Well-posedness and singularities of the water wave equations

Sijue Wu

Department of Mathematics

University of Michigan, Ann Arbor, USA

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2-D water waves, local well-posedness

- the formula for $-\frac{\partial P}{\partial \mathbf{n}}$
- the quasilinear equation in Riemann mapping coordinates
- the splash and splat singularity of Cordoba, Fefferman, et al.
- some brief remarks on water waves with the angled crests
- the quasilinear equation in coordinate invariant form

2-D water waves, global well-posedness

- A canonical equation: a natural normal form transformation and the coordinate change
- an almost global wellposedness result for small and smooth data

3-D water waves: local for arbitrary data, global for small data

Clifford analysis

References

2-D

- S. Wu *Well-posedness in Sobolev spaces of the full water wave problem in 2-D* Invent. Math. 130, 1997, pp. 39-72
- S. Wu *Almost global wellposedness of the 2-D full water wave problem* Invent. Math., 177, (2009) no.1 pp. 45-135.
- R. Kinsey & S. Wu *A priori estimates for two-dimensional water waves with angled crests* arXiv:1406.7573 (**Periodic setting**)

3-D

- S. Wu *Well-posedness in Sobolev spaces of the full water wave problem in 3-D* J. of the AMS. 12. no.2 (1999), pp. 445-495.
- S. Wu *Global wellposedness of the 3-D full water wave problem* Invent. Math. 184 (2011), no. 1, 125-220

Other formulations (for local wellposedness, arbitrary smooth data, with additional effects):

Christodoulou & Lindblad (2000), Lindblad (2005), Coutand & Shkoller (2007): Euler equation
D. Lannes (2005), Alazard, Burq & Zuily (2012) etc.:
Zakharov-Craig-Sulem equation
Ambrose & Masmoudi (2005):
Shatah & Zeng (2007): some geometric formulation

Other methods (for global wellposedness, small and smooth data):

Germain, Masmoudi & Shatah (2012), Ionescu & Pusateri (2013):
space-time resonance method
Alazard & Delort (2013): method of normal form
Hunt, Ifrim & Tataru (2014): modified energy method

The water wave equations

We assume that

- the air density is 0, the fluid density is 1. $(0, -g)$ is the gravity.
- the fluid is inviscid, incompressible, irrotational, the surface tension is zero. Let $\Omega(t)$ be the fluid domain, $\Sigma(t)$ be the interface at time t.

The motion of the fluid is described by

$$\begin{cases} \mathbf{v}_t + (\mathbf{v}\cdot\nabla)\mathbf{v} = (\mathbf{0}, -g) - \nabla P & \text{in } \Omega(t) \\ \operatorname{div}\mathbf{v} = 0, \qquad \operatorname{curl}\mathbf{v} = 0, & \text{in } \Omega(t) \\ P = 0, & \text{on } \Sigma(t) \end{cases} \tag{1}$$

$\mathbf{v}$ is the fluid velocity, P is the fluid pressure.

We assume that $\mathbf{v}(z,t) \to 0$ as $|z| \to \infty$, the surface $\Sigma(t)$ tends to the real line in the 2-D case or the xy-plane at infinity in the 3-D case.

Question:

For given position, velocity, is the water wave equation solvable for some finite time $[0, T]$, $T > 0$?

Remark 1: the problem with velocity $\mathbf{v}(z,t) \to c$ at ∞ can be reduced to one that tends to zero at infinity:
If $(\mathbf{v}, P)$, with $\Sigma(t) : z = z(\alpha, t)$ is a solution of (1), then
$\mathbf{V} = \mathbf{v}(z + (c, 0)t, t) - c$, $\mathbf{P} = P(z + (c, 0)t, t)$, with $\zeta = z(\alpha, t) - ct$ is also a solution of (1).
Remark 2: the following calculation extends to the periodic case, see

- R. Kinsey & S. Wu *A priori estimates for two-dimensional water waves with angled crests* arXiv:1406.7573

Basic facts on Cauchy integral:

- Fact 1: Let $\Omega \subset \mathbb{C}$ (unbounded, chord-arc), $f \in L^2(\partial\Omega)$. Then

$$\mathbb{P}_H f := \frac{1}{2}(I + \mathfrak{H})f$$

where

$$\mathfrak{H}g(\alpha) = \frac{1}{\pi i} \int \frac{z_\beta(\beta, t)}{z(\alpha, t) - z(\beta, t)} g(\beta)\, d\beta,$$

is the boundary value of a holomorphic function F on Ω, with $\lim_{z\to\infty} F(z) = 0$.

- Fact 2: Let $g \in L^2(\partial\Omega)$. g is the boundary value of a holomorphic function G on Ω if and only if

$$g = \mathfrak{H}g$$

- Fact 3: $\mathfrak{H}1 = 0$.

Basic facts of holomorphic functions

- If $f : \Omega_1 \to \Omega_2$, $g : \Omega_2 \to \mathbb{C}$ are holomorphic functions, then $g \circ f : \Omega_1 \to \mathbb{C}$ is holomorphic;
- If $f,\ g : \Omega \to \mathbb{C}$ are holomorphic, then fg is also holomorphic on Ω.

surface equation in Lagrangian coordinates

- Notation: we identify $(x, y) = x + iy$; let the gravity $(0, -g) = (0, -1) = -i$.
- Let $\Sigma(t) : z = z(\alpha, t)$, $\alpha \in \mathbb{R}$; α is the Lagrangian coordinate.

Equation of the free surface:

$$\begin{cases} z_{tt} + i = i\mathfrak{a} z_\alpha \\ \bar{z}_t = \mathfrak{H} \bar{z}_t \end{cases} \tag{2}$$

where $\mathfrak{H}$ is the Hilbert transform,

$$\mathfrak{a} = -\frac{\partial P}{\partial \mathbf{n}} \frac{1}{|z_\alpha|}.$$

Taking derivative to the first equation in (2), we get

$$\begin{cases}\bar{z}_{ttt} + i\mathfrak{a}\bar{z}_{t\alpha} = -i\mathfrak{a}_t\bar{z}_\alpha \\ \bar{z}_t = \mathfrak{H}\bar{z}_t\end{cases} \tag{3}$$

want to

- derive a formula for $-\frac{\partial P}{\partial \mathbf{n}}$ and $\mathfrak{a}$;
- show (3) is quasilinear

We will see two approaches:

- use the Riemann mapping;
- do not use Riemann mapping, derive in Lagrangian framework, this allows us to extend the derivation to 3-D.

surface equation in Riemann mapping framework

Let

$$\Phi = \Phi(z;t) : \Omega(t) \to P_-$$

be the Riemann mapping satisfying $\lim_{z\to\infty} \Phi_z(z;t) = 1$; here P_- is the lower half plane.

Define

- $h(\alpha,t) = \Phi(z(\alpha,t);t)$
- $h(h^{-1}(\alpha',t);t) = \alpha'; \quad f \circ g := f(g(\cdot,t);t) := U_g f$
- $Z(\alpha';t) := z(h^{-1}(\alpha',t);t) = \Phi^{-1}(\alpha';t)$, $\partial_{\alpha'} Z = Z_{,\alpha'}$
- $Z_t = z_t(h^{-1}(\alpha',t);t); \quad Z_{tt} = z_{tt}(h^{-1}(\alpha',t);t)$;
- $Z_{tt,\alpha'} = \partial_{\alpha'}\{z_{tt}(h^{-1}(\alpha',t);t)\}$;

we have

$$\lim_{\alpha'\to\pm\infty} Z_{,\alpha'} = \lim_{\alpha'\to\infty} (\Phi^{-1})_z(\alpha';t) = 1$$

surface equation in Riemann mapping framework

Equation of the free surface:

$$\begin{cases} Z_{tt} + i = iAZ_{,\alpha'} \\ \bar{Z}_t = \mathbb{H}\bar{Z}_t \end{cases} \tag{4}$$

where $\mathbb{H}$ is the Hilbert transform for P_-:

$$\mathbb{H}f(\alpha') = \frac{1}{\pi i}\int \frac{1}{\alpha' - \beta'} f(\beta')\, d\beta'$$

$$A \circ h = \mathfrak{a} h_\alpha.$$

the "quasilinear equation" (3) becomes

$$\begin{cases} \bar{Z}_{ttt} + iA\bar{Z}_{t,\alpha'} = \frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1}(\bar{Z}_{tt} - i) \\ \bar{Z}_t = \mathbb{H}\bar{Z}_t \end{cases} \tag{5}$$

- unknown functions: $u := \bar{Z}_t$, $w := \bar{Z}_{tt}$
- we know

$$U_{h^{-1}}\partial_t(f(h(\cdot,t);t)) = f_t + h_t \circ h^{-1} f_{\alpha'}$$

so

$$\bar{Z}_{tt} := U_{h^{-1}}\partial_t \bar{z}_t = (\partial_t + b\partial_{\alpha'})\bar{Z}_t; \quad \bar{Z}_{ttt} = (\partial_t + b\partial_{\alpha'})^2 \bar{Z}_t$$

Equation (5) becomes:

$$(\partial_t + b\partial_{\alpha'})^2 \bar{Z}_t + iA\partial_{\alpha'}\bar{Z}_t = \frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1}(\bar{Z}_{tt} - i) \tag{6}$$

need to find

$$A; \quad \frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1}; \quad b := h_t \circ h^{-1}$$

$$A = U_{h^{-1}}(\mathfrak{a} h_\alpha) = -\frac{1}{|Z_{,\alpha'}|}\frac{\partial P}{\partial \mathbf{n}}$$

Proposition

Let $A_1 := A|Z_{,\alpha'}|^2 := iZ_{,\alpha'}(\bar{Z}_{tt} - i)$. *Then*

$$A_1 = 1 - \Im[Z_t, \mathbb{H}]\bar{Z}_{t,\alpha'} = 1 + \frac{1}{2\pi}\int \frac{|Z_t(\alpha';t) - Z_t(\beta';t)|^2}{(\alpha'-\beta')^2}\, d\beta'$$

consequently

- $$A_1 \geq 1;$$
- $$-\frac{\partial P}{\partial \mathbf{n}} = \frac{A_1}{|Z_{,\alpha'}|} \geq 0; \quad A \geq c_0 > 0, \text{ if } \Sigma(t) \text{ is } C^{1,\gamma};$$
- $$A = \frac{|\bar{Z}_{tt} - i|^2}{A_1}; \quad \frac{1}{Z_{,\alpha'}} = i\frac{\bar{Z}_{tt} - i}{A_1}$$

Let $D_{\alpha'} f = \frac{1}{Z_{,\alpha'}} \partial_{\alpha'} f$; $D_\alpha g = \frac{1}{z_\alpha} \partial_\alpha g$; $\Im(x + iy) = y$.

Proposition

$$\frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1} = \frac{-\Im(2[Z_t, \mathbb{H}]\bar{Z}_{tt,\alpha'} + 2[Z_{tt}, \mathbb{H}]\bar{Z}_{t,\alpha'} - [Z_t, Z_t; D_{\alpha'}\bar{Z}_t])}{A_1}. \qquad (7)$$

where

$$[Z_t, Z_t; D_{\alpha'}\bar{Z}_t] = \frac{1}{\pi i} \int \left(\frac{Z_t(\alpha'; t) - Z_t(\beta'; t)}{\alpha' - \beta'}\right)^2 D_{\beta'}\bar{Z}_t(\beta'; t)\, d\beta'$$

Proof: Let $A_t \circ h = \mathfrak{a}_t h_\alpha$. want to derive the RHS of

$$Z_{,\alpha'}(\bar{Z}_{ttt} + iA\bar{Z}_{t,\alpha'}) = -iA_t|Z_{,\alpha'}|^2.$$

let $\bar{z}_t(\alpha, t) = F(z(\alpha; t); t)$.

- $\bar{z}_{tt} = (F_z \circ z)z_t + F_t \circ z$
- $\bar{z}_{ttt} = (F_{zz} \circ z)z_t^2 + 2(F_{tz} \circ z)z_t + (F_z \circ z)z_{tt} + F_{tt} \circ z.$
- $F_z \circ z = D_\alpha \bar{z}_t; \quad F_{zz} \circ z = D_\alpha^2 \bar{z}_t$; (b.v. of holo.)
- $F_{tz} \circ z = D_\alpha(\bar{z}_{tt} - (D_\alpha \bar{z}_t)z_t)$. (b.v. of holo.)
- $\bar{z}_{ttt} = (D_\alpha^2 \bar{z}_t)z_t^2 + 2z_t D_\alpha(\bar{z}_{tt} - (D_\alpha \bar{z}_t)z_t) + (D_\alpha \bar{z}_t)z_{tt} + F_{tt} \circ z.$
- Precomposing with h^{-1}, we have $\bar{Z}_{ttt} = (D_{\alpha'}^2 \bar{Z}_t)Z_t^2 + 2Z_t D_{\alpha'}(\bar{Z}_{tt} - (D_{\alpha'} \bar{Z}_t)Z_t) + (D_{\alpha'} \bar{Z}_t)Z_{tt} + F_{tt} \circ Z.$
- $iA\bar{Z}_{t,\alpha'} = (Z_{tt} + i)D_{\alpha'} \bar{Z}_t$

$$\begin{aligned} &Z_{,\alpha'}(\bar{Z}_{ttt} + iA\bar{Z}_{t,\alpha'}) \\ &= (\partial_{\alpha'} D_{\alpha'} \bar{Z}_t) Z_t^2 + 2Z_t \partial_{\alpha'}(\bar{Z}_{tt} - (D_{\alpha'} \bar{Z}_t) Z_t) + 2\bar{Z}_{t,\alpha'} Z_{tt} \\ &+ Z_{,\alpha'}(F_{tt} \circ Z) + i\bar{Z}_{t,\alpha'} \\ &= -iA_t |Z_{,\alpha'}|^2 \end{aligned} \tag{8}$$

$$\begin{aligned} [Z_t^2, \mathbb{H}] \partial_{\alpha'} D_{\alpha'} \bar{Z}_t + 2[Z_t, \mathbb{H}] \partial_{\alpha'}(\bar{Z}_{tt} - (D_{\alpha'} \bar{Z}_t) Z_t) + 2[Z_{tt}, \mathbb{H}] \bar{Z}_{t,\alpha'} \\ = (I - \mathbb{H}) \{-iA_t |Z_{,\alpha'}|^2\}. \end{aligned} \tag{9}$$

$$[f^2, \mathbb{H}] \partial_{\alpha'} g - 2[f, \mathbb{H}] \partial_{\alpha'}(fg) = -\frac{1}{\pi i} \int \left(\frac{f(\alpha') - f(\beta')}{\alpha' - \beta'}\right)^2 g(\beta') d\beta'. \tag{10}$$

$$\frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1} = \frac{A_t |Z_{,\alpha'}|^2}{A |Z_{,\alpha'}|^2} = \frac{A_t |Z_{,\alpha'}|^2}{A_1} \tag{11}$$

we have

Proposition

$$b := h_t \circ h^{-1} = 2\Re Z_t + \Re[Z_t, \mathbb{H}]\Big(\frac{1}{Z_{,\alpha'}} - 1\Big)$$

Proof:

- $h(\alpha, t) = \Phi(z(\alpha, t); t)$ implies
- $h_t(\alpha, t) = \Phi_z \circ z\, z_t + \Phi_t; \qquad h_\alpha = \Phi_z \circ z\, z_\alpha;$
- $h_t \circ h^{-1} = \frac{1}{Z_{,\alpha'}} Z_t + \Phi_t \circ \Phi^{-1}$
- $h_t \circ h^{-1}(\alpha', t) = \Re(I - \mathbb{H}) h_t \circ h^{-1} = \Re(I - \mathbb{H})\Big(\Big(\frac{1}{Z_{,\alpha'}} - 1\Big) Z_t\Big) + \Re(I - \mathbb{H}) Z_t$
- $h_t \circ h^{-1} = 2\Re Z_t + \Re[Z_t, \mathbb{H}]\Big(\frac{1}{Z_{,\alpha'}} - 1\Big)$

because $Z_t = -\mathbb{H} Z_t$, and $\mathbb{H}\Big(\frac{1}{Z_{,\alpha'}} - 1\Big) = \frac{1}{Z_{,\alpha'}} - 1$.

the quasilinear equation in Riemann mapping coordinate:

$$(\partial_t + b\partial_{\alpha'})^2 \bar{Z}_t + iA\partial_{\alpha'}\bar{Z}_t = \frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1}(\bar{Z}_{tt} - i) \tag{12}$$

where

$$\begin{aligned} &A = \frac{|\bar{Z}_{tt} - i|^2}{A_1}; \quad \frac{1}{Z_{,\alpha'}} = i\frac{\bar{Z}_{tt} - i}{A_1} \\ &A_1 = 1 - \Im[Z_t, \mathbb{H}]\bar{Z}_{t,\alpha'} = 1 + \frac{1}{2\pi}\int \frac{|Z_t(\alpha';t) - Z_t(\beta';t)|^2}{(\alpha' - \beta')^2}\, d\beta' \\ &b := h_t \circ h^{-1} = 2\Re Z_t + \Re[Z_t, \mathbb{H}]\Big(\frac{1}{Z_{,\alpha'}} - 1\Big) \\ &\frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1} = \frac{-\Im(2[Z_t, \mathbb{H}]\bar{Z}_{tt,\alpha'} + 2[Z_{tt}, \mathbb{H}]\bar{Z}_{t,\alpha'} - [Z_t, Z_t; D_{\alpha'}\bar{Z}_t])}{A_1}. \end{aligned} \tag{13}$$

$$i\partial_{\alpha'}\bar{Z}_t = i\partial_{\alpha'}\mathbb{H}\bar{Z}_t$$

Conclusions and Remarks:

- equation (12) is of hyperbolic type in the regime of smooth (C^2) interfaces, since $A \geq c_0 > 0$ in this case;— local wellposedness (with some further work) for arbitrary initial data in Sobolev spaces (of finite many derivatives);
- in particular, equation (12) is solvable for a short time for given velocity Z_t and acceleration Z_{tt} (in Sobolev spaces) that satisfies $(I - \mathbb{H})\bar{Z}_{tt} = [Z_t, \mathbb{H}] D_{\alpha'} \bar{Z}_t$.
- when the interface contains angled crests, $-\frac{\partial P}{\partial \mathbf{n}} \geq 0$, and $-\frac{\partial P}{\partial \mathbf{n}} = 0$ at the angled crests. — local existence of water waves with angled crests. see R. Kinsey & S. Wu *A priori estimates for two-dimensional water waves with angled crests* arXiv:1406.7573.
- equation (12) is an equation of velocity Z_t and acceleration Z_{tt}. — existence of splash and splat singularities of Cordoba, Fefferman et al.

Quasilinear equation in Lagrangian coordinate:

Purpose:

- to extend the result to 3-D;
- to have a coordinate invariant formulation.

Basic commutator identities

we have (see [Wu 2009])

Proposition

$$
\begin{aligned}
&[\partial_t, \mathfrak{H}]f = [z_t, \mathfrak{H}]\frac{f_\alpha}{z_\alpha} \\
&[\partial_t^2, \mathfrak{H}]f = [z_{tt}, \mathfrak{H}]\frac{f_\alpha}{z_\alpha} + 2[z_t, \mathfrak{H}]\frac{f_{t\alpha}}{z_\alpha} - \frac{1}{\pi i}\int \Big(\frac{z_t(\alpha,t) - z_t(\beta,t)}{z(\alpha,t) - z(\beta,t)}\Big)^2 f_\beta \, d\beta \\
&[\mathfrak{a}\partial_\alpha, \mathfrak{H}]f = [\mathfrak{a}z_\alpha, \mathfrak{H}]\frac{f_\alpha}{z_\alpha}, \quad \partial_\alpha \mathfrak{H} f = z_\alpha \mathfrak{H}\frac{f_\alpha}{z_\alpha} \\
&[\partial_t^2 - i\mathfrak{a}\partial_\alpha, \mathfrak{H}]f = 2[z_t, \mathfrak{H}]\frac{f_{t\alpha}}{z_\alpha} - \frac{1}{\pi i}\int \Big(\frac{z_t(\alpha,t) - z_t(\beta,t)}{z(\alpha,t) - z(\beta,t)}\Big)^2 f_\beta \, d\beta
\end{aligned}
\tag{14}
$$

$$\begin{cases} \bar{z}_{ttt} + i\mathfrak{a}\bar{z}_{t\alpha} = -i\mathfrak{a}_t\bar{z}_\alpha \\ \bar{z}_t = \mathfrak{H}\bar{z}_t \end{cases} \tag{15}$$

we apply $(I - \mathfrak{H})$ to the first equation, and get

$$\begin{aligned} (I - \mathfrak{H})(-i\mathfrak{a}_t\bar{z}_\alpha) &= [\partial_t^2 + i\mathfrak{a}\partial_\alpha, \mathfrak{H}]\bar{z}_{t\alpha} \\ &= 2[z_{tt}, \mathfrak{H}]\frac{\bar{z}_{t\alpha}}{z_\alpha} + 2[z_t, \mathfrak{H}]\frac{\bar{z}_{tt\alpha}}{z_\alpha} \\ &- \frac{1}{\pi i}\int \left(\frac{z_t(\alpha, t) - z_t(\beta, t)}{z(\alpha, t) - z(\beta, t)}\right)^2 \bar{z}_{t\beta}\, d\beta \end{aligned}$$

Let $\mathfrak{u} = \bar{z}_t$,

$$\mathfrak{u}_{tt} + \mathfrak{a}|z_\alpha|\nabla_{\mathbf{n}}\mathfrak{u} = \frac{\mathfrak{u}_t - i}{|\mathfrak{u}_t - i|}\mathfrak{a}_t|z_\alpha| \tag{16}$$

where

$$(I + \mathfrak{K}^*)(\mathfrak{a}_t|z_\alpha|) = \Re\Big(\frac{iz_\alpha}{|z_\alpha|}\{2[\bar{\mathfrak{u}}_t, \mathfrak{H}]\frac{\mathfrak{u}_\alpha}{z_\alpha} + 2[\bar{\mathfrak{u}}, \mathfrak{H}]\frac{\mathfrak{u}_{t\alpha}}{z_\alpha} - \frac{1}{\pi i}\int \Big(\frac{\bar{\mathfrak{u}}(\alpha,t) - \bar{\mathfrak{u}}(\beta,t)}{z(\alpha,t) - z(\beta,t)}\Big)^2 \mathfrak{u}_\beta \, d\beta\}\Big);$$

$\mathfrak{K}^*$ is the adjoint of the double layered potential operator:

$$\mathfrak{K}^* f(\alpha,t) = p.v. \int \Re\{\frac{-1}{\pi i}\frac{z_\alpha}{|z_\alpha|}\frac{|z_\beta(\beta,t)|}{(z(\alpha,t) - z(\beta,t))}\} f(\beta,t)\, d\beta;$$

$$\mathfrak{a}|z_\alpha| = |\mathfrak{u}_t - i|$$

- $\mathfrak{a}|z_\alpha| = -\frac{\partial P}{\partial \mathbf{n}} \geq c_0 > 0$ for C^{1+} interfaces;
- $\nabla_{\mathbf{n}}\mathfrak{u}$ is a positive operator;
- equation (16) is of hyperbolic type; —- local well-posedness.

Coordinate invariance of (16)

The quasilinear equation (16) is coordinate invariant:

- Notation:

$$U_k f(\alpha, t) = f \circ k(\alpha, t) = f(k(\alpha, t), t)$$

For fixed t, let $k = k(\alpha, t) : \mathbb{R} \to \mathbb{R}$ be a diffeomorphism, $k_\alpha > 0$,

$$k \circ k^{-1}(\alpha', t) = \alpha'$$

- Define:

$$\zeta = z \circ k^{-1}, \quad u = \bar{z}_t \circ k^{-1}, \quad w = \bar{z}_{tt} \circ k^{-1}$$

$$b = k_t \circ k^{-1}, \qquad A \circ k = \mathfrak{a} k_\alpha$$

Apply $U_{k^{-1}}$ to both sides of (15) or equivalently (16), we get

- $$(\partial_t + b\partial_{\alpha'})^2 u + iA\partial_{\alpha'} u = (\mathfrak{a}_t |z_\alpha|) \circ k^{-1} \frac{w - i}{|w - i|} \tag{17}$$
- where

$$(I + \mathcal{K}^*)((\mathfrak{a}_t |z_\alpha|) \circ k^{-1}) = \Re\Big(\frac{i\zeta_{\alpha'}}{|\zeta_{\alpha'}|}\{2[\bar{w}, \mathcal{H}]\frac{u_{\alpha'}}{\zeta_{\alpha'}} + 2[\bar{u}, \mathcal{H}]\frac{w_{\alpha'}}{\zeta_{\alpha'}}$$
$$- \frac{1}{\pi i}\int \Big(\frac{\bar{u}(\alpha', t) - \bar{u}(\beta', t)}{\zeta(\alpha', t) - \zeta(\beta', t)}\Big)^2 u_{\beta'}\, d\beta'\}\Big)$$

Global in time behavior for 2-D water waves

Question:

for small and smooth initial data, does the solution of the water wave equation (15) remain small and smooth for all time?

Main idea:

- Use a different coordinate system;
- Use the dispersive aspect of the quasilinear equation;
- Understand the nature of the quadratic nonlinearity of the water wave equation.

We work in the regime where the velocity, acceleration tend to zero, the interface tend to the real axis at the spatial infinity.

Linearize the water wave equation (16) about the flat interface (the trivial solution):

$$\partial_t^2 \mathfrak{u} + |\partial_\alpha|\mathfrak{u} = \text{nonlinear terms}$$

- Nonlinear term contains quadratic and higher order terms.
- The linear equation is globally well-posed.
- In $2-D$, the $L^1 \to L^\infty$ decay rate is $1/t^{1/2}$,
- In $3-D$, the $L^1 \to L^\infty$ decay rate is $1/t$.
- Nonlinear interaction can cause finite time blow up of solutions.
- Quadratic interaction is too strong.
- For small solution, the higher the order of the nonlinearity, the weaker the nonlinear interaction.

Heuristics

$$\partial_t^2 u + |\partial_\alpha| u = (\partial_t u)^{p+1}$$

Suppose we can prove for n-D water waves: for $i \le s - 10$,

$$|\partial^i \partial_t u(t)|_\infty \le (1+t)^{-\frac{n-1}{2}} E_s(t)^{1/2}$$

where ∂ is some kind of derivatives. Then the

Energy $E_s(t) = \sum_{|j| \le s} \int |\partial^j \partial_t u|^2 + |\partial^j |\partial_\alpha|^{1/2} u|^2 \, dx$ satisfies:

$$\frac{dE_s(t)}{dt} \le |\partial^i \partial_t u(t)|_\infty^p E_s(t) \le \frac{1}{(1+t)^{p/2}} E_s(t)^{1+p/2} \qquad \text{2-D}$$

$$\frac{dE_s(t)}{dt} \le |\partial^i \partial_t u(t)|_\infty^p E_s(t) \le \frac{1}{(1+t)^{p}} E_s(t)^{1+p/2} \qquad \text{3-D}$$

Heuristics..

Therefore

$$E_s(T)^{-p/2} \geq E_s(0)^{-p/2} - \int_0^T (1+t)^{-\frac{p}{2}}\, dt \qquad \text{2-D}$$

$$E_s(T)^{-p/2} \geq E_s(0)^{-p/2} - \int_0^T (1+t)^{-p}\, dt \qquad \text{3-D}$$

To obtain $E_s(T) < \infty$, we need

$$\int_0^T (1+t)^{-\frac{(n-1)p}{2}}\, dt < E_s(0)^{-p/2} \qquad \text{n-D}$$

Idea: look for possible cancellations in the quadratic terms.

The method of normal form: Let

$$\mathfrak{v} = \mathfrak{u} + K(\mathfrak{u})$$

Find an appropriate nonlinear form $K(\mathfrak{u})$ so that (hopefully) $\mathfrak{v}$ satisfies an equation

$$\mathfrak{v}_{tt} + |\partial_\alpha|\mathfrak{v} = F_1(z, \mathfrak{u}, \mathfrak{u}_t, \mathfrak{u}_{tt}, \mathfrak{u}_\alpha)$$

with F_1 consisting of only cubic and higher order terms, and

$$\|\mathfrak{v}\| \approx \|\mathfrak{u}\|$$

- Poincare

Poincare's method of normal form for ODE:

- ODE system

$$u_t = Au + Q(u) + C(u) \tag{18}$$

- $C(u)$ cubic and higher order terms. Let
-

$$v = u + B(u, u) = u + u^T M u \tag{19}$$

- Want to find the matrix M so that the equation for v contains no quadratic terms. Calculate:
-

$$\begin{aligned} v_t &= u_t + B(u_t, u) + B(u, u_t) \\ &= Au + Q(u) + B(Au, u) + B(u, Au) + C(u) \end{aligned}$$

-

$$Av = Au + AB(u, u)$$

Want: $v_t = Av + C(u)$

- Require:

$$Q(u) + B(Au, u) + B(u, Au) = AB(u, u)$$

  A system of linear equations.
- One shouldn't expect to find the $B(u, u)$ always.
- For Klein-Gordon equation:
- Shatah: generalize the ODE idea to the Klein-Gordon equation, with a bilinear $K(\mathfrak{u}) = B(\mathfrak{u}, \mathfrak{u})$.

Water wave

$$\mathfrak{v} = \mathfrak{u} + K(\mathfrak{u})$$

with a bilinear $K(\mathfrak{u})$ doesn't work for the water wave equation.

- There is a small divisor in the Fourier symbol of K: Let

$$K(\mathfrak{u})(x) = \int e^{ix\cdot(\xi+\eta)} m(\xi,\eta)\hat{\mathfrak{u}}(\xi)\hat{\mathfrak{u}}(\eta)\, d\xi d\eta$$

$$m(\xi,\eta) \approx \frac{1}{\xi} B(\xi,\eta) \quad \text{near } \xi = 0$$

  $\lim_{\xi\to 0} B(\xi,\eta) \neq 0$
- This amounts to such difficulties as proving

$$\|f\|_{L^2(\mathbb{R}^d)} \lesssim \|\nabla f\|_{L^2(\mathbb{R}^d)}$$

- It is impossible.

However (from the right formulation of the water wave equation) we notice that the small divisor could be related to the coordinate system we use.

Consider only the part without small divisor — the partial transformation:

- The partial transformation doesn't quite work, since it gives rise to an equation with poor structures (depending on $\mathfrak{u}$).
- The partial transformation and the resulting equation are not coordinate invariant.

- Let $z = x + iy = z(\alpha, t)$: equation of the interface at time t.
- We find the projection $(I - \mathfrak{H})$ partially works:

Proposition

The quantity

$$\Pi = (I - \mathfrak{H})(z - \bar{z}) = 2i(I - \mathfrak{H})y$$

satisfies the equation:

$$\begin{aligned}&(\partial_t^2 - i\mathfrak{a}\partial_\alpha)\Pi \\ &= \frac{4}{\pi}\int \frac{(z_t(\alpha,t) - z_t(\beta,t))(y(\alpha,t) - y(\beta,t))}{|z(\alpha,t) - z(\beta,t)|^2} z_{t\beta}\, d\beta \\ &+ \frac{1}{\pi i}\int (\frac{z_t(\alpha,t) - z_t(\beta,t)}{z(\alpha,t) - z(\beta,t)})^2 (z_\beta - \bar{z}_\beta)\, d\beta\end{aligned} \tag{20}$$

- The right hand side of (20) consists of cubic & higher order terms.
- The left hand side of (20) still contains quadratic terms.
- The structure of (20) is coordinate invariant.
- $(I - \mathfrak{H})$ is fully nonlinear
- $\mathfrak{H}f(\alpha, t) = \frac{1}{\pi i} \int \frac{z_\beta(\beta,t)}{z(\alpha,t)-z(\beta,t)} f(\beta, t)\, d\beta$

Idea:

Look for an appropriate change of coordinates k.

- Chain rule:

$$U_{k^{-1}}(\partial_t^2 - i\mathfrak{a}\partial_\alpha)\Pi = [(\partial_t + b\partial_{\alpha'})^2 - iA\partial_{\alpha'}]\Pi \circ k^{-1}$$

- Q: Is there a change of variables k so that

$$b = k_t \circ k^{-1}, \quad A - 1 = \mathfrak{a}k_\alpha \circ k^{-1} - 1$$

are quadratic?

Answer:

Yes.

- Let $h = h(\alpha, t) = \Phi(z(\alpha, t), t)$
- $\Phi(\cdot, t) : \Omega(t) \to P_-$ be the Riemann mapping.
- Let

$$k(\alpha, t) = 2x(\alpha, t) - h(\alpha, t)$$

  then

$$b = k_t \circ k^{-1}, \quad A - 1 = \mathfrak{a} k_\alpha \circ k^{-1} - 1$$

  are consisting of only quadratic and higher order terms.

Remark: The crucial fact for the coordinate change k is that:

- $k(\cdot, t) : \mathbb{R} \to \mathbb{R}$ is such that $\bar{z} - k$ is the trace of a holomorphic function in the fluid region; that is

$$(I - \mathfrak{H})(\bar{z} - k) = 0$$

Let

$$\mathcal{H}f(\alpha') = U_{k^{-1}}\mathfrak{H}U_k f(\alpha') = \frac{1}{\pi i}\int \frac{\zeta_{\beta'}(\beta',t)}{\zeta(\alpha',t)-\zeta(\beta',t)} f(\beta')\,d\beta'$$

Proposition

Let $b = k_t \circ k^{-1}$ *and* $A = (\mathfrak{a}k_\alpha)\circ k^{-1}$. *We have*

$$(I-\mathcal{H})b = -[z_t\circ k^{-1}, \mathcal{H}]\frac{\bar\zeta_{\alpha'}-1}{\zeta_{\alpha'}}$$

$$(I-\mathcal{H})A = 1 + i[z_t\circ k^{-1}, \mathcal{H}]\frac{(\bar z_t\circ k^{-1})_{\alpha'}}{\zeta_{\alpha'}} + i[z_{tt}\circ k^{-1}, \mathcal{H}]\frac{\bar\zeta_{\alpha'}-1}{\zeta_{\alpha'}}$$

How does it work? Let

$$\zeta = z \circ k^{-1} = \mathfrak{x} + \mathfrak{y}, \quad u = \bar{z}_t \circ k^{-1}, \quad w = \bar{z}_{tt} \circ k^{-1}$$

We have

$$\begin{aligned}
&((\partial_t + b\partial_{\alpha'})^2 - iA\partial_{\alpha'})\{(I - \mathcal{H})(\zeta - \alpha')\} \\
&= \frac{4}{\pi} \int \frac{(\bar{u}(\alpha', t) - \bar{u}(\beta', t))(\mathfrak{y}(\alpha', t) - \mathfrak{y}(\beta', t))}{|\zeta(\alpha', t) - \zeta(\beta', t)|^2} \bar{u}_{\beta'} \, d\beta' \\
&+ \frac{1}{\pi i} \int (\frac{\bar{u}(\alpha', t) - \bar{u}(\beta', t)}{\zeta(\alpha', t) - \zeta(\beta', t)})^2 (\zeta_{\beta'} - \bar{\zeta}_{\beta'}) \, d\beta'
\end{aligned}$$

$$(I - \mathcal{H})(\bar{\zeta} - \alpha') = 0$$

Let $\psi(\alpha,t)=\phi(z(\alpha,t),t)$ be the trace of the velocity potential ϕ on $\Sigma(t)$.

Proposition

Let $\Lambda=(I-\mathfrak{H})\psi$, $\mathfrak{v}=\partial_t\Pi$. *We have*

$$(\partial_t^2-i\mathfrak{a}\partial_\alpha)\Lambda=-[z_t,\mathfrak{H}\frac{1}{z_\alpha}+\bar{\mathfrak{H}}\frac{1}{\bar z_\alpha}](\bar z_\alpha z_{tt})+[z_t,\bar{\mathfrak{H}}](\bar z_t\frac{z_{t\alpha}}{\bar z_\alpha})+z_t[z_t,\mathfrak{H}]\frac{\bar z_{t\alpha}}{z_\alpha}$$
$$-2[z_t,\mathfrak{H}]\frac{z_t\cdot z_{t\alpha}}{z_\alpha}+\frac{1}{\pi i}\int(\frac{z_t(\alpha,t)-z_t(\beta,t)}{z(\alpha,t)-z(\beta,t)})^2 z_t\cdot z_\beta\,d\beta$$

$$(\partial_t^2-i\mathfrak{a}\partial_\alpha)\mathfrak{v}=\partial_t\{(\partial_t^2-i\mathfrak{a}\partial_\alpha)\Pi\}+i\mathfrak{a}_t\partial_\alpha(I-\mathfrak{H})(z-\bar z)$$

$$2\bar u-U_{k-1}\mathfrak{v},\quad 2\mathfrak{y}_{\alpha'}-\Im\partial_{\alpha'}U_{k-1}\Pi,\quad 2\bar w-U_{k-1}\partial_t\mathfrak{v}$$

are each at least of quadratic order.

Conclusion:

- The nonlinearity of the 2-D water wave equation is of cubic and higher orders when working in the right coordinate system with the right quantities.

Remark:

1. The projection $(I - \mathfrak{H})$ handles partly the bounded part in the formal bilinear normal form calculation. $(I - \mathfrak{H})$ is fully nonlinear and is coordinate invariant.
2. The coordinate change handles partly the small divisor part in the formal bilinear normal form calculation.
3. Difference between $\partial_t^2 + |\partial_\alpha|$ and $\partial_t^2 - i\partial_\alpha$.

- Let ϕ be the velocity potential: $\nabla\phi = \mathbf{v}$, $(x, \eta(x, t))$ be the interface.
- The water wave equation is Hamiltonian in term of the variables $(\eta, \xi = \phi(x, \eta))$ (Zakharov).
- W. Craig et. al: Birkhoff normal form transformation procedure for periodic water wave (discrete spectrum): there is no quadratic resonance in non-zero frequencies. (moment conditions can be imposed in the periodic case, so there is no need to consider the zero frequency.)

The small divisor occurs at the zero frequency for our variables $(\eta, \mathbf{v}) = (\eta, \nabla\phi)$ (non-periodic):

- assuming $\phi = \nabla^{-1}\mathbf{v}$ localized would resolve the singularity.
- but in physically relevant situations ϕ need not be localized.
- We resolve this difficulty by choosing the right coordinates.

Statement of the Theorems.

Theorem (2-D water wave)

Assume that

- *initial height function is* $y_0(x) = \epsilon f(x)$
- *initial velocity is* $\mathbf{v}_0(x, y) = \epsilon g(x, y)$

$f \in L^2(\Sigma(0))$, $g \in L^2(\Omega(0))$, *and finitely many derivatives of* f *and* g *are in* L^2.

There is $\epsilon_0 > 0$, *such that if* $0 \leq \epsilon \leq \epsilon_0$, *there is a unique smooth solution of the 2-D water wave equation for a time period* $[0, e^{c/\epsilon}]$. *During this time, the solution remains smooth and small.*

Remark:

- Smallness in steepness $\partial_x y_0(x)$ of the initial interface and initial surface velocity are part of the assumption.

Theorem (3-D water wave)

Let $\xi_0 = (x, y, z_0(x, y))$ *be the initial interface,* $\mathfrak{u}_0 = \mathfrak{u}_0(x, y)$ *be the initial velocity along the interface* $\Sigma(0)$. *Assume that*

- $(\partial_x z_0, \partial_y z_0) = \epsilon\,(f(x, y), g(x, y))$ *(steepness)*
- $\mathfrak{u}_0 = \epsilon\, u(x, y)$

$(f, g) \in L^2(\Sigma(0))$, $u \in L^2(\Sigma(0))$, *and finitely many derivatives of* (f, g) *and* u *are in* L^2.

There is $\epsilon_0 > 0$, *such that if* $0 \le \epsilon \le \epsilon_0$, *there is a unique smooth solution of the 3-D water wave equation globally in time. During this time, the solution remains smooth and small.*

Remark:

- No smallness assumptions on the initial height function z, and the initial energy $\int_{\Omega(0)} |\mathbf{v}_0|^2$ can be infinite.

Remark:

- The total kinetic energy can be infinity in both the 2d and 3d cases: $\epsilon \times \infty$:
  2d: $\epsilon \times \infty$ in the constant wave motion direction;
  3d: $\epsilon \times \infty$ in depth.
- The wave motion is localized in the L^2 sense.
- Periodic case: existence for $t \in [0, c/\epsilon^2]$, assuming slope is of size ϵ.
- There is no small solitary waves in infinite depth 2-d and 3-d water waves.

Clifford analysis

- Use the framework of the algebra of quaternions.

  Let $\{1, e_1, e_2, e_3\}$ be the basis of $\mathcal{C}(V_2)$, s. t.
- $e_i^2 = -1, e_i e_j = -e_j e_i, i \neq j, e_3 = e_1 e_2.$
- $\mathcal{D} = \partial_x e_1 + \partial_y e_2 + \partial_z e_3$
- $F : \Omega \subset \mathbb{R}^3 \to \mathcal{C}(V_2)$ is called Clifford analytic if $\mathcal{D}F = 0$.
- $F = \sum_{i=1}^3 f_i e_i$ is Clifford analytic in Ω iff $\mathrm{div} F = 0$, $\mathrm{curl} F = 0$.
- F analytic in Ω iff $F = \mathfrak{H}F$, where
- $\mathfrak{H}g(\alpha, \beta) = p.v. \iint K(\xi(\alpha', \beta') - \xi(\alpha, \beta)) \, (\xi'_{\alpha'} \times \xi'_{\beta'}) \, g(\alpha', \beta') \, d\alpha' d\beta'.$
  $\partial\Omega : \xi = \xi(\alpha, \beta), \quad K(\xi) = -2\mathcal{D}\Gamma(\xi) = -\frac{2}{\omega_3} \frac{\xi}{|\xi|^3}.$

3-D water wave, local in time

Let $\Sigma(t) : \xi = \xi(\alpha, \beta, t)$ be the interface in Lagrangian coordinates (α, β).

The surface equation for 3-D water waves is

$$\begin{cases} \xi_{tt} + e_3 = \mathfrak{a} N \\ \xi_t = \mathfrak{H} \xi_t \end{cases}$$

where $N = \xi_\alpha \times \xi_\beta$ is the normal pointing out of the fluid domain $\Omega(t)$, $\mathfrak{a} = -\frac{1}{|N|} \frac{\partial P}{\partial \mathbf{n}}$.
Quasilinear equation:

$$\begin{cases} \xi_{ttt} - \mathfrak{a} N_t = \mathfrak{a}_t N \\ \xi_t = \mathfrak{H} \xi_t \end{cases}$$

$$-N_t = |N| \nabla_{\mathbf{n}} \xi_t$$

3-D water wave, global in time

The projection $(I - \mathfrak{H})$ still works.

- Let $\Sigma(t) : \xi = (x(\alpha,\beta,t), y(\alpha,\beta,t), z(\alpha,\beta,t))$ be the interface in Lagrangian coordinates (α,β),
- Let $\pi = (I - \mathfrak{H})z$.

$$\begin{aligned}(\partial_t^2 - \mathfrak{a}N \times \nabla)\pi \\ = \iint K(\xi' - \xi)\,(\xi_t - \xi'_t) \times (\xi'_{\beta'}\partial_{\alpha'} - \xi'_{\alpha'}\partial_{\beta'})\overline{\xi'_t}\, d\alpha' d\beta' \\ - \iint K(\xi' - \xi)\,(\xi_t - \xi'_t) \times (\xi'_{t\beta'}\partial_{\alpha'} - \xi'_{t\alpha'}\partial_{\beta'}) z'\, d\alpha' d\beta' e_3 \\ - \iint \partial_t K(\xi' - \xi)\,(\xi_t - \xi'_t) \times (\xi'_{\beta'}\partial_{\alpha'} - \xi'_{\alpha'}\partial_{\beta'}) z'\, d\alpha' d\beta' e_3\end{aligned} \tag{21}$$

Coordinate change

$k = \xi - (I + \mathfrak{H}) z e_3 + \mathfrak{K} z e_3$ Here $\mathfrak{K}$ is the double layered potential operator.

Applying coordinate change k^{-1} gives for $\chi = \pi \circ k^{-1}$,

$$
\begin{aligned}
&((\partial_t + b \cdot \nabla_\perp)^2 - A\mathcal{N} \times \nabla)\chi \\
&= \iint K(\zeta' - \zeta)\,(u - u') \times (\zeta'_{\beta'}\partial_{\alpha'} - \zeta'_{\alpha'}\partial_{\beta'})\overline{u'}\, d\alpha' d\beta' \\
&- \iint K(\zeta' - \zeta)\,(u - u') \times (u'_{\beta'}\partial_{\alpha'} - u'_{\alpha'}\partial_{\beta'})\mathfrak{z}'\, d\alpha' d\beta' e_3 \\
&- \iint ((u' - u) \cdot \nabla) K\,(u - u') \times (\zeta'_{\beta'}\partial_{\alpha'} - \zeta'_{\alpha'}\partial_{\beta'})\mathfrak{z}'\, d\alpha' d\beta' e_3
\end{aligned}
$$

- The first term on the right hand side is of type $(I + \mathfrak{H})Q$,
- Q quadratic. The left hand side is of type $(I - \mathfrak{H})$, or is analytic in the air region.
- The projection $(I - \mathfrak{H})$ applies to the first term on the right turns it into cubic.
  Let $\zeta = \xi \circ k^{-1} = P + \lambda$, $P = (\alpha, \beta)$.
- The left hand side $= \partial_t^2 \chi - \mathcal{N} \times \nabla \chi +$cubic terms $=$
  $\partial_t^2 \chi - (e_2 \partial_\alpha - e_1 \partial_\beta)\chi - \partial_\beta \lambda \partial_\alpha \chi + \partial_\alpha \lambda \partial_\beta \chi +$cubic terms.

- $\mathcal{N} \times \nabla\chi = (e_2\partial_\alpha - e_1\partial_\beta)\chi + \partial_\beta\lambda\partial_\alpha\chi - \partial_\alpha\lambda\partial_\beta\chi$.
- $\partial_\alpha\lambda\partial_\beta\chi - \partial_\beta\lambda\partial_\alpha\chi$ is a null form.

- $\partial_\alpha\lambda\partial_\beta\chi - \partial_\beta\lambda\partial_\alpha\chi$ is one of the null forms treated by Klainerman for the wave equation.
- $\partial_\alpha\lambda\partial_\beta\chi - \partial_\beta\lambda\partial_\alpha\chi$ is a null form for the water wave equations as well.

- Use method of vector fields.
- For 2D water wave equation: the operator $\partial_t^2 - i\partial_\alpha$ has invariant vector fields ∂_t, ∂_α, $L_0 = \frac{1}{2}t\partial_t + \alpha\partial_\alpha$, $\Omega_0 = \alpha\partial_t + \frac{1}{2}ti$.
- For 3D water wave equation: the operator $\partial_t^2 - e_2\partial_\alpha + e_1\partial_\beta$ has invariant vector fields ∂_t, $\partial_\alpha, \partial_\beta$, $L_0 = \frac{1}{2}t\partial_t + \alpha\partial_\alpha + \beta\partial_\beta$, $\varpi = \alpha\partial_\beta - \beta\partial_\alpha - \frac{1}{2}e_3$ and Ω_{01}, Ω_{02}.
- Combine a $L^2 \to L^\infty$ decay estimate and the energy estimates we prove the energy remains bounded for all time for 3D water wave and almost all time for 2D water wave.

- We get almost global wellposedness for 2-D water wave and global wellposedness for 3-D water wave.
- Alazard & Delort (2013): Global well-posedness for 2-D for similar small and smooth data.
- Ionescu & Pusateri (2013): Global well-posedness for 2-D for data with some further restrictions on small frequency waves.