System Identification for linear quantum systems and spin networks

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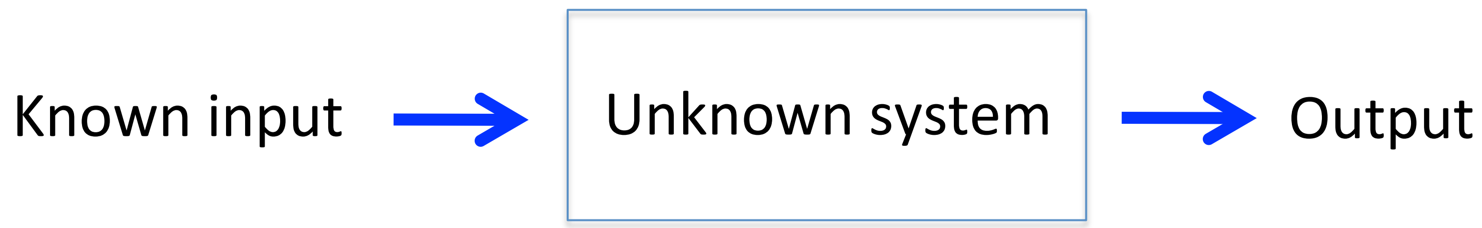
System Identification ?



A set of input-output data is given to us. The purpose of system identification is, using those data set, to construct a mathematical model of the system. Then the problem is called the non-parametric identification.

If we already know the mathematical model of the system and it contains some unknown parameters, then the problem is called the parametric identification.

The most successful case for classical system ID:

We assume the following linear parametric model:

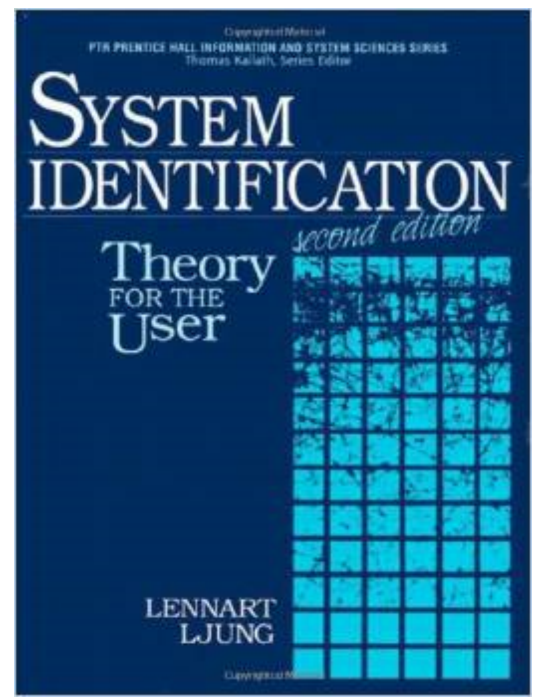
$$
\begin{aligned}
d\boldsymbol{x}(t) &= A\boldsymbol{x}(t)dt + B\boldsymbol{u}(t)dt \\
d\boldsymbol{y}(t) &= C\boldsymbol{x}(t)dt + D\boldsymbol{u}(t)dt
\end{aligned}
$$

$$\boldsymbol{u}(t) \rightarrow \boxed{\boldsymbol{x}(t)} \rightarrow \boldsymbol{y}(t)$$

Problem:

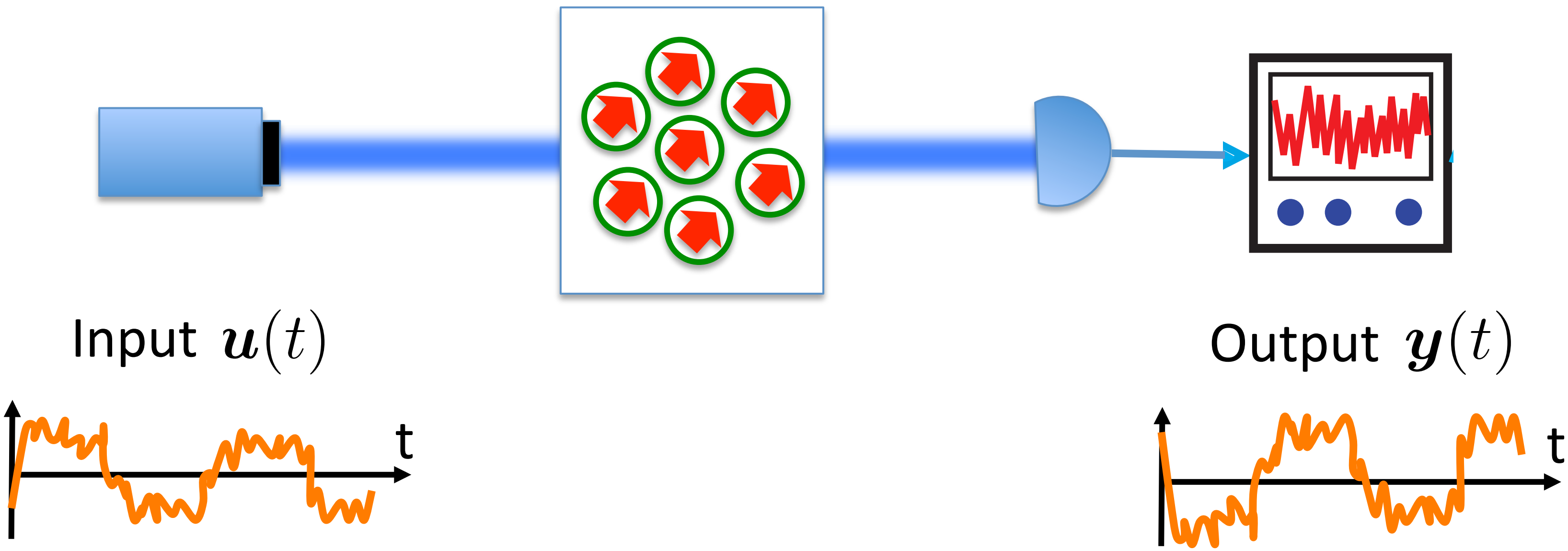
$$\boxed{\text{Identify } (A, B, C, D) \text{ using the input } \boldsymbol{u}(t) \text{ and the output } \boldsymbol{y}(t)}$$

There are a lot of efficient methods for linear and even nonlinear system ID in the classical case !

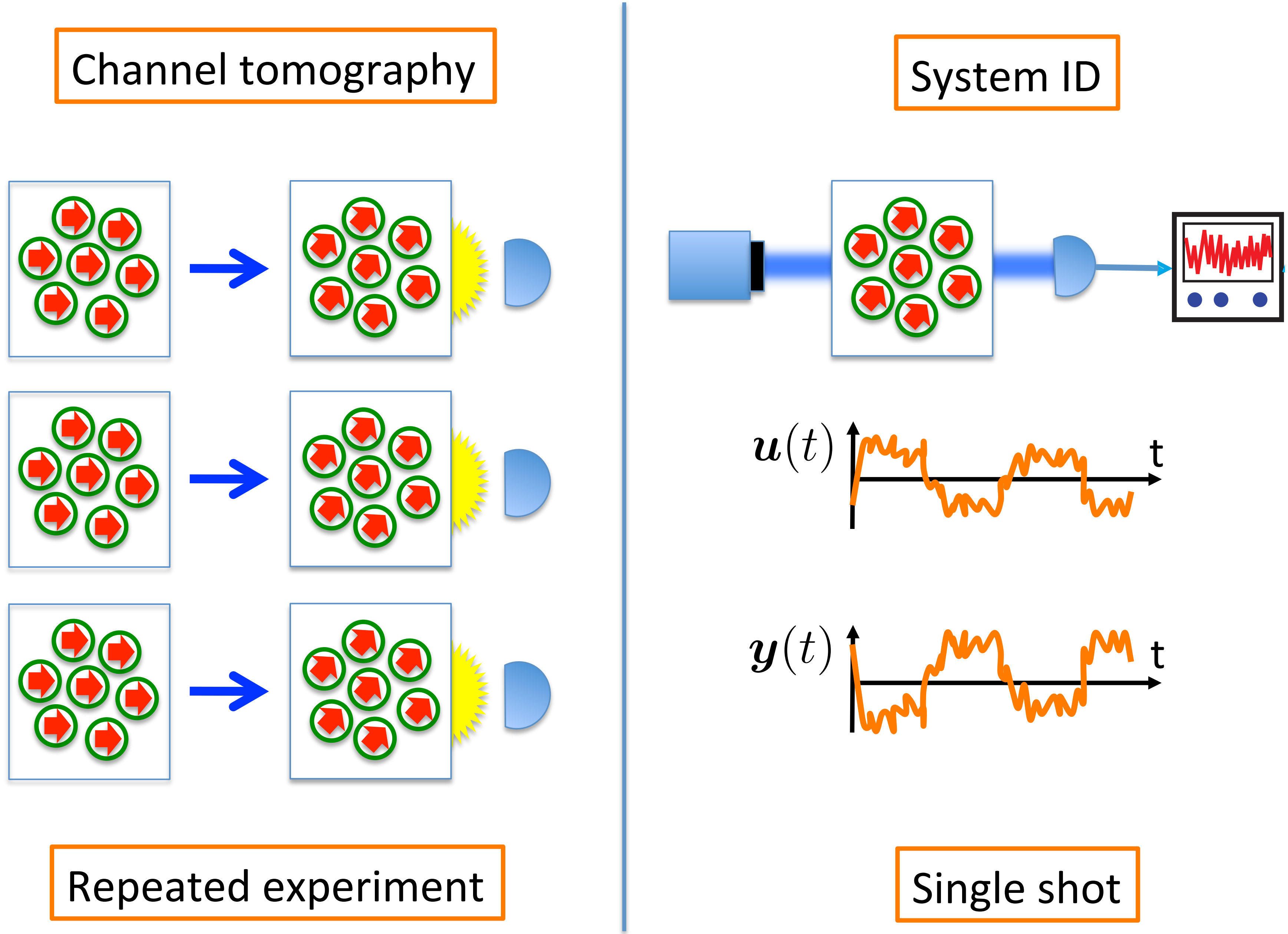


In the classical case, system ID is usually formulated based on a dynamical input-output setup.

→ We formulate quantum system ID problems in the (Markov) dynamical input-output setup.



Use $\boldsymbol{u}(t)$ and $\boldsymbol{y}(t)$ to identify the parametric model.



Channel tomography
System ID
$u(t)$
t
$y(t)$
t
Repeated experiment
Single shot

Contents

1. Identification for linear systems --- analytic approach

   Identifiability --- What can we identify?

   ID method --- How to identify?

   Statistics --- How much can we identify?

2. Identification for nonlinear systems --- numerical approach

   Structure identification of spin networks

Contents

1. Identification for linear systems --- analytic approach

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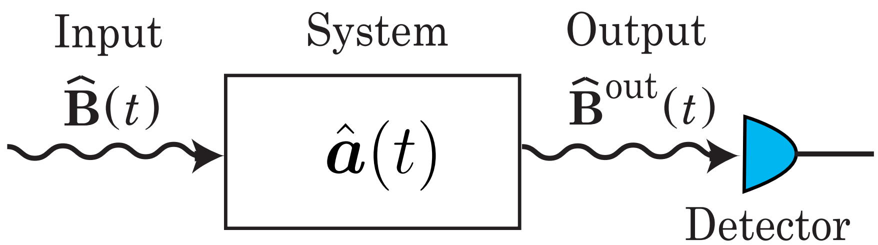
Ref: M. Guta and NY, Linear quantum system identification, Arxiv and Proceedings of 2013 IEEE CDC.

Quantum (passive) linear systems



$$d\hat{\boldsymbol{a}}(t) = A\hat{\boldsymbol{a}}(t)dt - C^\dagger d\hat{\mathbf{B}}(t)$$

$$d\hat{\mathbf{B}}^{\rm out}(t) = C\hat{\boldsymbol{a}}(t)dt + d\hat{\mathbf{B}}(t)$$

System-field coupling

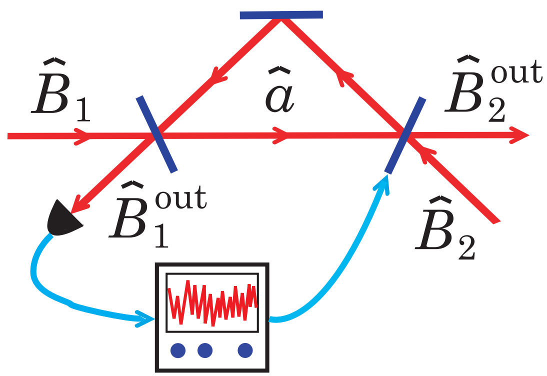
$$A := -i\Omega - \frac{1}{2}C^\dagger C$$

System Hamiltonian

1. The system matrices have specific structure characterized by (Ω, C)

2. The input and output are served by optical fields, not by abstract ``signals”.

Example 1: Optical resonator (mode cleaning cavity)



$$d\hat{a} = (-i\omega - \kappa)\hat{a}dt - \sqrt{\kappa}d\hat{B}_1 - \sqrt{\kappa}d\hat{B}_2$$

$$d\hat{B}_1^{\mathrm{out}} = \sqrt{\kappa}\hat{a}dt + d\hat{B}_1$$

$$d\hat{B}_2^{\mathrm{out}} = \sqrt{\kappa}\hat{a}dt + d\hat{B}_2$$

$$\Omega = \omega$$

$$C^\dagger = [\sqrt{\kappa}, \sqrt{\kappa}]$$

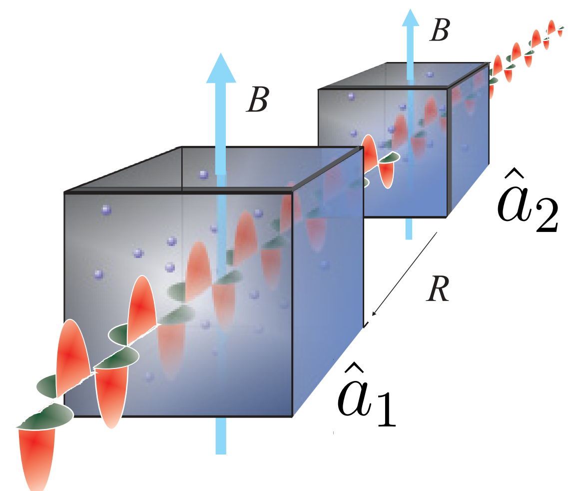
Example 2: Atomic ensembles generating entanglement

$$d\hat{\boldsymbol{a}} = -\frac{\kappa}{2} Y \hat{\boldsymbol{a}} dt - \sqrt{\kappa} Y^{1/2} d\hat{\mathbf{B}}$$

$$d\hat{\mathbf{B}}^{\text{out}} = \sqrt{\kappa} Y^{1/2} \hat{\boldsymbol{a}} dt + d\hat{\mathbf{B}}$$

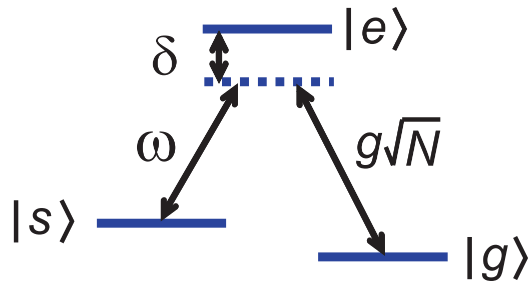
$$Y = \begin{bmatrix} \cosh(2r) & -\sinh(2r) \\ -\sinh(2r) & \cosh(2r) \end{bmatrix}$$

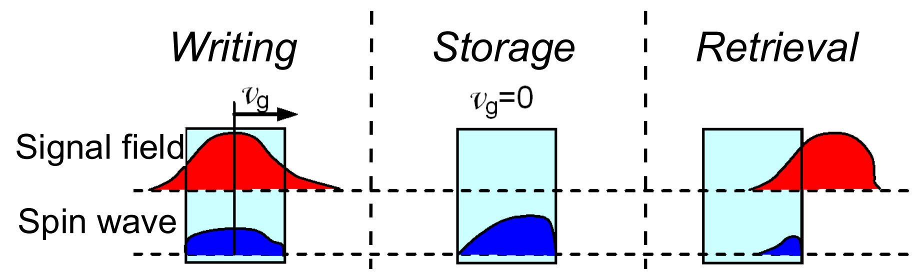
$$\boxed{\Omega = 0 \qquad C = \sqrt{\kappa} Y^{1/2}}$$



Muschik, Polzik, Cirac, PRA 2011

Example 3: Quantum memory





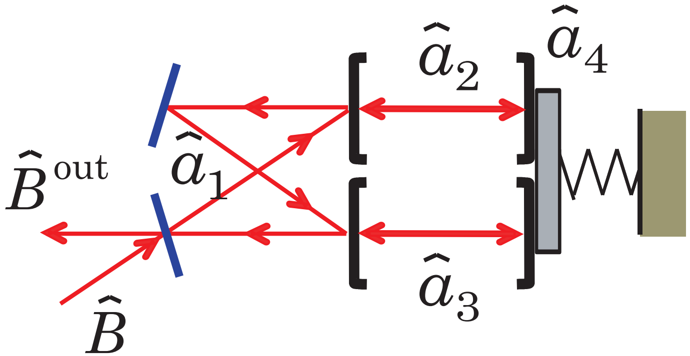
Gorshkov et al, PRA 2007

$$d\hat{\boldsymbol{a}} = \begin{bmatrix} -\kappa & ig\sqrt{N} & 0 \\ ig\sqrt{N} & -i\delta & i\omega \\ 0 & i\omega^* & 0 \end{bmatrix} \hat{\boldsymbol{a}} dt - \begin{bmatrix} \sqrt{2\kappa} \\ 0 \\ 0 \end{bmatrix} d\hat{B}$$

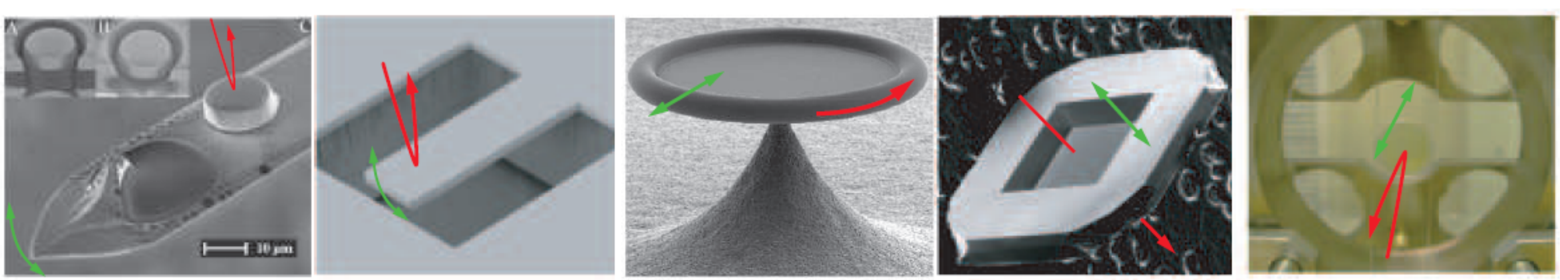
$$d\hat{B}^{\text{out}} = \sqrt{2\kappa}\hat{a}_1 dt + d\hat{B}$$

$$\boxed{\Omega(\theta) = \begin{bmatrix} 0 & \theta_1 & 0 \\ \theta_1 & 0 & \theta_2 \\ 0 & \theta_2 & 0 \end{bmatrix} \qquad C = [\sqrt{2\kappa},\ 0,\ 0]}$$

Example 4: Opto-mechanical oscillator

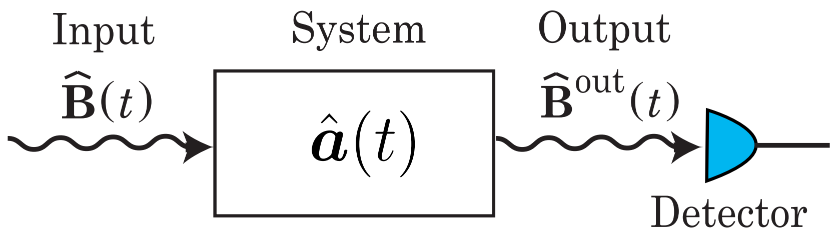


$$\Omega(\theta) = \begin{bmatrix} 0 & \theta_1 & \theta_2 & 0 \\ \theta_1 & 0 & 0 & \theta_3 \\ \theta_2 & 0 & 0 & \theta_4 \\ 0 & \theta_3 & \theta_4 & 0 \end{bmatrix} \qquad C = [\sqrt{2\kappa},\ 0,\ 0,\ 0]$$



Kippenberg and Vahala, Opt. Express 2007

System identification for linear systems



$$d\hat{\boldsymbol{a}}(t) = A\hat{\boldsymbol{a}}(t)dt - C^\dagger d\hat{\mathbf{B}}(t)$$
$$d\hat{\mathbf{B}}^{\mathrm{out}}(t) = C\hat{\boldsymbol{a}}(t)dt + d\hat{\mathbf{B}}(t)$$

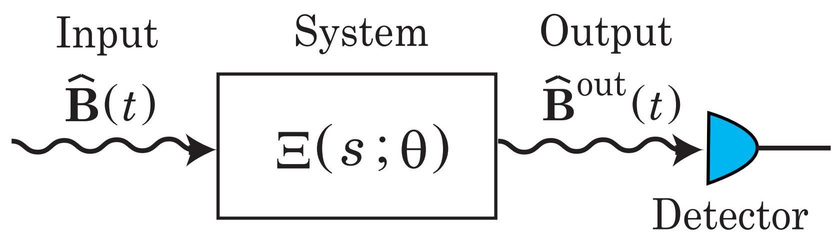
$$A := -i\Omega - \frac{1}{2}C^\dagger C$$

The system contains some unknown parameters θ, hence

$$(\Omega, C) = (\Omega(\theta), C(\theta))$$

→ We want to identify (estimate) θ, using the input-output data.

System identification for linear systems



$$\boxed{\begin{aligned} d\hat{\boldsymbol{a}}(t) &= A\hat{\boldsymbol{a}}(t)dt - C^\dagger d\hat{\mathbf{B}}(t) \\ d\hat{\mathbf{B}}^{\rm out}(t) &= C\hat{\boldsymbol{a}}(t)dt + d\hat{\mathbf{B}}(t) \end{aligned}}$$

$$A := -i\Omega - \frac{1}{2}C^\dagger C$$

In the Laplace domain, the input and output are connected by

$$\mathcal{L}[\hat{\boldsymbol{b}}^{\rm out}](s) = \Xi(s)\mathcal{L}[\hat{\boldsymbol{b}}](s)$$

$\boxed{\Xi(s) := I_m - C(sI - A)^{-1}C^\dagger}$ --- Transfer function

--- We can at most determine $\Xi(s) = \Xi(s;\theta)$ from only the input and output data.

Contents

1. Identification for linear systems --- analytic approach

Identifiability --- What can we identify?

An identifiable system:
the parameters can be identified in principle uniquely

ID method --- How to identify?

Statistics --- How much can we identify?

Ref: D'Alessandro, IEEE TAC 2005
Burgarth and Yuasa, PRL 2012

System identifiability for linear systems

We can at most determine $\Xi(s) := I_m - C(sI - A)^{-1}C^{\dagger}$ from only the input and output data.

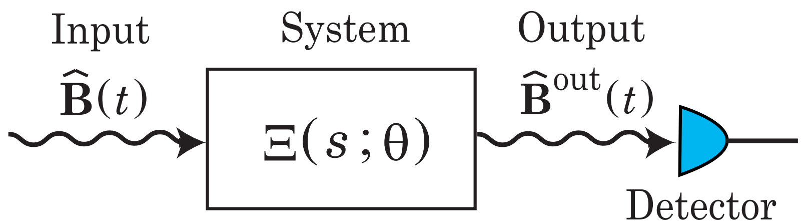
But the following fact holds for general linear systems:

$$
\begin{aligned}
dx(t) &= Ax(t)dt + Bu(t)dt, \\
dy(t) &= Cx(t)dt + Du(t)dt,
\end{aligned}
$$

Two minimal systems (A, B, C, D) and (A', B', C', D') have the same transfer function iff there exists an invertible matrix T

s.t. $A' = TAT^{-1}, \quad B' = TB, \quad C' = CT^{-1}, \quad D' = D$

System identifiability for linear systems



Definition: The parameter θ is identifiable if

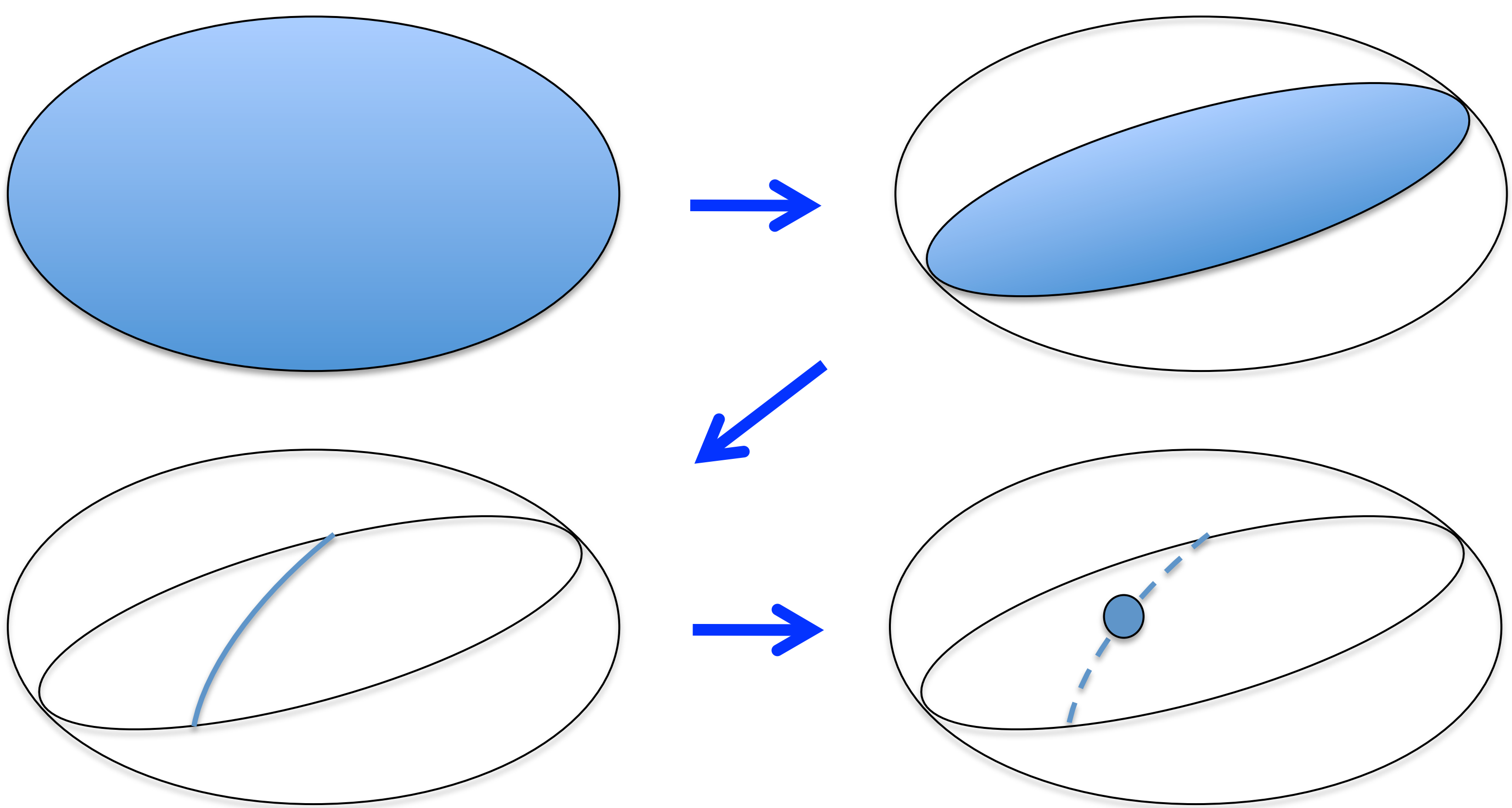
$$\Xi(s;\theta) = \Xi(s;\theta') \quad \forall s \longrightarrow \theta = \theta'$$

We can at most determine $\Xi(s) := I_m - C(sI - A)^{-1}C^\dagger$ from only the input and output data.

$\longrightarrow$ We need some prior knowledge to satisfy the above condition.

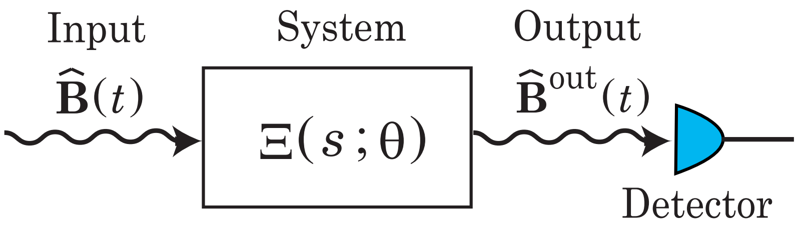
Region colored with blue : the set of θ generating the same transfer function (equivalent class)

We want to reduce the size of equiv. class using prior knowledge !



$\theta \longleftrightarrow \Xi(s;\theta)$

Equivalent class of linear passive quantum systems



The system's transfer function: $\Xi(s) := I_m - C(sI - A)^{-1}C^\dagger$

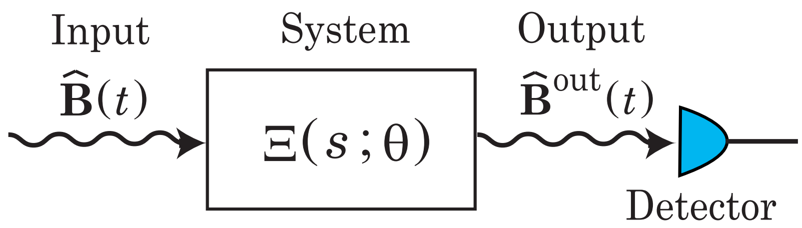
$$A := -i\Omega - \frac{1}{2}C^\dagger C$$

Two minimal quantum linear systems (Ω_1, C_1) and (Ω_2, C_2) have the same transfer function iff there exists a unitary matrix U

s.t. $\Omega_2 = U\Omega_1 U^\dagger, \quad C_2 = C_1 U^\dagger$

Freedom is reduced from 2n^2 to n^2-2n !

Identifiability test



The system's transfer function: $\Xi(s) := I_m - C(sI - A)^{-1}C^\dagger$

$$A := -i\Omega - \frac{1}{2}C^\dagger C$$

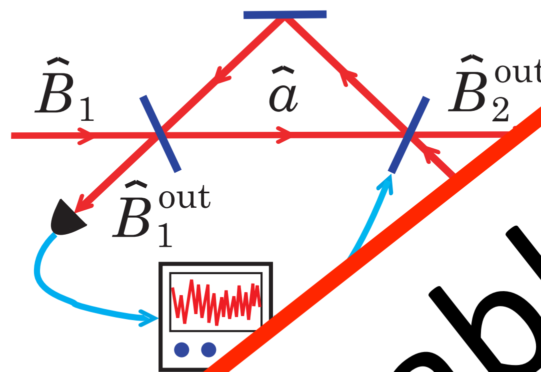
The definition becomes:

The parameter θ is identifiable if

$$\Omega(\theta') = U\Omega(\theta)U^\dagger, \quad C(\theta') = C(\theta)U^\dagger \longrightarrow \theta = \theta'$$

Solve these equations with the help of some prior knowledge about $(\Omega, C) = (\Omega(\theta), C(\theta))$

Example 1: Optical resonator (mode cleaning cavity)



Identifiable !

$$d\hat{a} = (-i\omega \text{ [illegible] } B_1 - \sqrt{\kappa} d\hat{B}_2$$

$$d\hat{B}_1^{\mathrm{out}} = \sqrt{\text{[illegible]}} B_1$$

$$d\hat{B}_2^{\mathrm{out}} = \sqrt{\kappa}\hat{a} \text{ [illegible] } + d\hat{B}_2$$

$$\Omega = \omega$$

$$C^\dagger = [\sqrt{\kappa}, \sqrt{\kappa}]$$

Example 2: Atomic ensembles generating entanglement

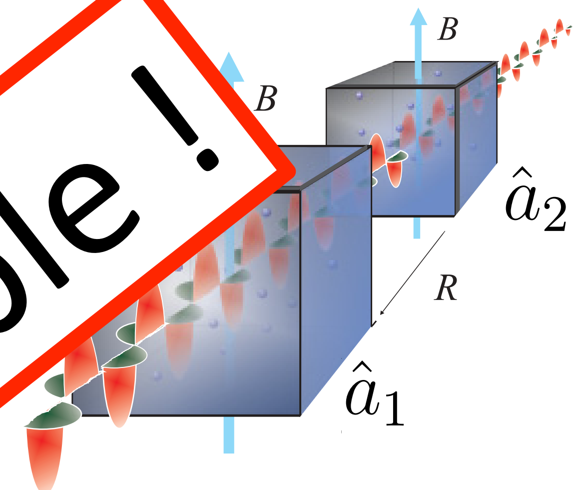
Identifiable !

$$d\hat{\boldsymbol{a}} = -\frac{\kappa}{2} Y \hat{\boldsymbol{a}} dt - \sqrt{\kappa} Y^{1/2} \ldots$$

$$d\hat{\mathbf{B}}^{\mathrm{out}} = \sqrt{\kappa} Y^{1/2} \ldots$$

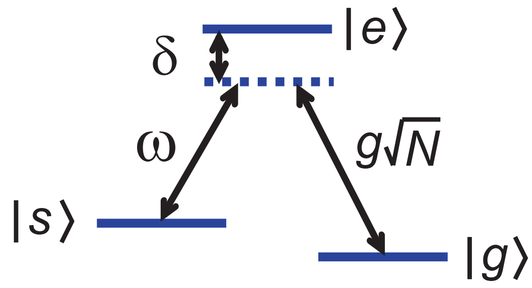
$$Y = \left[\ldots (2r) \quad \ldots (2r) \right]$$

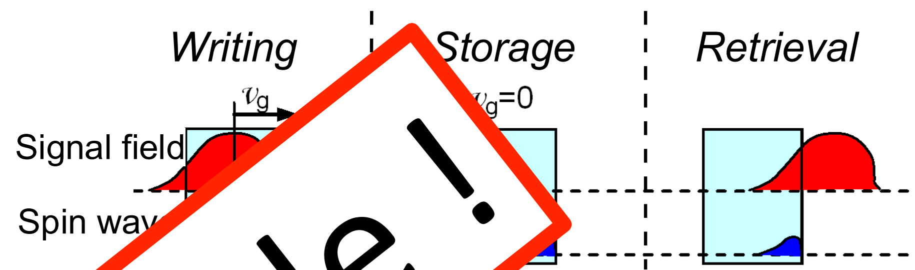
$$\boxed{\Omega = 0 \qquad C = \sqrt{\kappa} Y^{1/2}}$$



Muschik, Polzik, Cirac, PRA 2011

Example 3: Quantum memory





Gorshkov et al, PRA 2007

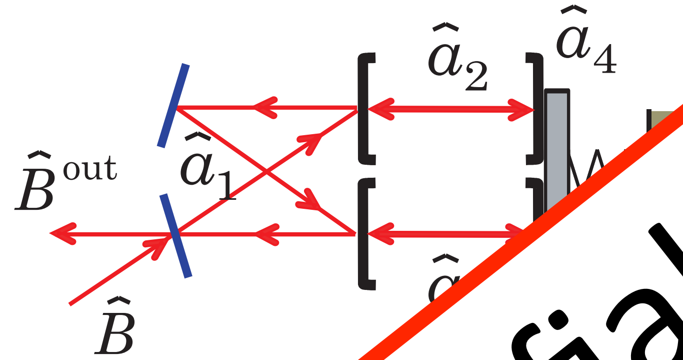
Identifiable !

$$d\hat{\boldsymbol{a}} = \begin{bmatrix} -\kappa & ig\sqrt{N} \\ ig\sqrt{N} & \\ 0 & \end{bmatrix} \qquad \begin{bmatrix} \sqrt{2\kappa} \\ 0 \\ 0 \end{bmatrix} d\hat{B}$$

$$d\hat{B}^{\mathrm{out}} = \sqrt{2\kappa}\hat{a}_1 d$$

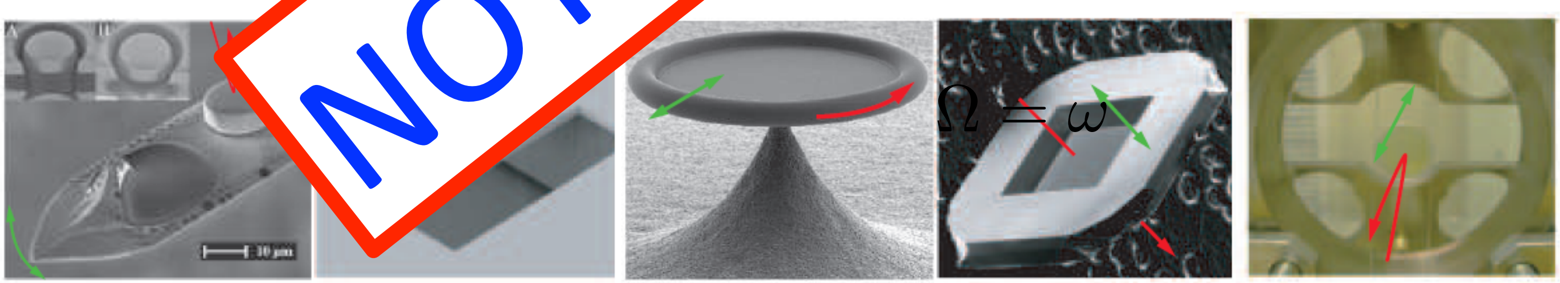
$$\boxed{\Omega(\theta) = \begin{bmatrix} 0 & \theta_1 & 0 \\ \theta_1 & 0 & \theta_2 \\ 0 & \theta_2 & 0 \end{bmatrix} \qquad C = [\sqrt{2\kappa},\ 0,\ 0]}$$

Example 4: Opto-mechanical oscillator



$$\Omega(\theta) = \qquad C = [\sqrt{2\kappa},\ 0,\ 0,\ 0]$$

NOT Identifiable !

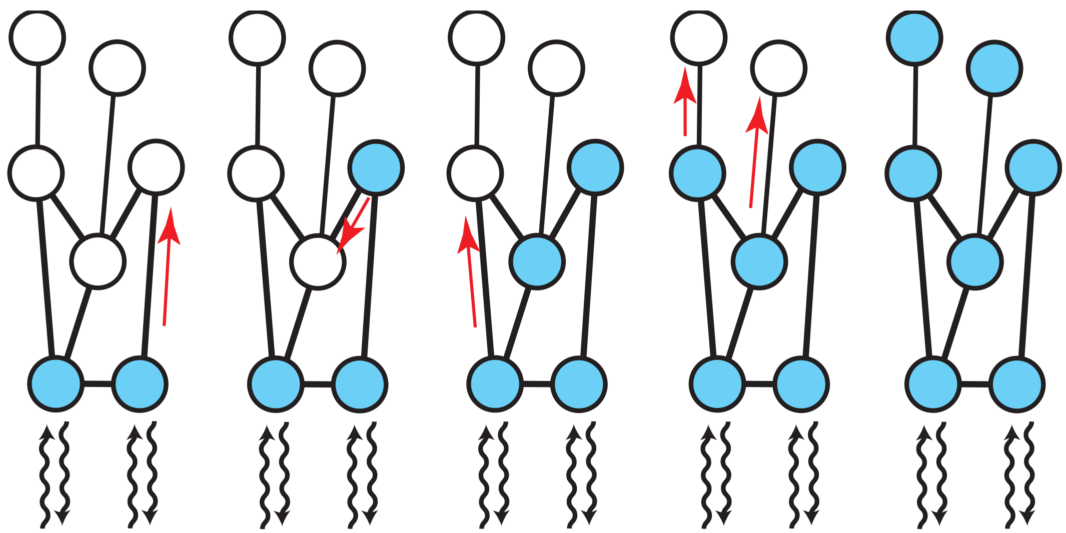


Kippenberg and Vahala, Opt. Express 2007

It should be good if we have a general condition that guarantees the identifiability.

Theorem:

Suppose that the linear quantum system has the infection property. Then, it is always identifiable.



(a) (b) (c) (d) (e)

Ref: Burgarth and Yuasa, PRL 2012

Contents

1. Identification for linear systems --- analytic approach

Identifiability --- What can we identify?

ID method --- How to identify?

Statistics --- How much can we identify?

Input-output relation: $\mathcal{L}[\hat{\boldsymbol{b}}^{\rm out}](s) = \Xi(s)\mathcal{L}[\hat{\boldsymbol{b}}](s)$

(SISO case) Assume that the transfer function is constructed using the input-output data as follows:

$$\Xi(s) = 1 + \frac{c_{n-1}s^{n-1} + \cdots + c_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}$$

A typical realization of the system's dynamical equation:

$$d\hat{\boldsymbol{x}}(t) = A_0\hat{\boldsymbol{x}}(t)dt + B_0 d\hat{\mathbf{B}}(t) \qquad d\hat{\mathbf{B}}^{\rm out}(t) = C_0\hat{\boldsymbol{x}}(t)dt + d\hat{\mathbf{B}}(t)$$

$$A_0 = \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & 0 & 1 \\ -a_0 & -a_1 & & -a_{n-1} \end{bmatrix} \qquad B_0 = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} \qquad C_0 = [c_0, \cdots, c_{n-1}]$$

But these matrices do not satisfy the quantum constraint characterized by (Ω, C). How can we reconstruct them ?

Algorithm (general case):

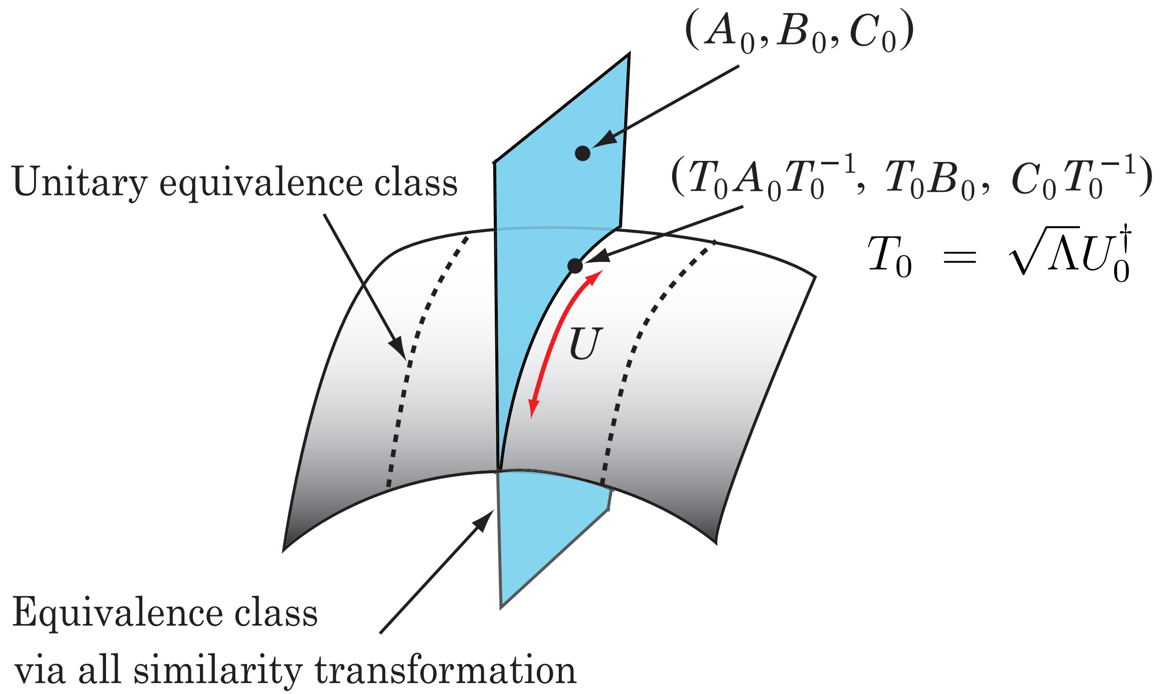
1. Solve $XA_0 + A_0^\dagger X + C_0^\dagger C_0 = 0$

2. Diagonalize $X = U_0 \Lambda U_0^\dagger$

3. Then we can reconstruct!

$$\Omega = U\Omega_0 U^\dagger \qquad \Omega_0 = \frac{i}{2}\left[\sqrt{\Lambda}U_0^\dagger A_0 U_0 \sqrt{\Lambda^{-1}} - \sqrt{\Lambda^{-1}}U_0^\dagger A_0^\dagger U_0 \sqrt{\Lambda}\right]$$

$$C = (C_0 U_0 \sqrt{\Lambda^{-1}})U^\dagger$$



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Quantum parameter estimation

Problem:

We want to estimate the unknown parameters θ contained in our state ρ_θ

Quantum Cramer-Rao bound:

There exists an unbiased estimator $\tilde{\theta}$ satisfying

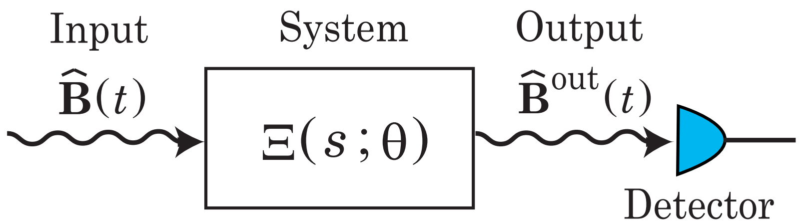
$$\lim_{n\to\infty} n \cdot \mathbb{E}_\theta[(\tilde{\theta}_n - \theta)^2] = F(\rho_\theta)^{-1}$$

$F(\theta) = \mathrm{Tr}(\rho_\theta \mathrm{L}_\theta^2)$ is the Q Fisher information obtained from

$$\frac{d\rho_\theta}{d\theta} = \frac{1}{2}(L_\theta \rho_\theta + \rho_\theta L_\theta)$$

Unbiased: $\mathbb{E}_\theta(\tilde{\theta}) = \theta$

What is the optimal input maximizing Fisher information?



``Standard" case: Coherent laser field with several amp. and freq.

Input: $|\vec{\mathbf{z}},\vec{\omega}\rangle_{\rm in} = |\mathbf{z}_1;\omega_1\rangle \otimes \cdots \otimes |\mathbf{z}_p;\omega_p\rangle$

Output: $|\vec{\mathbf{z}}_\theta,\vec{\omega}\rangle_{\rm out} = |\Xi_\theta(i\omega_1)\mathbf{z}_1;\omega_1\rangle \otimes \cdots \otimes |\Xi_\theta(i\omega_p)\mathbf{z}_p;\omega_p\rangle$

Total energy $E=\sum_i |\mathbf{z}_i|^2$ is upper bounded.

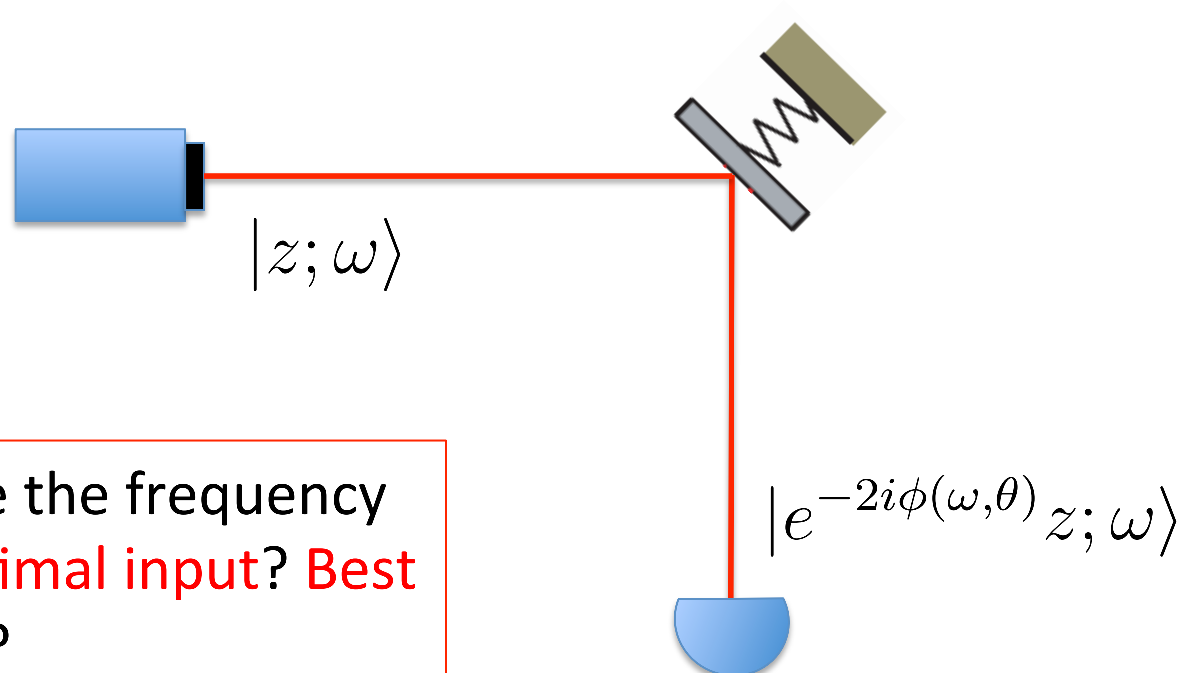
$$\rightarrow \quad F(\theta) = \sum_{i=1}^p F_i(\theta) = 4\sum_{i=1}^p \left\| \frac{d\Xi_\theta(i\omega_i)}{d\theta}\mathbf{z}_i \right\|^2 \le \underline{4E\left\| \frac{d\Xi_\theta(i\omega)}{d\theta}\mathbf{w} \right\|^2}$$

Maximize this over ω and $\mathbf{w}$ ($\|\mathbf{w}\|=1$) !

Example: 1-dim SISO system with $\Omega = \theta$ and $C = c$

$$\Xi_\theta(i\omega) = \frac{i\omega - i\theta - c^2/2}{i\omega - i\theta + c^2/2} = -\exp(-2i\phi(\omega,\theta))$$

$$\phi(\omega,\theta) = \arctan\left(\frac{2\omega - 2\theta}{c^2}\right)$$ --- ω-dependent phase shift



We want to estimate the frequency of the oscillator. Optimal input? Best achievable accuracy?

Example: 1-dim SISO system with $\Omega = \theta$ and $C = c$

$$F(\theta;\omega) = 16E\left|\frac{d\phi(\omega,\theta)}{d\theta}\right|^2 = 16E\left|\frac{2c^2}{c^4+4(\omega-\theta)^2}\right|^2 \leq \frac{64E}{c^4}$$

- Optim. freq is given by $\omega_{\mathrm{opt}} = \theta$, which is though unknown.

→ As in the classical case, we need the M-sequence input, which is now served by coherent laser fields.

- $\mathbb{E}_\theta[(\tilde{\theta}-\theta)^2] \geq \dfrac{c^4}{64E}$ --- Standard Quantum Limit

- C should be small.

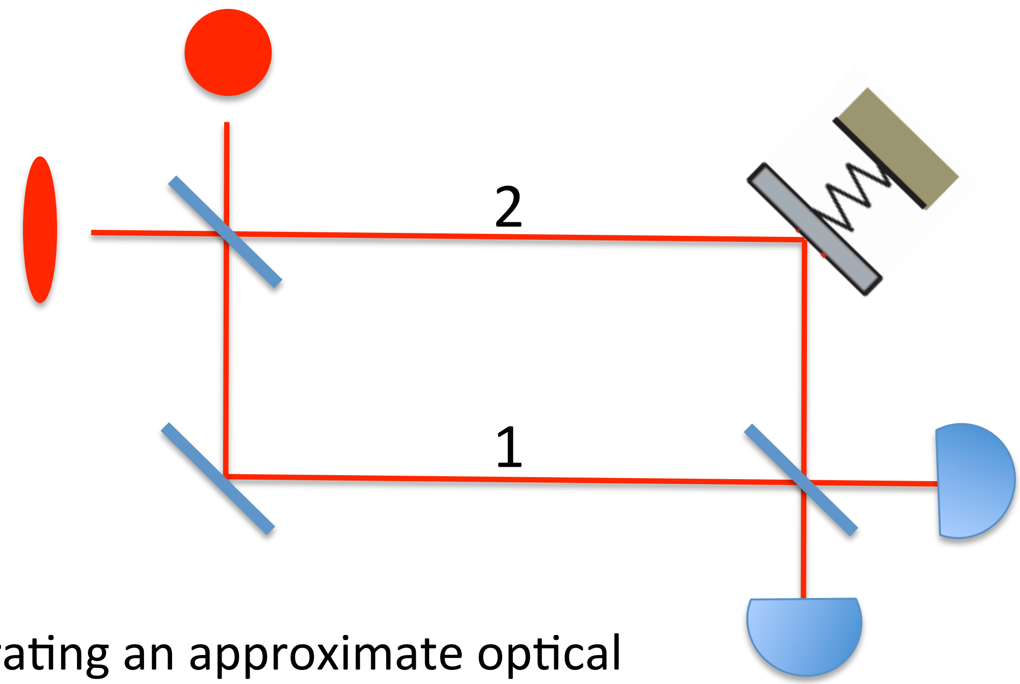
→ There is a tradeoff between stability and accuracy.

Quantum metrology

In Q mechanics, we can take some ``non-standard" inputs.

NOON state : $|\psi\rangle = \frac{1}{\sqrt{2}}\left(|n;\omega\rangle_1|0\rangle_2 + |0\rangle_1|n;\omega\rangle_2\right)$

n : Photon number



A schematic generating an approximate optical NOON state (e.g. H. Hoffman, PRA 2007)

Example: 1-dim SISO system with $\Omega = \theta$ and $C = c$

Input: $|\psi\rangle = \frac{1}{\sqrt{2}}(|n;\omega\rangle|0\rangle + |0\rangle|n;\omega\rangle)$ (In this case, $E = n/2$)

Output: $|\psi_\theta\rangle_{\mathrm{out}} = \frac{1}{\sqrt{2}}(|n;\omega\rangle|0\rangle + |0\rangle \otimes \Xi_\theta(i\omega)^n|n;\omega\rangle)$

$$= \frac{1}{\sqrt{2}}\left(|n;\omega\rangle|0\rangle - e^{-2in\phi(\omega,\theta,c)}|0\rangle|n;\omega\rangle\right)$$

Then we have $F(\theta) = 16E^2\left|\frac{d\phi(\omega,\theta,c)}{d\theta}\right|^2 \leq \frac{64E^2}{c^4}$

- $\mathbb{E}_\theta[(\tilde{\theta} - \theta)^2] \geq \frac{c^4}{64E^2}$ --- Heisenberg Limit
- Optim. freq is given by $\omega_{\mathrm{opt}} = \theta$

$\longrightarrow$ As before, we need the M-sequence, which in this case is served by a different quantum state!

Contents

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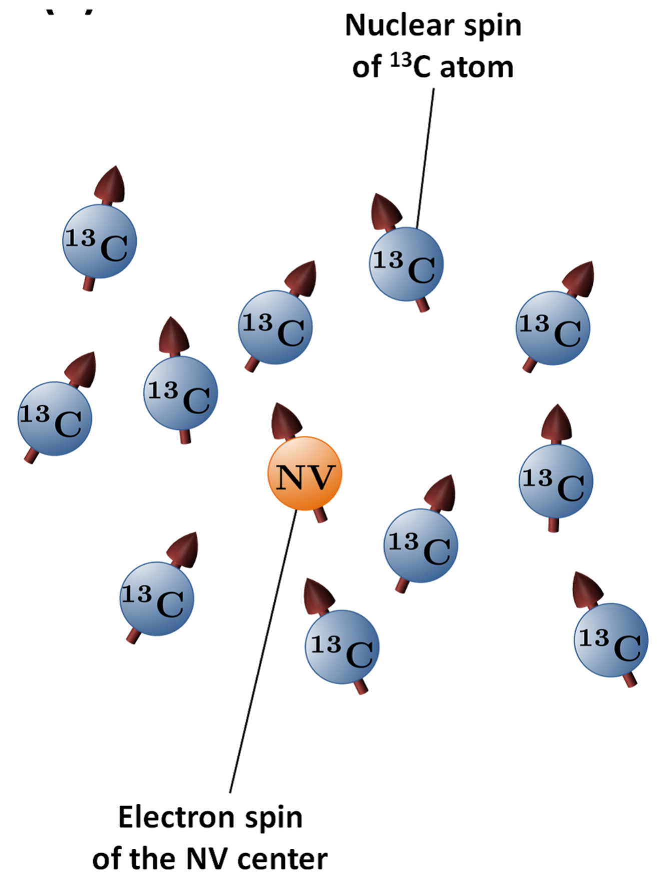
Structure identification of spin networks

Ref: Y. Kato and NY, New J. Phys. **16** 023024, 2014

Problem: Want to identify a limited-access spin network.

Nitrogen – Vacancy (NV) center in diamond is a hybrid quantum spin network composed of:

- Electron spin in NV
- Nuclear spins in carbon 13



Ref:

Coherent dynamics of coupled electron and nuclear spin qubits in diamond, L. Childress et al. Science 314, 281 (2006)

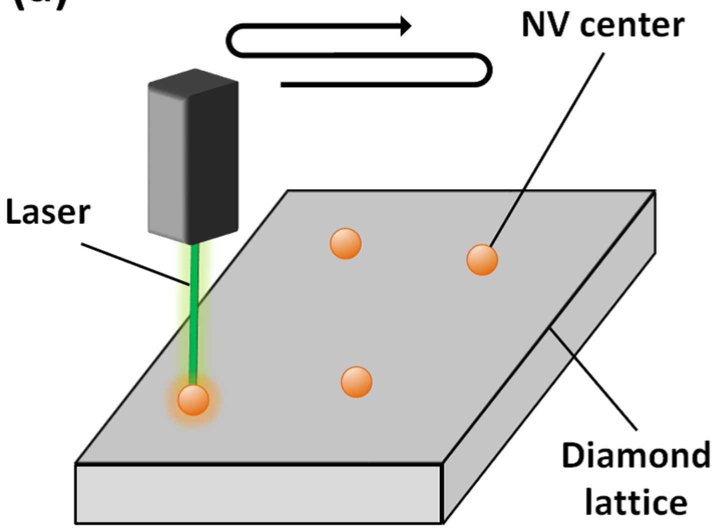
Issue:

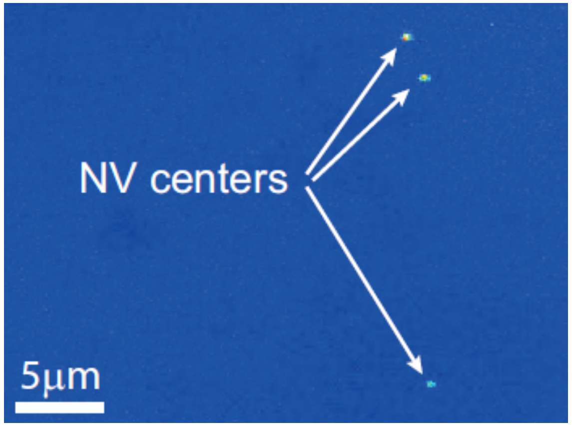
- Only the NV center is accessible.
- Carbon atoms are generated randomly.

The structure of the system composed of NV and C is unknown.

The location of NVs can be identified:

(a)





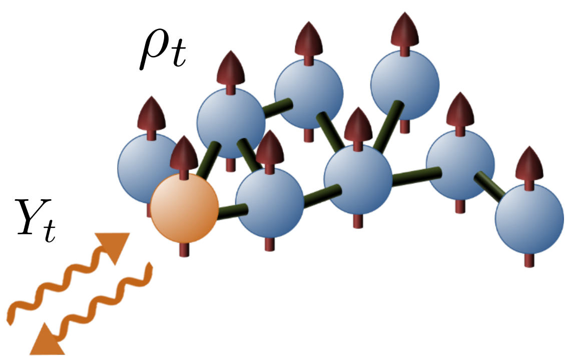
T. van der Sar, Ph.D Thesis, Delft Univ. of Technology, 2012

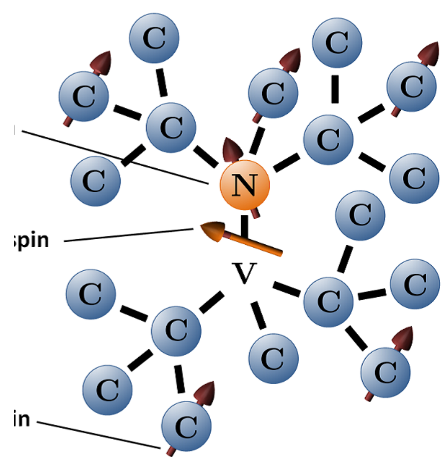
System setup:

1. We consider a spin network connected by the XY-type interaction Hamiltonian (with graph structure G).

$$H = \sum_{(j,k)\in E(G)} \lambda_{jk}(\sigma_j^x \otimes \sigma_k^x + \sigma_j^y \otimes \sigma_k^y)$$

2. We are allowed to measure only a single node.





System setup:

3. We perform a continuous measurement of σ^z of the accessible spin. Then our conditional state obeys:

$$\boxed{\begin{aligned} d\rho_t &= -i[H,\rho_t]dt + \gamma\mathcal{D}[c]\rho_t dt + \sqrt{\gamma}\mathcal{H}[c]\rho_t dW_t \\ dY_t &= \sqrt{\gamma}\,\mathrm{Tr}[(c+c^\dagger)\rho_t]dt + dW_t \end{aligned}}$$

Stochastic master equation

where

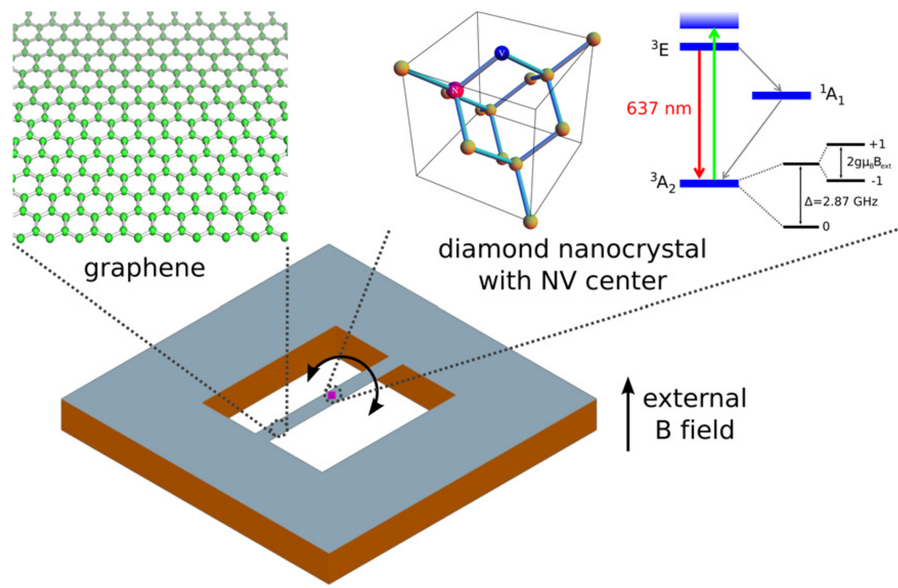
$$c = \sigma^z \otimes I^{\otimes(N-1)}$$

$$\mathcal{D}[c]\rho = c\rho c^\dagger - \frac{1}{2}(c^\dagger c\rho + \rho c^\dagger c)$$

$$\mathcal{H}[c]\rho = c\rho + \rho c^\dagger - [(c+c^\dagger)\rho]\rho$$

γ - Measurement strength

dW_t - Standard Winner increment

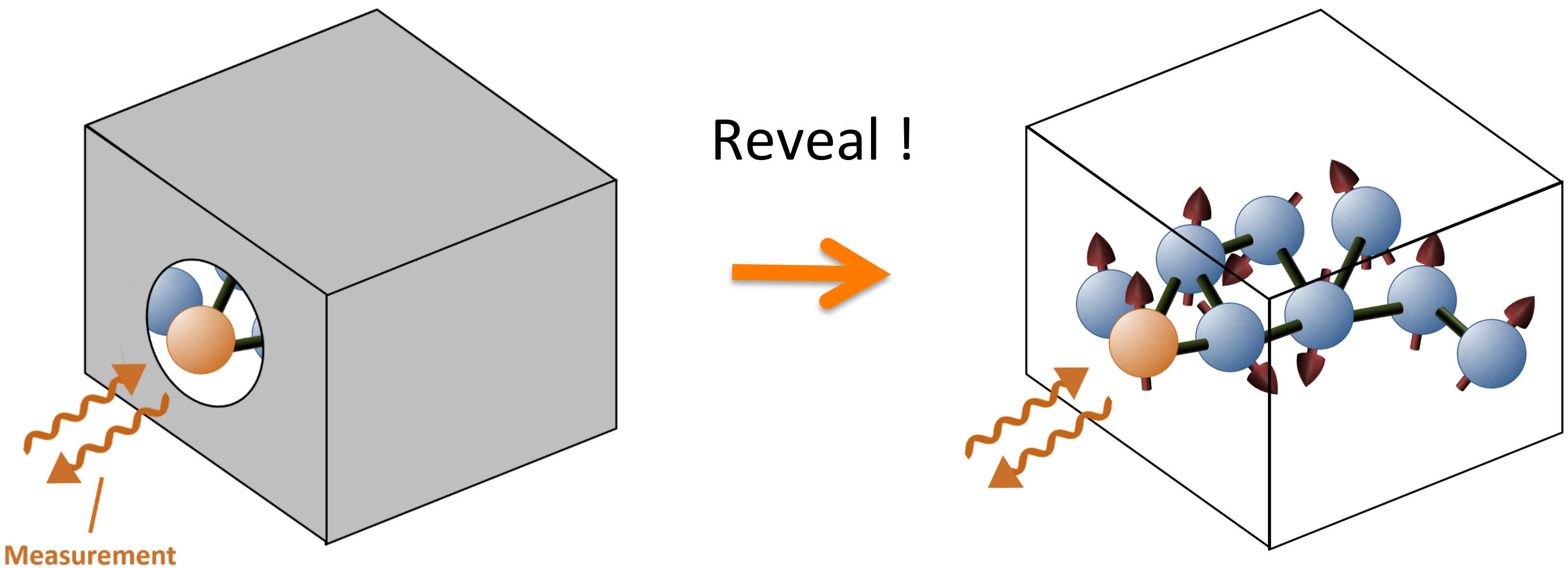


B D'Urso et al. New J. Physics 2011

Problem:

For a limited-access spin network, with its Hamiltonian unknown, how can we identify its structure ?

$$H = \sum_{(j,k)\in E(G)} \lambda_{jk}(\sigma_j^x \otimes \sigma_k^x + \sigma_j^y \otimes \sigma_k^y)$$



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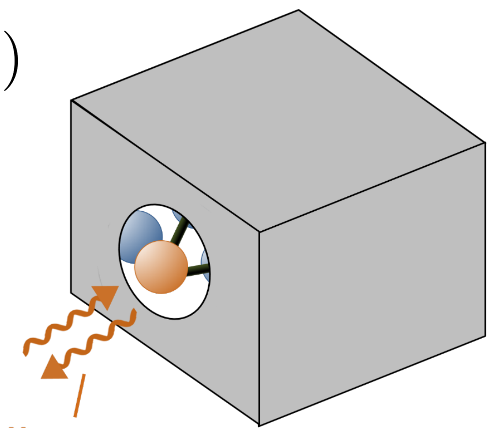


Graph structure ``G" is unknown, but

Assumption 1:

λ_{jk} are all distributed around the known value λ

Assumption 2:

The maximum number of the nodes is known.

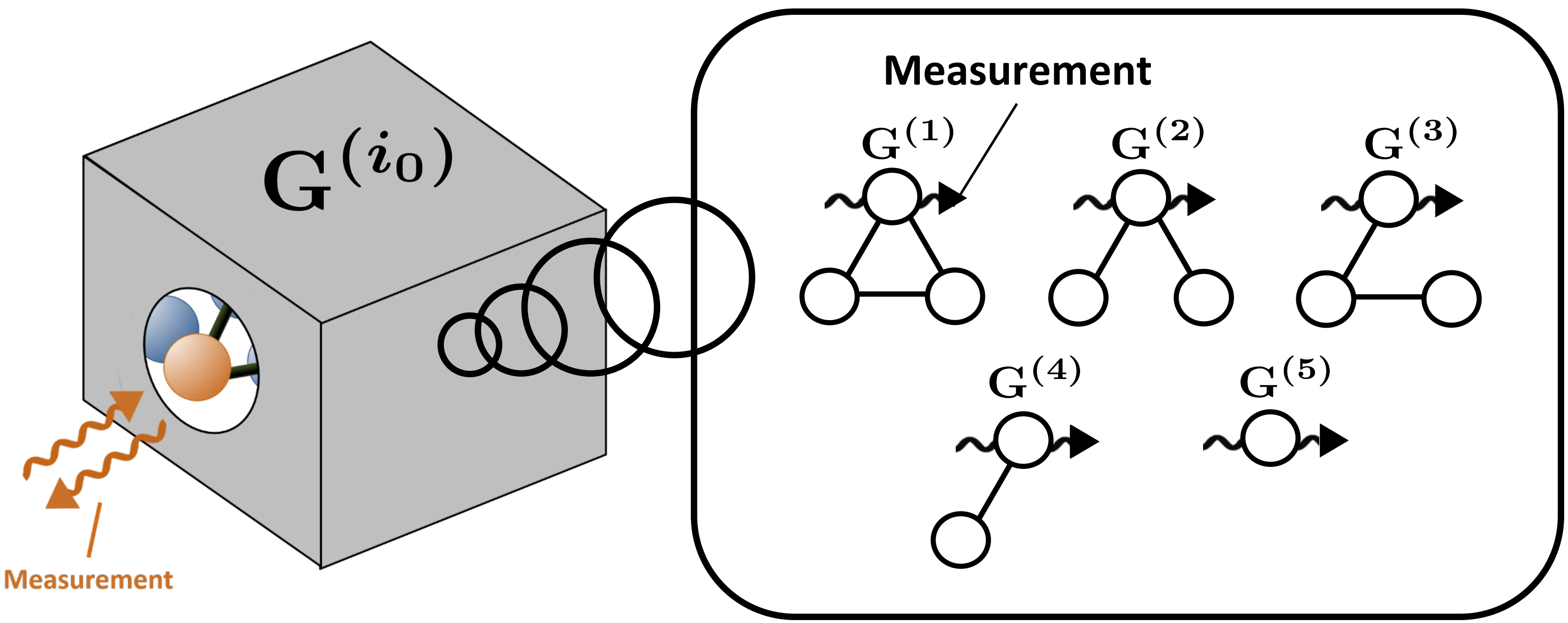
(Note) Once the structure is identified, then λ_{jk} can be estimated by the method developed in [Burgarth, NJP 2009]

Problem

Choose the true graph $\mathbf{G}^{(i_0)}$ from all possible nominal graphs $\mathbf{G}^{(1)}, \ldots, \mathbf{G}^{(m)}$

$G^{(i)}$ is driven by $H^{(i)} = \sum_{(j,k)\in E(G^{(i)})} \lambda \left(\sigma_j^x \otimes \sigma_k^x + \sigma_j^y \otimes \sigma_k^y\right)$

Example: Maximum number of the nodes = 3.



Idea: Update $p_t^{(i)} = \mathbb{P}(\{G = G^{(i)}\} \,|\, \mathcal{Y}_t)$ as well as $\rho_t^{(i)}$

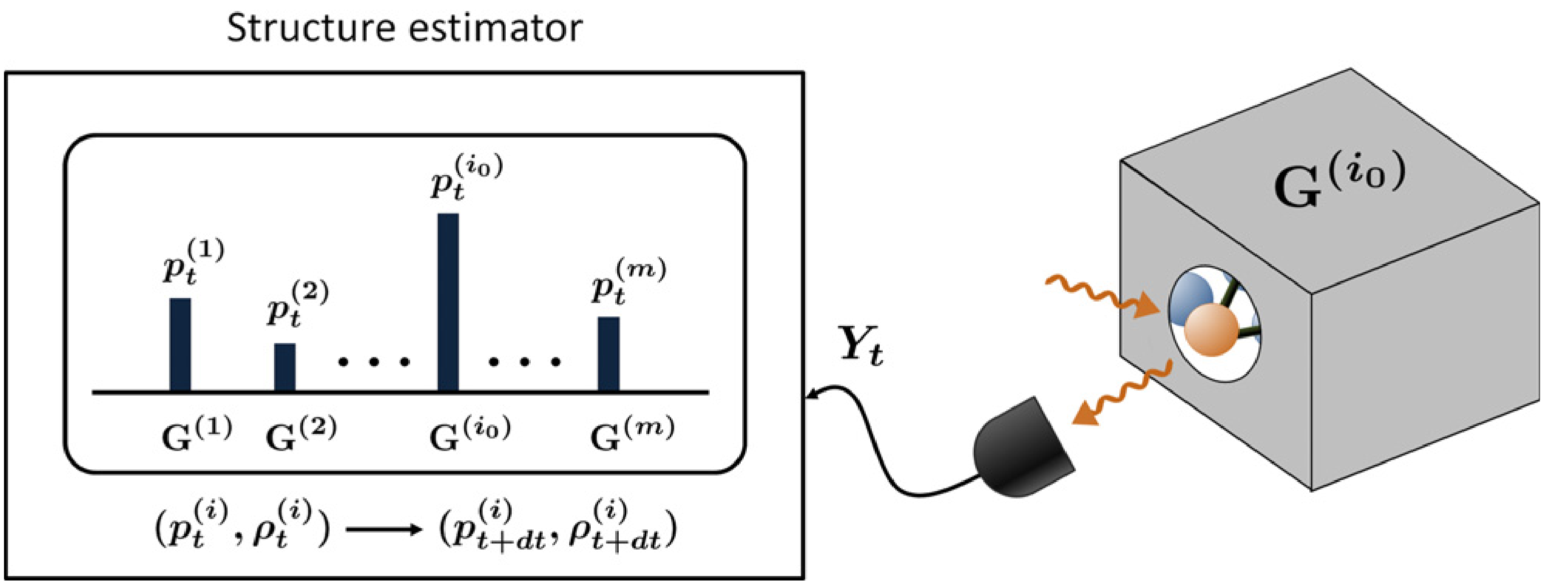
$$
\begin{aligned}
\mathrm{d}\rho_t^{(i)} = -\mathrm{i}\left[H^{(i)}, \rho_t^{(i)}\right] \mathrm{d}t + \gamma \mathcal{D}[c]\rho_t^{(i)}\,\mathrm{d}t \\
+ \sqrt{\gamma}\mathcal{H}[c]\rho_t^{(i)} \left(\mathrm{d}Y_t - 2\sqrt{\gamma}\ \mathrm{Tr}\left(c\rho_t^{(i)}\right)\ \mathrm{d}t\right)
\end{aligned}
$$

$$
\begin{aligned}
p_{t+\mathrm{d}t}^{(i)} &= \frac{\mathbb{P}(\mathcal{Y}_{t+\mathrm{d}t} \,|\, \{G = G^{(i)}\})\, p_t^{(i)}}{\sum_i \mathbb{P}(\mathcal{Y}_{t+\mathrm{d}t} \,|\, \{G = G^{(i)}\})\, p_t^{(i)}} \\
&= \frac{1}{\mathcal{N}} \exp\!\left[-\frac{1}{2\mathrm{d}t}\left(\mathrm{d}Y_t - 2\sqrt{\gamma}\ \mathrm{Tr}(c\rho_t^{(i)})\,\mathrm{d}t\right)^2\right] p_t^{(i)} \\
&= \frac{\left(1 + 2\sqrt{\gamma}\ \mathrm{Tr}(c\rho_t^{(i)})\,\mathrm{d}Y_t\right) p_t^{(i)}}{1 + 2\sqrt{\gamma}\sum_i\ \mathrm{Tr}(c\rho_t^{(i)})\, p_t^{(i)}\mathrm{d}Y_t} = \frac{\left(1 + 2\sqrt{\gamma}\ \mathrm{Tr}(c\rho_t^{(i)})\,\mathrm{d}Y_t\right) p_t^{(i)}}{1 + 2\sqrt{\gamma}\ \mathrm{Tr}(c\tilde{\rho}_t)\,\mathrm{d}Y_t} \\
&= p_t^{(i)} + 2\sqrt{\gamma}\{\mathrm{Tr}(c\rho_t^{(i)}) - \mathrm{Tr}(c\tilde{\rho}_t)\}(\mathrm{d}Y_t - 2\sqrt{\gamma}\ \mathrm{Tr}(c\tilde{\rho}_t)\,\mathrm{d}t)\, p_t^{(i)}
\end{aligned}
$$

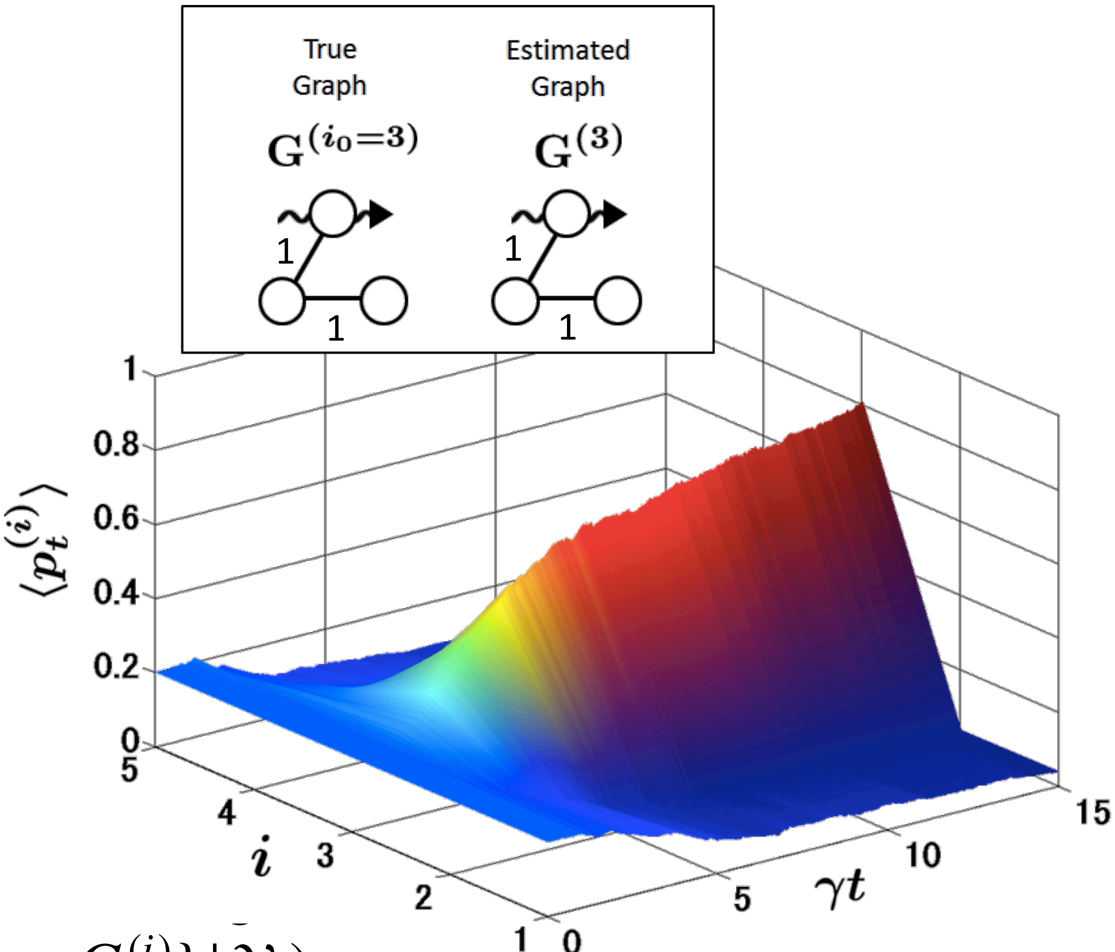
Idea: Update $p_t^{(i)} = \mathbb{P}(\{G = G^{(i)}\} \mid \mathcal{Y}_t)$ as well as $\rho_t^{(i)}$

$$\begin{aligned} d\rho_t^{(i)} &= -\mathrm{i}\left[H^{(i)}, \rho_t^{(i)}\right] dt + \gamma \mathcal{D}[c]\rho_t^{(i)} dt \\ &\quad + \sqrt{\gamma}\mathcal{H}[c]\rho_t^{(i)} \left(dY_t - 2\sqrt{\gamma}\ \mathrm{Tr}\left(c\rho_t^{(i)}\right) dt\right) \\ dp_t^{(i)} &= 2\sqrt{\gamma}\left\{\mathrm{Tr}\left(c\rho_t^{(i)}\right) - \mathrm{Tr}(c\tilde{\rho}_t)\right\} p_t^{(i)}(dY_t - 2\sqrt{\gamma}\ \mathrm{Tr}(c\tilde{\rho}_t)\, dt) \end{aligned}$$

$$\tilde{\rho}_t := \sum_{i=1}^m p_t^{(i)} \rho_t^{(i)}$$

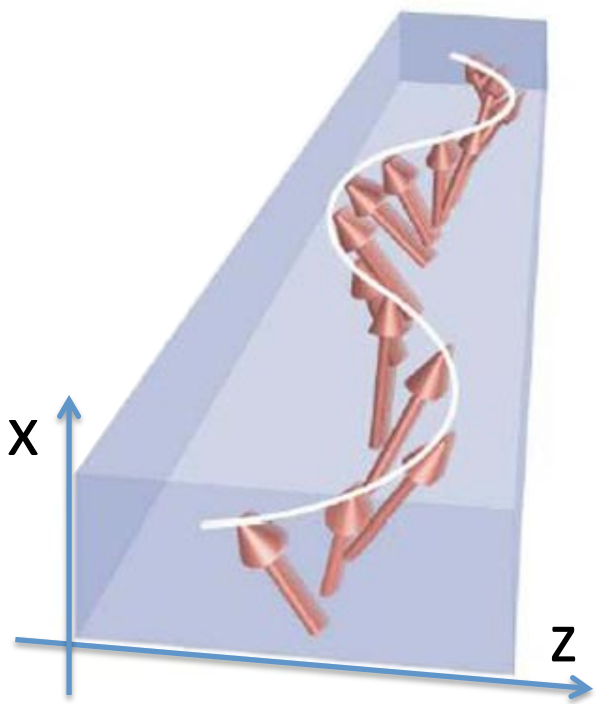


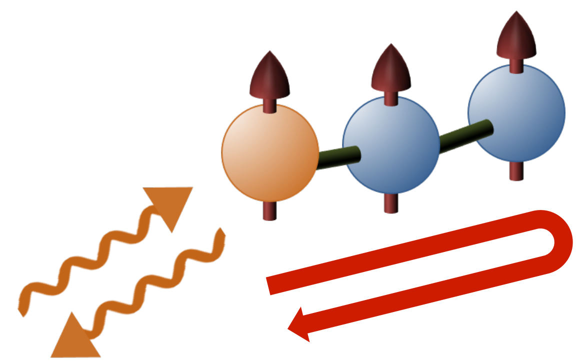
Example: Max number of the nodes = 3.



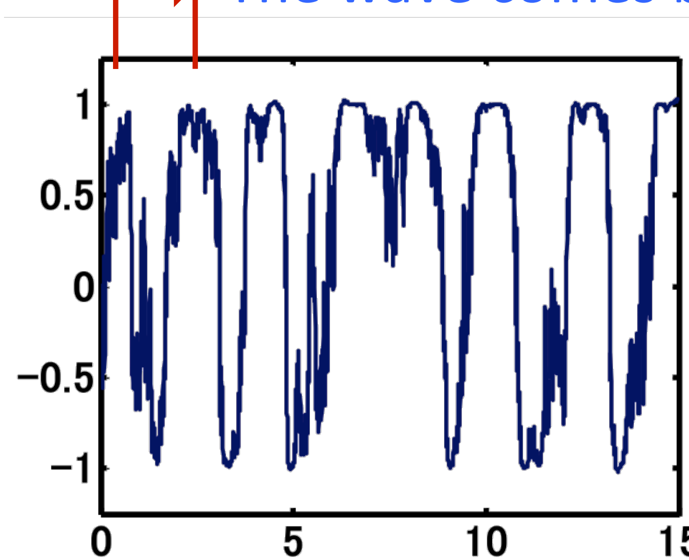
$$p_t^{(i)} = \mathbb{P}(\{G = G^{(i)}\} \mid \mathcal{Y}_t)$$

Physical interpretation: We are observing an edge of the spin wave (with measurement back-action).





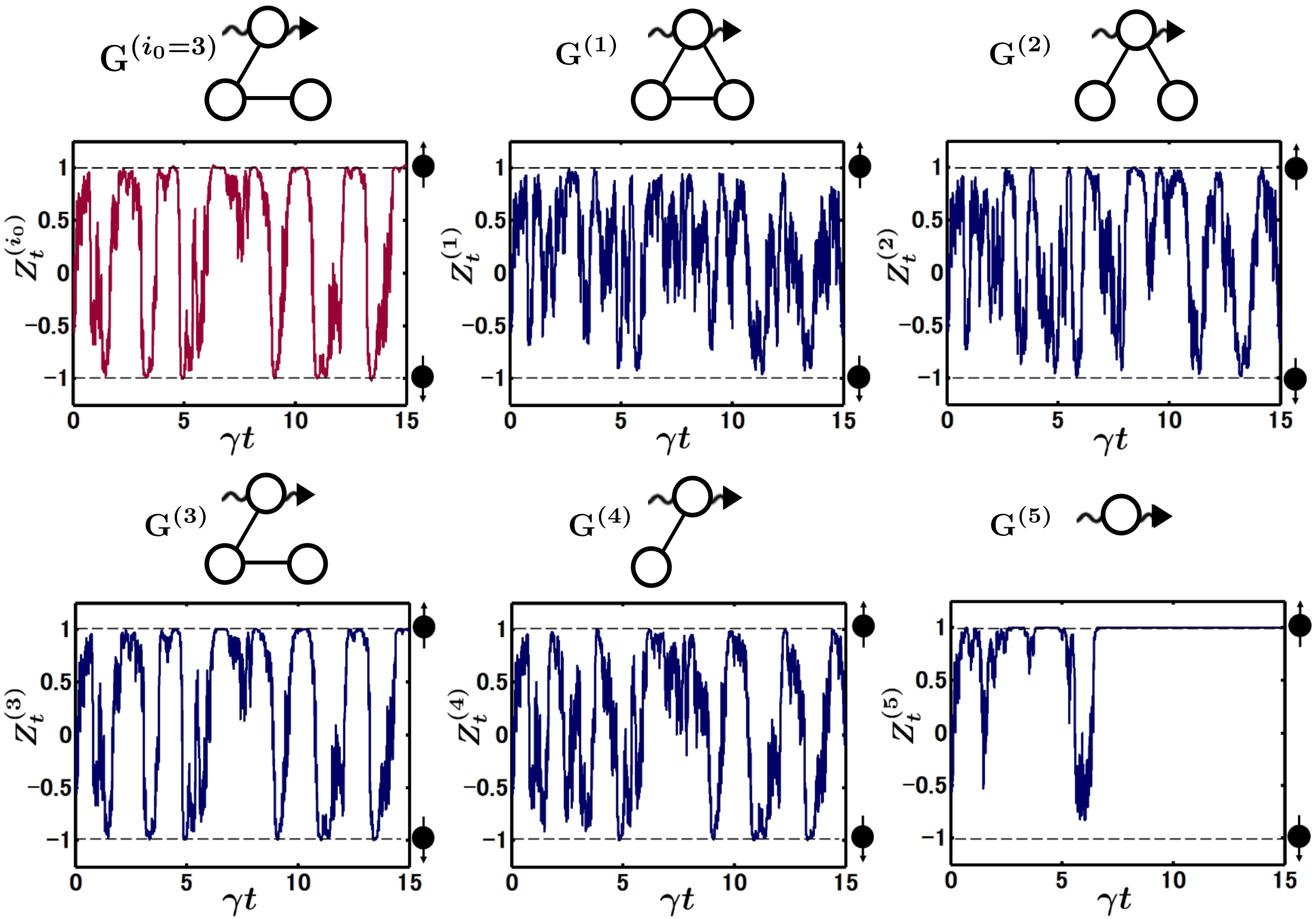
The wave comes back !



$$Z_t^{(i_0)} = \mathrm{Tr}(c\rho_t^{(i_0)})$$

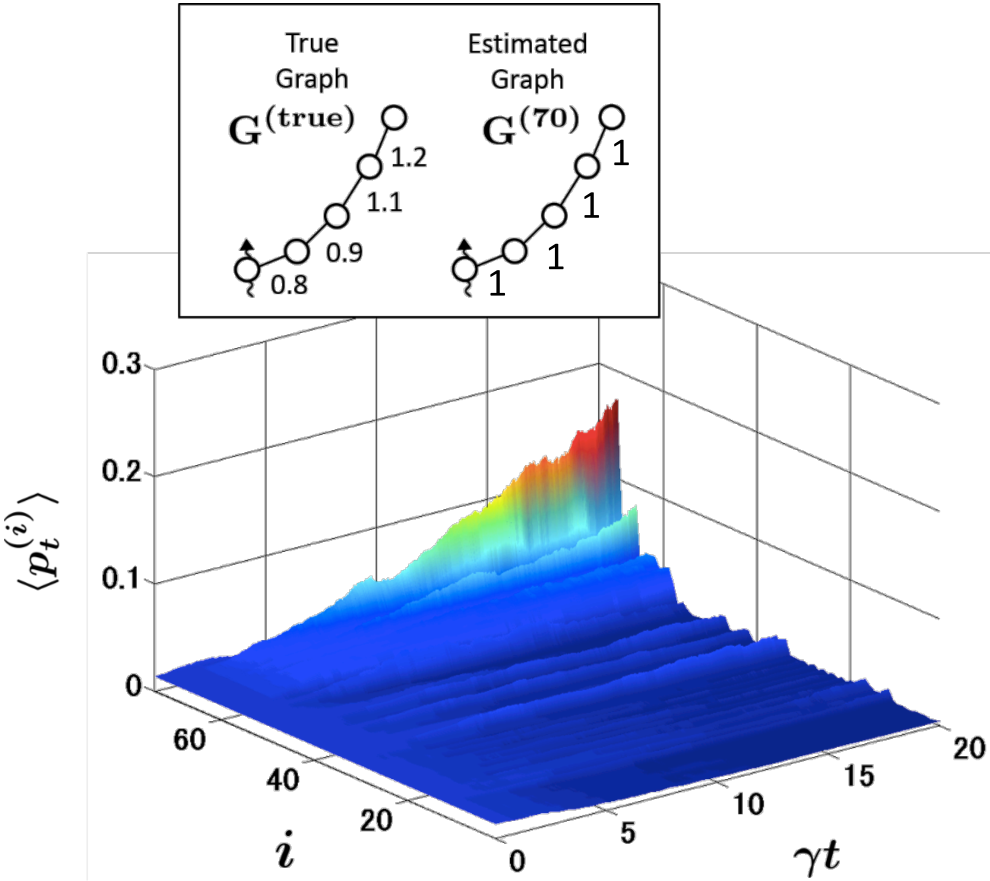
$$c = \sigma^z \otimes I^{\otimes(N-1)}$$

$$Z_t^{(i)} = \mathrm{Tr}(c\rho_t^{(i)})$$



Example: Max number of the nodes = 5.

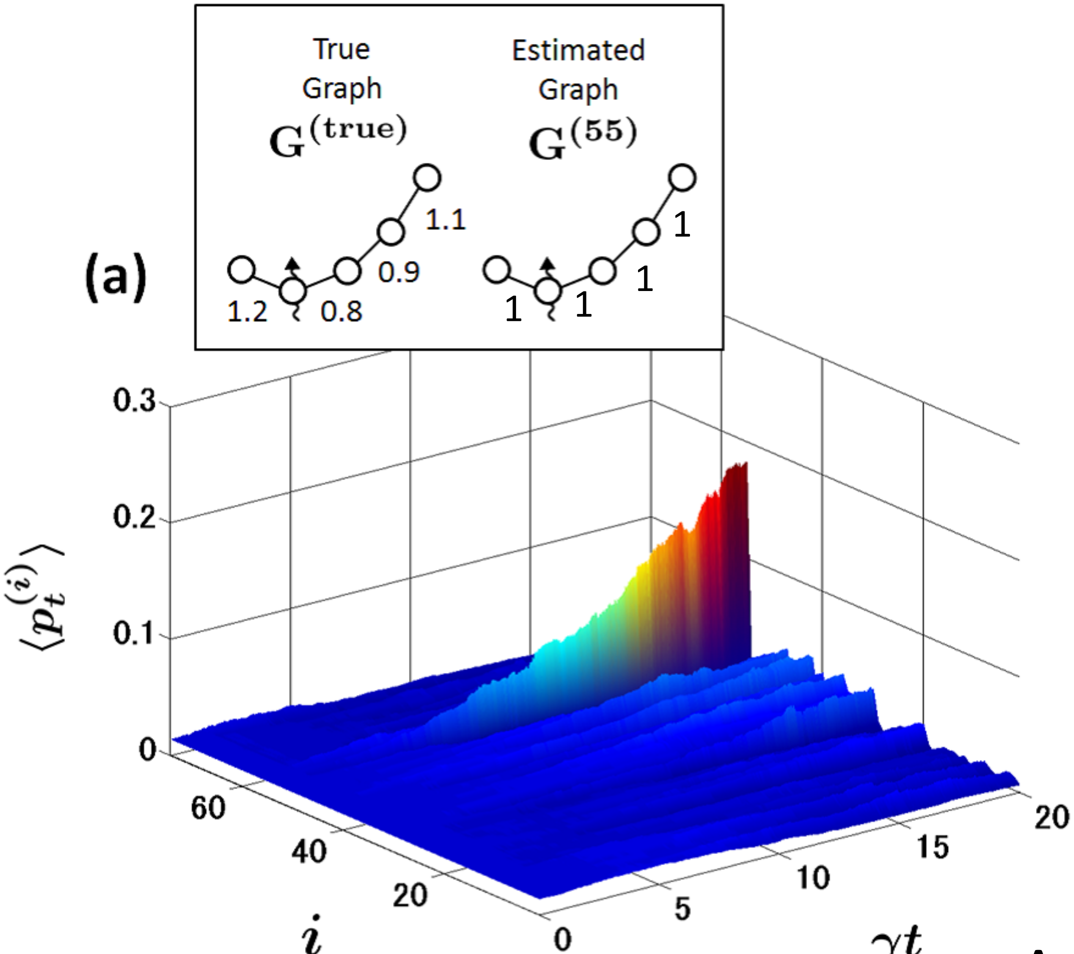
True coupling strength λ_{jk} is distributed around $\lambda = 1$



$$p_t^{(i)} = \mathbb{P}(\{G = G^{(i)}\} \,|\, \mathcal{Y}_t)$$

Averaged over **10** sample paths

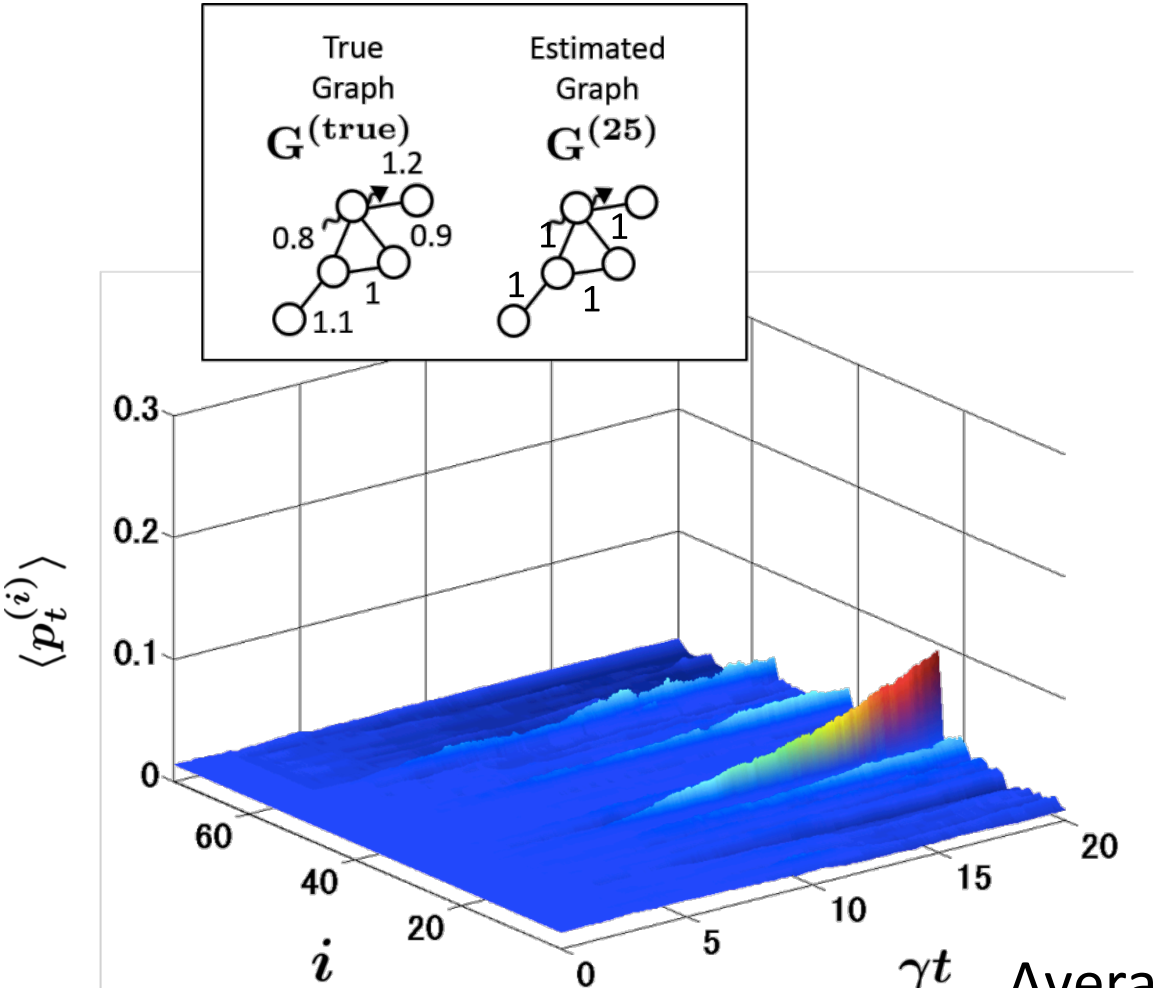
Example: Max number of the nodes = 5. λ_{jk} is around $\lambda = 1$



$$p_t^{(i)} = \mathbb{P}(\{G = G^{(i)}\} \mid \mathcal{Y}_t)$$

Averaged over **50** sample paths

Example: Max number of the nodes = 5. λ_{jk} is around $\lambda = 1$



Averaged over
50 sample paths

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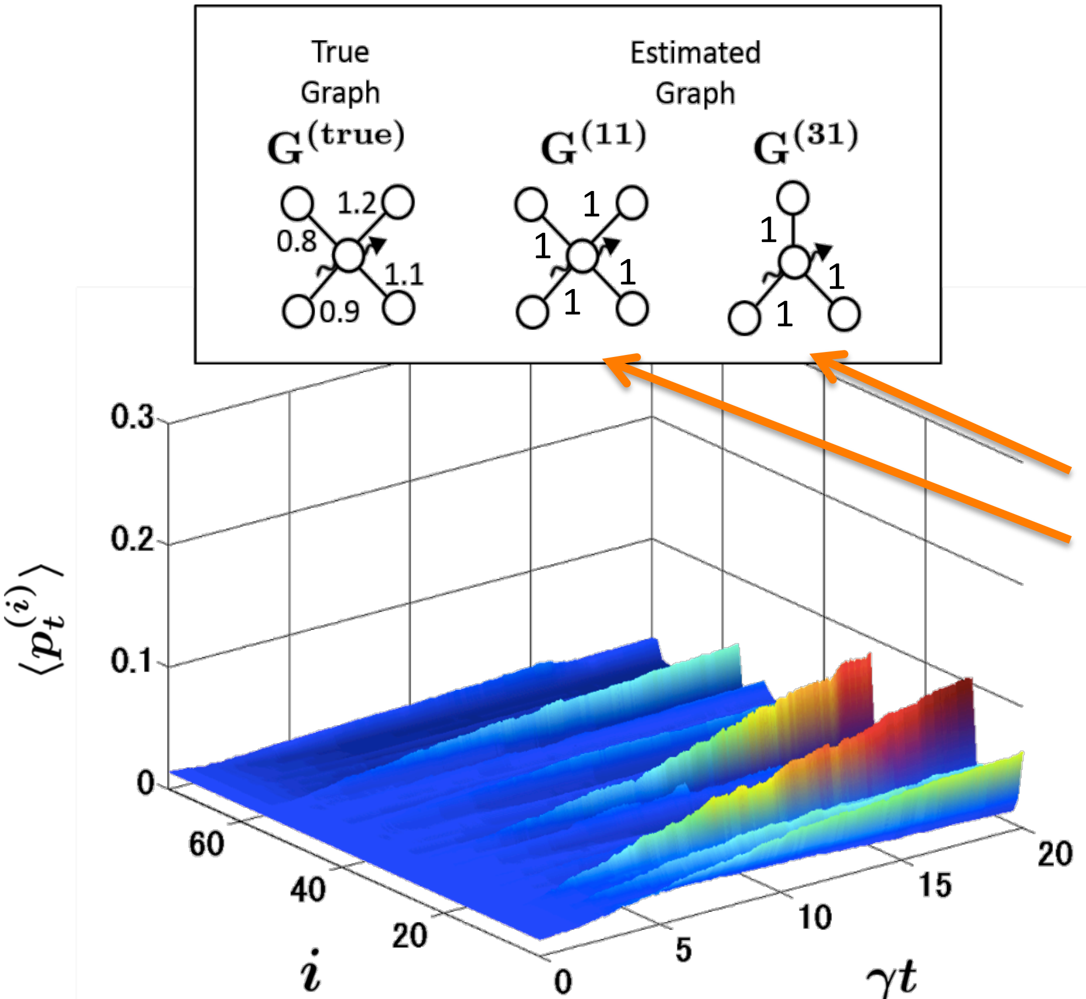
Example: Max number of the nodes = 5. λ_{jk} is around $\lambda = 1$

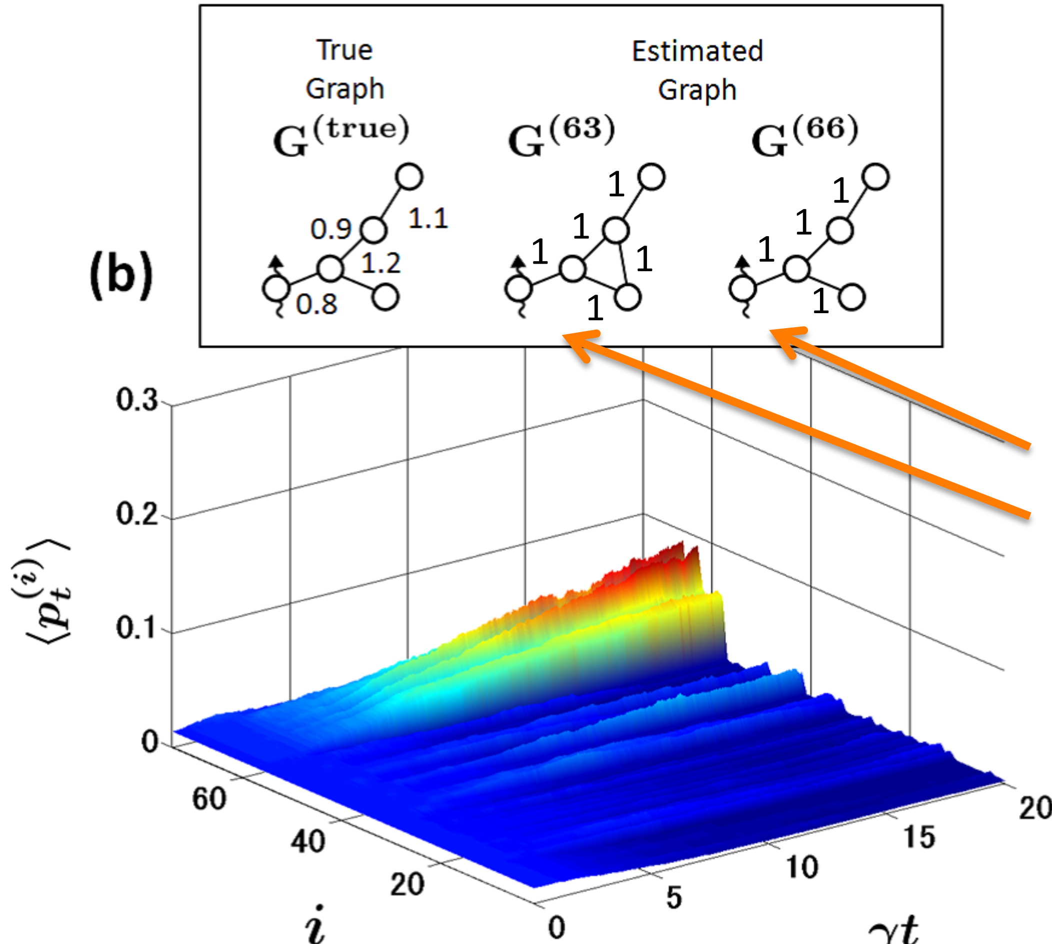


Maybe they are indistinguishable.

$p_t^{(i)} = \mathbb{P}(\{G = G^{(i)}\} \mid \mathcal{Y}_t)$

Averaged over **50** sample paths

Example: Max number of the nodes = 5. λ_{jk} is around $\lambda = 1$



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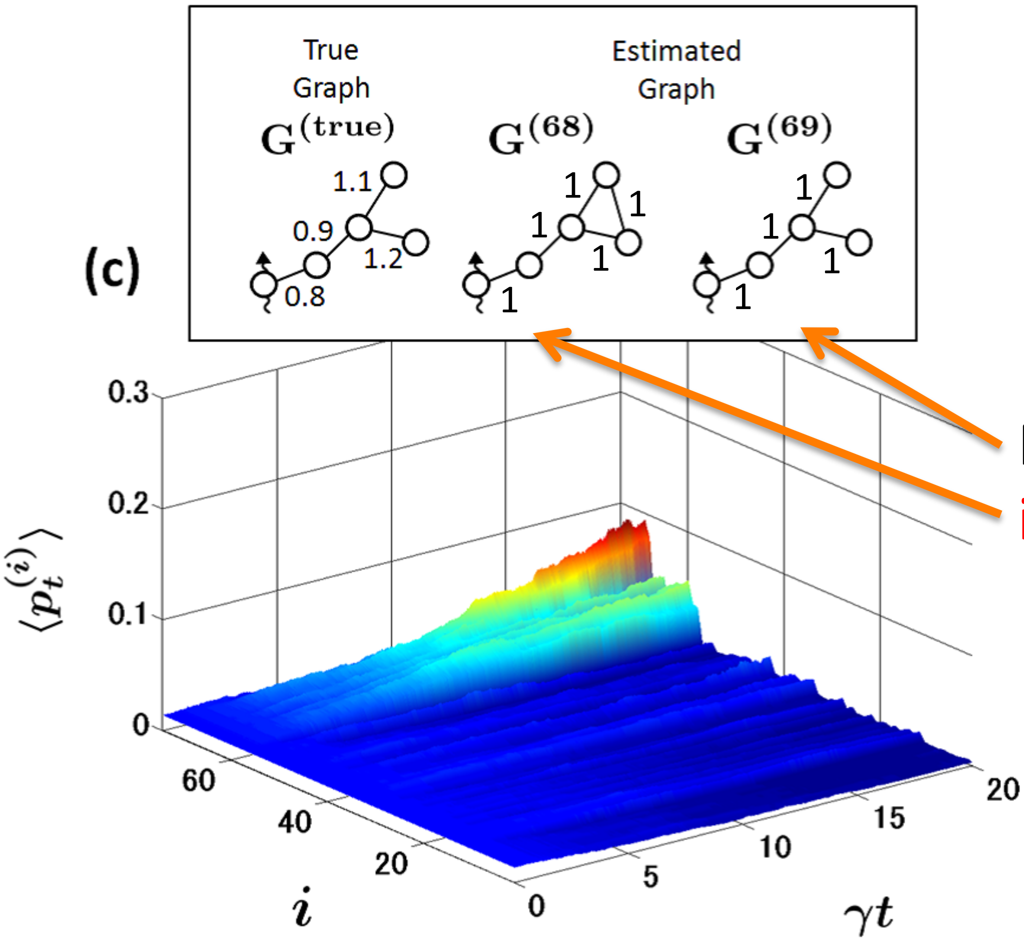
$$p_t^{(i)} = \mathbb{P}(\{G = G^{(i)}\} \mid \mathcal{Y}_t)$$

Averaged over **50** sample paths

<u>Example</u>: Max number of the nodes = 5. λ_{jk} is around $\lambda = 1$



Maybe they are indistinguishable.

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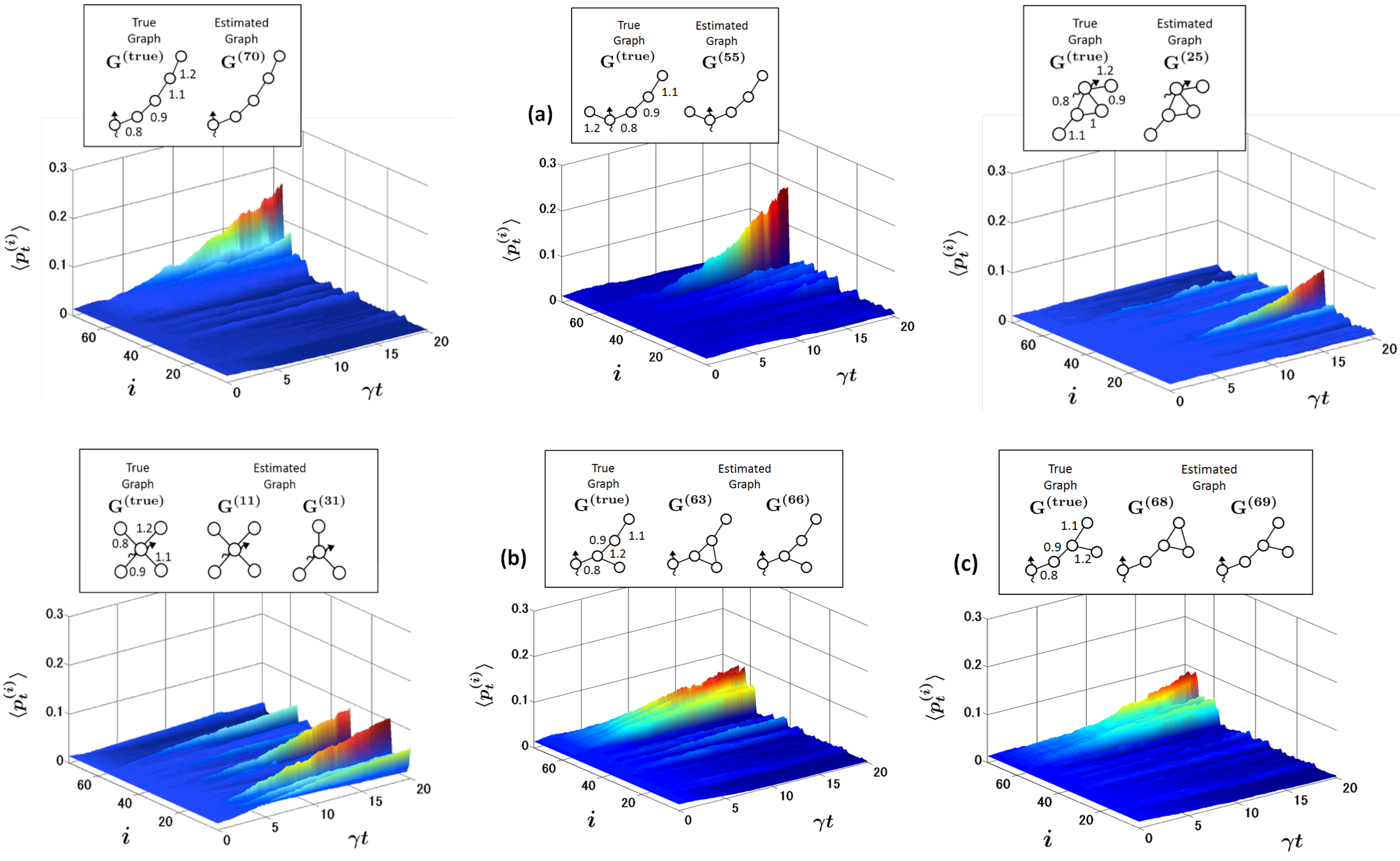
Averaged over **50** sample paths



If the system is identifiable, then the convergence is unique?

Summary

1. Identification for linear systems --- analytic approach

Identifiability --- What can we identify?

ID method --- How to identify?

Statistics --- How much can we identify?

There are many classical results applicable to quantum case, but some interesting new aspects arise in quantum metrology.

2. Identification for nonlinear systems --- numerical approach

Structure identification of spin networks

Lack of theory. Structure identification in classical case?