j=q+1}^{i} \frac{1}{\sqrt{m(n-m-q+1)}} \left( S_{j, m}(\phi) - \frac{m}{n} S_n(\phi) \right) R_j,$$

where $S_{j, m}(\phi) = \sum_{r=j}^{j+m-1} Y_r$ and $(R_i)_{i=1}^n$ are i.i.d. $N(0, 1)$ independent of $(X_i)_{i=1}^n$. Under $H_0$ and conditional on $(X_i)_{i=1}^n$, we have

$$\max_{q+1 \leq i \leq n-m+1} \left| \Phi_{i, m}(\phi) - \frac{i}{n-m+1} \Phi_{n-m+1, m}(\phi) \right| \Rightarrow \sup_{0 \leq t \leq 1} |U(t) - tU(1)|.$$
Simulation: accuracy under stationarity

Scenarios: (a). AR(1) model with AR-coefficient $a$. (b). ARMA(1,1) model with AR-coefficient $a$ and MA coefficient 0.5. (c). ARMA(1,1) model with AR-coefficient $a$ and MA coefficient $-0.6$.

$$\alpha = 0.05$$

<table>
<thead>
<tr>
<th>Case</th>
<th>$a$</th>
<th>LW1</th>
<th>LW2</th>
<th>SN</th>
<th>RB1</th>
<th>RB2</th>
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<tbody>
<tr>
<td>(a)</td>
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<td>4.1</td>
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<td>5.1</td>
<td>3.4</td>
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<tr>
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<td>9.4</td>
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$$\alpha = 0.1$$

<table>
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<th>RB1</th>
<th>RB2</th>
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<td>9.3</td>
<td>14.4</td>
<td>8.5</td>
<td>7.5</td>
</tr>
<tr>
<td>(b)</td>
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<td>9.5</td>
<td>10.7</td>
<td>10.6</td>
<td>8.9</td>
</tr>
<tr>
<td>(b)</td>
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<td>9.3</td>
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<td>4.1</td>
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<tr>
<td>(c)</td>
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<td>8.3</td>
<td>3.4</td>
<td>7.7</td>
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<tr>
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Table 1. Simulated type I error rates (in %) for the five methods with nominal levels 5% and 10% under stationary models (a), (b) and (c). Series length $n = 200$. 
### Table 2. Simulated type I error rates (in %) for the five methods with nominal levels 5% and 10% under stationary models (a), (b) and (c). Series length $n = 500$. 

<table>
<thead>
<tr>
<th>Case</th>
<th>$\alpha$</th>
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<th>LW2</th>
<th>SN</th>
<th>RB1</th>
<th>RB2</th>
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</thead>
<tbody>
<tr>
<td>(a)</td>
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<td>6.6</td>
<td>5.8</td>
<td>5.1</td>
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<td>5.6</td>
</tr>
<tr>
<td>(a)</td>
<td>0.5</td>
<td>6.9</td>
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<td>5.9</td>
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<tr>
<td>(a)</td>
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<td>6.2</td>
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<tr>
<td>(b)</td>
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<td>4.8</td>
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<td>4.3</td>
</tr>
<tr>
<td>(b)</td>
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<td>5.2</td>
<td>5.0</td>
<td>5.7</td>
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<td>4.6</td>
</tr>
<tr>
<td>(b)</td>
<td>0.8</td>
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<td>6.3</td>
<td>6.8</td>
<td>3.3</td>
<td>5.3</td>
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<tr>
<td>(c)</td>
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<td>1.5</td>
<td>4.0</td>
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<td>1.4</td>
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<tr>
<td>(c)</td>
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<td>3.8</td>
<td>5.5</td>
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<td>3.4</td>
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<tr>
<td>(c)</td>
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<td>18.9</td>
<td>6.8</td>
<td>6.0</td>
<td>19.0</td>
<td>6.0</td>
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</tbody>
</table>

### Accuracy under stationarity

$$\alpha = 0.05$$

$$\alpha = 0.1$$
Simulation: accuracy under non-stationarity

Case (I):
\[ X_i = V(t_i)X_i^*, \quad i = 1, 2, \ldots, n, \text{ where } V(t) = 1 + 4I\{t > 0.75\}, \]
\{X_i^*\} is a zero-mean AR(1) process with AR coefficient 0.5.
Simulation: accuracy under non-stationarity

- **Case (I):**
  \[ X_i = V(t_i)X_i^*, \quad i = 1, 2, \cdots, n, \text{ where } V(t) = 1 + 4I\{t > 0.75\}, \]
  \( \{X_i^*\} \) is a zero-mean AR(1) process with AR coefficient 0.5.

- **Case (II):**
  \[ X_i = G_0(t_i, \mathcal{F}_i) \text{ for } t_i \leq 1/3 \text{ and } X_i = G_1(t_i, \mathcal{F}_i) \text{ for } t_i > 1/3, \]
  where
  \[ G_0(t, \mathcal{F}_i) = 0.5G_0(t, \mathcal{F}_{i-1}) + \epsilon_i, \quad G_1(t, \mathcal{F}_i) = -0.5G_1(t, \mathcal{F}_{i-1}) + \epsilon_i, \]
  and \( \epsilon_i \)'s are i.i.d. \( N(0, 1) \).
Simulation: accuracy under non-stationarity

- Case (I):
  \[ X_i = V(t_i)X_i^*, \quad i = 1, 2, \ldots, n, \text{ where } V(t) = 1 + 4I\{t > 0.75\}, \]
  \{X_i^*\} is a zero-mean AR(1) process with AR coefficient 0.5.

- Case (II):
  \[ X_i = G_0(t_i, F_i) \text{ for } t_i \leq 1/3 \text{ and } X_i = G_1(t_i, F_i) \text{ for } t_i > 1/3, \]
  where
  \[ G_0(t, F_i) = 0.5G_0(t, F_{i-1}) + \varepsilon_i, \quad G_1(t, F_i) = -0.5G_1(t, F_{i-1}) + \varepsilon_i, \]
  and \(\varepsilon_i's\) are i.i.d. \(N(0, 1)\).

- Case (III):
  \[ X_i = G_0(t_i, F_i), \]
  where
  \[ G_0(t, F_i) = 0.75\cos(2\pi t)G_0(t, F_{i-1}) + \varepsilon_i, \]
  and \(\varepsilon_i's\) are i.i.d. \(N(0, 1)\).
Simulation: accuracy under non-stationarity

Case (I):

\[ X_i = V(t_i)X_i^*, \quad i = 1, 2, \cdots, n, \text{ where } V(t) = 1 + 4I\{t > 0.75\}, \]

\( \{X_i^*\} \) is a zero-mean AR(1) process with AR coefficient 0.5.

Case (II): \( X_i = G_0(t_i, F_i) \) for \( t_i \leq 1/3 \) and \( X_i = G_1(t_i, F_i) \) for \( t_i > 1/3 \), where

\[ G_0(t, F_i) = 0.5G_0(t, F_{i-1}) + \varepsilon_i, \quad G_1(t, F_i) = -0.5G_1(t, F_{i-1}) + \varepsilon_i, \]

and \( \varepsilon_i \)'s are i.i.d. \( N(0, 1) \).

Case (III): \( X_i = G_0(t_i, F_i) \), where

\[ G_0(t, F_i) = 0.75 \cos(2\pi t)G_0(t, F_{i-1}) + \varepsilon_i, \]

and \( \varepsilon_i \)'s are i.i.d. \( N(0, 1) \).

Case (IV): \( X_i = G_0(t_i, F_i) \) for \( t_i \leq 0.8 \) and \( X_i = G_1(t_i, F_i) \) for \( t_i > 0.8 \), where

\[ G_0(t, F_i) = 0.75 \cos(2\pi t)G_0(t, F_{i-1}) + \varepsilon_i, \quad G_1(t, F_i) = (0.5 - t)G_1(t, F_{i-1}) + \varepsilon_i \]

and \( \varepsilon_i \)'s are i.i.d. \( N(0, 1) \).
Simulation: accuracy under non-stationarity

Table 3. Simulated type I error rates (in %) for the five methods with nominal levels 5% and 10% under non-stationary models I - IV. Series lengths $n = 200$ and 500.

<table>
<thead>
<tr>
<th>Model</th>
<th>LW1</th>
<th>LW2</th>
<th>SN</th>
<th>RB1</th>
<th>RB2</th>
<th>LW1</th>
<th>LW2</th>
<th>SN</th>
<th>RB1</th>
<th>RB2</th>
</tr>
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Power comparison

\[(A)\] : 
\[e_i = 0.5e_{i-1} + \varepsilon_i \text{ where } \varepsilon_i \sim \text{i.i.d. } N(0, 1).\]

\[(B)\] : \(\{e_i\}\) follow the following PLS AR(1) model 
\[e_i = G_0(t_i, \mathcal{F}_i) \text{ for } t_i \leq 1/2 \text{ and } e_i = G_1(t_i, \mathcal{F}_i) \text{ for } t_i > 1/2, \]
where 
\[G_0(t, \mathcal{F}_i) = 0.75 \cos(2\pi t)G_0(t, \mathcal{F}_{i-1}) + \varepsilon_i, \quad G_1(t, \mathcal{F}_i) = (0.5 - t)G_1(t, \mathcal{F}_{i-1}) + \varepsilon_i.\]

Alternative:
\[\mu_i = \delta[t_i I\{0 \leq t_i \leq 0.5\} + (t_i - 1)I\{0.5 < t_i \leq 1\}],\]
Figure: Left panel: simulated rejection rates for Case (A). Right panel: simulated rejection rates for Case (B). The horizontal lines represent the nominal level $\gamma = 0.1$. 
Argentina rainfall data

Figure: Upper panel: Time series plot of the Argentina rainfall data. Lower panel: Fitted marginal standard deviation of Argentina rainfall data.

Yearly rainfall in Tucumán Province, Argentina in millimeters from 1884 to 1996.
Classic long-run variance normalization: \( p \)-value < 0.1%
Robust bootstrap: \( p \)-value = 2%.
Figure: Change of weekly U.S. 1-year Treasury constant maturity rate from January 4, 1962 to September 10, 1999.

Classic long-run variance normalization for mean: $p$-value 12%
Robust bootstrap for mean: $p$-value 22%.
Classic long-run variance normalization for ACVF1: $p$-value 13%
Robust bootstrap for ACVF1: $p$-value 18%.
REFERENCES
Thank you!