On best possible rates of approximation by lattice orbits on homogeneous varieties

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Newton Institute, Cambridge :

Groups, number theory and dynamics

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Based on joint work with

Anish Ghosh and Alex Gorodnik

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• Let $\|g\|$ denote a natural gauge on G, namely a continuous, non-negative and proper function from G to $\mathbb{R}_+$.

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• with $\gamma \in \Gamma$ satisfying $\|\gamma\| < \left(\frac{1}{\epsilon}\right)^{\zeta}$, and $\zeta = \zeta(x, x_0) < \infty$.

Thus $\zeta(x, x_0)$ gives a rate of approximation of a general point $x_0 \in G/H$ by the Γ-orbit of x.

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• Problem II : Explicit bounds. Establish an upper bound and a lower bound for the rate of approximation, and explicate their dependence on G, H, and Γ explicitly.

• Problem III : Optimality. Give a simple, easily verifiable and widely applicable criterion for when the upper and lower bounds coincide, giving rise to the optimal rate of approximation by lattice orbits on the homogeneous space.

Scope of the problem : some instances

• $G(\mathbb{R})$ a real algebraic group defined over $\mathbb{Q}$, $H(\mathbb{R})$ an algebraic subgroup, $\Gamma = G(\mathbb{Z})$ the lattice of integral points. This includes natural Diophantine approximation problems on homogeneous affine varieties, as well as on homogeneous projective varieties.

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• G is an S-algebraic $\mathbb{Q}$-group, H a closed subgroup, $\Gamma = G(\mathbb{Z}[S^{-1}])$. This includes for example $G = G(\mathbb{R}) \times G(\mathbb{Q}_p)$ and $H = G(\mathbb{Q}_p)$, namely approximation in the connected group $G(\mathbb{R})$ by the dense subgroup $G(Z[\frac{1}{p}])$.

• $G(\mathbb{A})$ and $H(\mathbb{A})$ are groups of rational adéles, $\Gamma = G(\mathbb{Q})$. This includes the problem of intrinsic diophantine approximation, namely by rational points lying on the algebraic variety $G(\mathbb{R})/H(\mathbb{R})$ itself.

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• We note that a main assumption in our approach is that G is non-amenable.

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• Kleinbock and Merrill (2013) have established the best possible exponent for rational approximation on the unit spheres in any dimension $n \geq 2$, together with an analog of Khinchine's theorem (and even sharper results). More recently [FKMS] considered general quadratic varieties.

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• and in the local field case, take the standard valuation on the field, and the standard maximum norm on the linear space F^n, and on $M_n(F)$.

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• $\kappa(x, x_0)$ is a $\Gamma \times \Gamma$-invariant function, hence almost surely a constant κ when the action is ergodic. κ depends on G, Γ and V, but not on the norms chosen on F^n and $M_n(F)$.

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• Taking the resulting two inhomogeneous equations mod 1, we conclude that for every $x_0 = (u_0, v_0) \in \mathbb{T}^2$, for almost every $x = (u, v) \in \mathbb{T}^2$, and for every ϵ sufficiently small, there are integers a, b, c, d with

$$\|(au + bv, cu + dv) - (u_0, v_0)\| < \epsilon$$

such that $ad - bc = 1$, and

$$\max\{|a|, |b|, |c|, |d|\} < \frac{B}{\epsilon} \cdot \log\left(\frac{1}{\epsilon}\right)^{2+\eta}$$

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• For the corresponding approximation result using algebraic integers in other imaginary quadratic number fields, it is possible to give upper estimates for the exponent κ, but its exact value remains an open problem.

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- H is the fixed point set of the involution $(g_1, g_2) \mapsto (g_2, g_1)$ and $V = G/H$ is a semisimple symmetric space.
- Consider $F = \mathbb{R}, \mathbb{C}, \mathbb{Q}_p$, and every irreducible lattice Γ of $G = SL_3(F) \times SL_3(F)$ (e.g. $F = \mathbb{R}$, $\Gamma = SL_3(\mathbb{Z}[\sqrt{p}])$, p prime).
- Then the exponent of Diophantine approximation of Γ on $V_k(F)$ is given by $\kappa = 4/3$, in all cases.
- The best possible exponent for irreducible lattices in $SL_2(F) \times SL_2(F)$ is $\kappa = 3/2$. Whether it is achieved by any irreducible lattice remains an open problem.

Lower bound for the Diophantine exponent

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- Define the empirical growth parameter for such points :

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• Let d denote a bound for the local growth parameter of the invariant measure m_V on the affine subvariety $V = G/H \subset F^n$, namely $m_V(\{\|v - v_0\| < \epsilon\} \geq C_\eta \epsilon^{d+\eta}$, for all $\eta > 0$.

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• Theorem 1. d/a is a lower bound on the Diophantine exponent κ : it is impossible to approximate points on $V = G/H$ as above by points in lattice orbits any faster, namely using matrices of smaller norm.

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Deriving an upper bound

• An upper bound on the Diophantine exponent requires other considerations, namely

• spectral estimates in the automorphic representation $L^2(\Gamma \setminus G)$ leading to a quantitative mean ergodic theorem for H,

• dynamical arguments exploiting the speed of equidistribution, in the form of a shrinking target argument.

• Consider the intersection of norm balls with the stability group H, namely $H_T = \{h \in H \,;\, \|h\| < T\}$.

• Consider the invariant probability measure $m_{\Gamma \setminus G}$ on $Y = \Gamma \setminus G$ and define averaging operators $\pi_Y(\beta_T) : L^2(\Gamma \setminus G) \to L^2(\Gamma \setminus G)$, given by

The quantitative mean ergodic theorem

$$\pi_Y(\beta_T)f(y) = \frac{1}{m_H(H_T)} \int_{h\in H_T} f(yh)dm_H(h)\ ,\ y \in \Gamma \setminus G\,.$$

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• Assume that the quantitative mean ergodic theorem for the averaging operators $\pi_Y^0(\beta_T)$ holds, namely :

• there exists $\theta > 0$ such that

$$\|\pi_Y(\beta_T)f - \int_Y fdm\|_{L^2(\Gamma\setminus G)} \le C(\eta)m_H(H_T)^{-\theta+\eta}\|f\|_{L^2(\Gamma\setminus G)}$$

for every $\eta > 0$, suitable $C(\eta)$, and $t \ge t_\eta$.

Spectral gaps

• The foregoing estimate holds in great generality. Indeed, the representation of G is $L^2_0(\Gamma \setminus G)$, typically has a spectral gap.

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• Note also that the quantitative mean ergodic theorem for the averaging operators $\pi_Y(\beta_T)$ implies of course the ergodicity of H on $\Gamma \setminus G$. By the duality principle for homogeneous spaces, the ergodicity of Γ on G/H follows, and in particular, almost every Γ-orbit in $V = G/H$ is dense.

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- Conclusion : if $2\theta = 1$ then the lower and upper bounds for the Diophantine exponent coincide !

Best possible rate of approximation

- Corollary 1. If the rate of convergence in the mean ergodic theorem for the averaging operators β_T acting on $L^2_0(\Gamma \setminus G)$, is as fast as the inverse of the square root of the volume of H_T, then the rate of Diophantine approximation of Γ-orbits on the variety $V = G/H$ is best possible, and the Diophantine exponent is given by $\kappa = \frac{d}{a}$, the a-priori pigeon-hole bound.

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- Corollary 2. If the stability group H is semi simple and non-compact, and the restriction of the automorphic representation $\pi^0_{G/\Gamma}$ to H is a tempered representation of H, then the Diophantine exponent of the irreducible lattice Γ of G in its action on G/H is best possible, and is given by $\kappa = \frac{d}{a}$.

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• It is a surprising and useful fact that the foregoing condition of "subgroup temperedness" holds in considerable generality for a large class of triples (G, H, Γ).

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• When G is semi simple and had property T, there are bounds on the K-finite matrix coefficients of G in general unitary representations, and these can be restricted to a subgroup H and are often in $L^{2+\eta}(H)$ so the restriction is tempered.

• For example, this holds for (the images of) all the irreducible representations $SL_2(\mathbb{R}) \to SL_n(\mathbb{R})$, as observed by Margulis (1995).

• Unitary representations of semi simple groups have matrix coefficients in $L^{2k}(G)$ for some k. Restricting a representation of G^k to the diagonally embedded copy of G yields matrix coefficients which are in $L^{2+\eta}(G)$, so the diagonally embedded subgroup is tempered.

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• For some lattices and their low level congruence subgroups the Selberg eigenvalue conjecture is known to hold, so that $L_0^2(\Gamma \setminus G)$ is known to be a tempered representation of G. This holds for example for $SL_2(\mathbb{Z}) \subset SL_2(\mathbb{R})$ and $SL_2(\mathbb{Z}[i]) \subset SL_2(\mathbb{C})$.

The duality principle

• Theorem 2 is an instance of the general method of duality in homogenous dynamics, which aims to establish properties of the Γ-orbits in G/H by using properties of the H-orbits in $\Gamma \setminus G$.

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• A general quantitative duality principle has been developed in joint work with Alex Gorodnik. It yields conclusions which are considerably more precise than just the existence of a rate of approximation by Γ-orbits.

• For example, it is possible to prove quantitative mean and pointwise ergodic theorems for the discrete averages supported on orbit points when ordered by a norm, although the optimality of the rate is compromised.

The method of duality

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• The quantitative method of duality applies in considerable generality, for all locally compact groups, closed subgroups H, and discrete lattices Γ,

• subject only to natural and necessary assumptions about

1. the growth of the sets H_T and the lattice points in their vicinity,
2. the spectral theory of H in $L^2_o(\Gamma \setminus G)$,
3. the local behavior of the invariant measure $m_{G/H}$ on the homogeneous space $G/H = V$.