Modulation and water waves – Part 1

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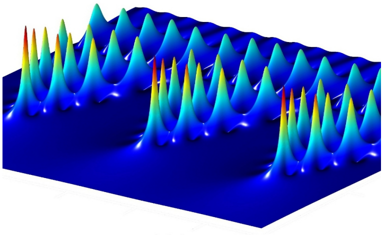
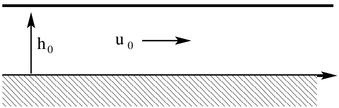


Modulation of uniform flows and KdV



$$Q = h_0 u_0 \quad \text{(mass flux)}$$

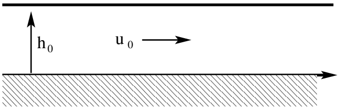
$$R = gh_0 + \tfrac{1}{2}u_0^2 \quad \text{(total head, Bernoulli energy)}\,.$$

Modulate the uniform flow

$$u = u_0 + \varepsilon q(X,T,\varepsilon)\,, \quad X = \varepsilon x\,,\ T = \varepsilon t\ (SWEs)$$

$$u = u_0 + \varepsilon^2 q(X,T,\varepsilon)\,, \quad X = \varepsilon x\,,\ T = \varepsilon^3 t\ (KdV)\,.$$

Modulation of uniform flows and KdV



$$\begin{aligned} Q &= h_0 u_0 \quad \text{(mass flux)} \\ R &= g h_0 + \tfrac{1}{2} u_0^2 \quad \text{(total head, Bernoulli energy)}\,. \end{aligned}$$

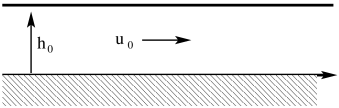
The flow is critical when

$$\left.\frac{\partial Q}{\partial u_0}\right|_{R \text{ fixed}} = 0\,.$$

Computing,

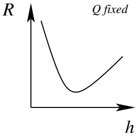
$$Q\Big|_{R \text{ fixed}} = h_0 u_0 = \frac{u_0}{g}\left(R - \tfrac{1}{2} u_0^2\right).$$

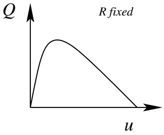
Differentiate with respect to u_0.



Computing,

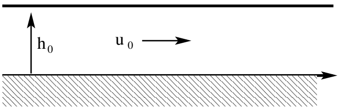
$$\left.\frac{\partial Q}{\partial u_0}\right|_{R\ \mathrm{fixed}} = \frac{1}{g}\left(R - \frac{3}{2}u_0^2\right) = \frac{1}{g}\left(gh_0 - u_0^2\right) \,.$$





What does second derivative $\left.\frac{\partial^2 Q}{\partial u_0^2}\right|_{R\ \mathrm{fixed}}$ at the maximum mean?

Modulation of uniform flows and KdV



Modulate the uniform flow:

$$u = u_0 + \varepsilon^2 q(X, T, \varepsilon)\,, \quad X = \varepsilon x\,,\ T = \varepsilon^3 t\,.$$

Claim: q satisfies the KdV equation with

$$2\mathcal{M}_u \frac{\partial q}{\partial T} + \left.\frac{\partial^2 Q}{\partial u_0^2}\right|_{R\ \text{fixed}} q\frac{\partial q}{\partial X} + \mathscr{K}\frac{\partial^3 q}{\partial X^3} = 0\,.$$

Modulation of the mass CLAW for the full water wave problem

$$M_t + Q_x = 0\,,$$

where $\mathcal{M}$ is M evaluated on the uniform flow.

Symmetry, modulation, and KdV

• basic state: (h_0, u_0)
• conservation laws: $Q_x = 0$ and $R_x = 0$

• symmetry of water wave problem: $\phi \mapsto \phi + \gamma$, for all $\gamma \in \mathbb{R}$
• the uniform flow is a symmetry induced solution:
$\phi(x, y, t) = u_0 x + \phi_0$
• if the system is generated by a Lagrangian then symmetry implies the conservation law

• strategy: modulate basic state, use connection between symmetry and CLAW to get modulation equation

Symmetry, modulation, KdV

- Lagrangian $\int\int\int L\,\mathrm{d}x\mathrm{d}z\mathrm{d}t$ (e.g. Luke's Lagr)
- One-parameter symmetry group $\Rightarrow$ CLAW $A_t + B_x = 0$
- basic state: $\widehat{Z}(\theta, k)$ with $\theta = kx + \theta_0$
- modulate: $Z(x,t) = \widehat{Z}(\theta + \varepsilon\psi, k + \varepsilon^2 q) + \varepsilon^3 W(\theta, X, T)$
- Evaluate CLAW on basic state: $\mathscr{A}(k)$ and $\mathscr{B}(k)$
- If $\mathscr{B}'(k) = 0$ (criticality) then KdV emerges

$$2\mathscr{A}_k q_T + \mathscr{B}_{kk} q q_X + \mathscr{K} q_{XXX} = 0\,.$$

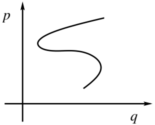
- $\mathscr{K}$: Krein signature, sign of momentum flux, dispersion relation, ...

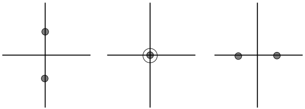
Planar homoclinic bifurcation

Consider a parameter-dependent planar Hamiltonian system

$$-\dot{p} = \frac{\partial H}{\partial q}\,, \quad \dot{q} = \frac{\partial H}{\partial p}\,, \quad \text{with} \quad H(q,p,\alpha)\,.$$

Suppose there is a family of equilibria $(q_0(\alpha), p_0(\alpha))$.





Suppose further that at some value $\alpha = \alpha_0$ there is a saddle-centre transition of eigenvalues.

Planar homoclinic bifurcation

At the saddle-centre transition, there is a Jordan chain associated with the zero eigenvalue

$$\mathbf{L}\xi_1 = 0 \quad \text{and} \quad \mathbf{L}\xi_2 = \mathbf{J}\xi_1 \,.$$

The linear system can be transformed to (symplectic) Jordan normal form

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{pmatrix} \widetilde{q} \\ \widetilde{p} \end{pmatrix}_t = \begin{bmatrix} 0 & 0 \\ 0 & s \end{bmatrix} \begin{pmatrix} \widetilde{q} \\ \widetilde{p} \end{pmatrix} , \quad (s = \pm 1 \text{ is a symplectic sign}) \,.$$

$(\widetilde{q}, \widetilde{p})$ are the transformed (q, p).

Planar homoclinic bifurcation – nonlinear theory

Introduce a nonlinear normal form transformation up to quadratic order. Again calling the new coordinates $\widetilde{q}(t)$ and $\widetilde{p}(t)$, they satisfy

$$
\begin{aligned}
-\widetilde{p}_t &= I - \tfrac{1}{2}\kappa\widetilde{q}^2 + \cdots \\
\widetilde{q}_t &= s\,\widetilde{p} + \cdots
\end{aligned}
$$

where $I = c(\alpha - \alpha_0)$ is an unfolding parameter and

$$
\kappa = \langle \xi_1, D^3H(q_0,p_0)(\xi_1,\xi_1)\rangle
$$

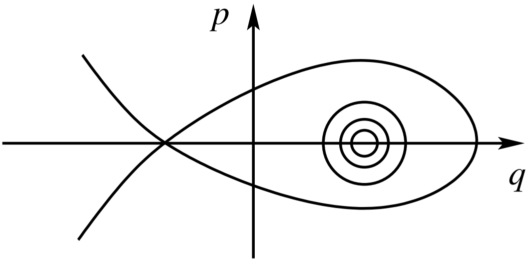
Nonlinear normal form found in textbooks ...

- ARNOLD ET AL, Dyn Sys III; MEYER & HALL, Ham Sys

Planar homoclinic bifurcation and curvature

Generates the familiar homoclinic (fish) diagram.

$$
\begin{aligned}
-\widetilde{p}_t &= I - \tfrac{1}{2}\kappa\widetilde{q}^2 + \cdots \\
\widetilde{q}_t &= s\,\widetilde{p} + \cdots
\end{aligned}
$$



Claim: more to the story – the coefficient κ is a curvature

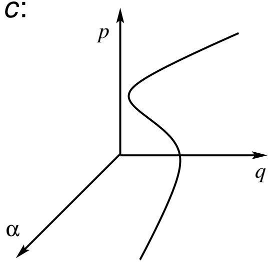
Normal form via modulation

The normal form can be interpreted as a modulation equation and the coefficient κ emerges from the modulation

$$
\begin{aligned}
-\widetilde{p}_t &= I - \tfrac{1}{2}\alpha''\widetilde{q}^2 + \cdots \\
\widetilde{q}_t &= s\,\widetilde{p} + \cdots
\end{aligned}
$$

α is a function of a parameter c:
lift to (p, q, α) space
How to choose the c?

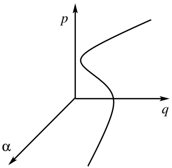
$$c = \frac{\partial H}{\partial \alpha}$$



Modulation and curvature

Claim: the family of equilibria satisfy

$$H_q = 0$$
$$H_p = 0$$
$$H_\alpha = c$$



Saddle-centre at $c = c_0 \Leftrightarrow \alpha'(c_0) = 0$

Nonlinear coefficient κ equals the curvature $\alpha''(c_0)$.

Lift: equilibria $\nearrow$ relative equilibria

$$\left.\begin{aligned} -\dot{p} &= H_q \\ \dot{q} &= H_p \\ -\dot{\alpha} &= 0 \end{aligned}\right\} \text{ standard lift}$$

add in $\dot{\theta} = H_\alpha$ (symplectic lift)

Symplectic lifted system has symmetry! $\gamma \mapsto \theta + \gamma \quad \forall \gamma \in \mathbb{R}$.

The equilibrium becomes a RE, $\theta(t) = ct + c_0$

RE satisfy

$$H_q = 0\,, \quad H_p = 0\,, \quad H_\alpha = c\,.$$

Normal form via modulation

Hamiltonian system on $\mathbb{R}^4$

$$\mathbf{J}Z_t = \nabla H(Z)\,, \quad Z \in \mathbb{R}^4\,. \tag{1}$$

One-parameter abelian symmetry group with RE:

$$\widehat{Z}(\theta, c) \quad \text{where} \quad \theta = ct + \theta_0\,.$$

Evaluate conserved quantity, $\alpha_t = 0$ on RE: $\alpha(c)$

Modulate (ansatz)

$$Z(t) = \widehat{Z}(\theta + \varepsilon\phi, c + \varepsilon^2 q) + \varepsilon^3 W(\theta, T, \varepsilon)\,, \tag{2}$$

where $\phi(T, \varepsilon)$ and $T = \varepsilon t$.

Normal form via modulation

Substitute the ansatz

$$Z(t) = \widehat{Z}(\theta + \varepsilon\phi, c + \varepsilon^2 q) + \varepsilon^3 W(\theta, T, \varepsilon)\,, \tag{3}$$

into $\mathbf{J}Z_t = \nabla H(Z)$, expand everything in powers of ε

- ε^2: $q = \phi_T$
- ε^3: solvability for W_1 requires $\alpha'(c_0) = 0$ for some c_0
- ε^4: solve equation for W_2
- ε^5: solvability for W_3 if and only if

$$\alpha''(c_0) q q_T - s q_{TTT} = 0\,.$$

Normal form via modulation

Substitution of the ansatz, and equating terms up to ε^5 to zero gives

$$q = \phi_T \quad \text{and} \quad \alpha''(c_0)qq_T - sq_{TTT}\,,$$

or, integrating the second equation and calling the constant of integration I,

$$\begin{aligned} -\dot{I} &= 0 \\ -\dot{p} &= I - \tfrac{1}{2}\alpha''(c_0)q^2 \\ \dot{\phi} &= q \\ \dot{q} &= sp\,. \end{aligned}$$

Now reduce back to the planar system, giving the modulation characterisation of the planar normal form.

Normal form via modulation

$$
\begin{aligned}
-\dot{p} &= I - \tfrac{1}{2}\alpha''(c_0)q^2 \\
\dot{q} &= sp \\
-\dot{I} &= 0 \\
\dot{\phi} &= q
\end{aligned}
$$

The first two components recover the normal form, with a new interpretation of the coefficient of nonlinearity as a curvature.

Note that the spectrum of the linear system (the "dispersion relation") is not computed: the saddle-centre is predicted by $\alpha'(c_0) = 0$.

Summary: saddle-centre, lift, RE, curvature, ...

Saddle-centre to homoclinic bifurcation

- Dynamical systems approach: compute eigenvalues, identify saddle-centre transition, normal form transformations, analyze normal form
- Geometric approach: lift equilibria to RE, generates function $\alpha(c)$, $\alpha'(c) = 0$ then implies saddle-centre, $\alpha''(c)$ gives coefficient of normal form, analyze normal form.
- Concomitantly, can modulate a family of RE, and generate conditions and normal form for homoclinic bifurcation
- $\alpha''(c_0)qq_T - sq_{TTT} = 0$ with T replaced by X is steady KdV

KdV via RE and modulation

Where does the KdV equation come from?

Most widely used approach: the dispersion relation $\Rightarrow$ KdV equation

For some system of PDEs, suppose the dispersion relation to leading order is

$$\omega = -c_0 k + a k^3 + \cdots \,.$$

replace $\mathrm{i}\omega$ by ∂_t and $\mathrm{i}k$ by ∂_x and add in nonlinearity

$$0 = \eta_t + c_0\eta_x + a\eta_{xxx} + \begin{cases} b\eta^2 \\ b\eta\eta_x \\ b\eta_x^2 \,. \end{cases}$$

Symmetry argument leads to the middle choice. Now compute coefficients. *The most difficult calculation is the coefficient b of the nonlinearity.*

Derivation of KdV for water waves

- Assume shallow water: $h_0/L \to 0$. The limiting equation is the SWEs.
- "amplitude balances dispersion" – introduce an amplitude parameter take amplitude and h_0/L to zero in appropriate ratio. Limiting equation is a two way Boussinesq SWE.
- Now uni-directionalise: split the Boussinesq equation into a left-running and right-running component. Result is a pair of KdV equations. (hidden assumption of criticality)

— Shallow water is neither necessary nor sufficient for emergence of the KdV equation —

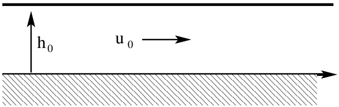
Emergence of KdV by modulating background state

- New observation: KdV emerges due to modulation of the background state
- The resulting KdV equation takes a universal form

$$2\mathscr{A}'(k)q_T + \mathscr{B}''(k)qq_X + \mathscr{K}\, q_{XXX} = 0$$

- Coefficients – including nonlinearity – reduced to an elementary calculation.
- KdV arises due to a critical point of a family of RE – in the classic case the RE is a uniform flow.

The KdV equation in shallow water hydrodynamics



The KdV equation in shallow water is

$$2\mathscr{A}'(u_0)q_T + \mathscr{B}''(u_0)qq_X + \mathscr{K}\,q_{XXX} = 0\,,$$

where, relative to a laboratory frame,

$$\mathscr{A}(u_0) = h_0 = \frac{1}{g}(R - \tfrac{1}{2}u_0^2) \quad \text{and} \quad \mathscr{A}'(u_0) = -\frac{u_0}{g}\,,$$

and

$$\mathscr{B}(u_0) = h_0 u_0 = \frac{u_0}{g}(R - \tfrac{1}{2}u_0^2) \quad \text{and} \quad \mathscr{B}''(u_0) = -3\frac{u_0}{g}\,.$$

KdV in shallow water continued

Substituting

$$-2\frac{u_0}{g}q_T - 3\frac{u_0}{g}qq_X + \frac{1}{3}h_0^3 q_{XXX} = 0\,,$$

and computing $\mathscr{K} = h_0^3/3$. Now let $q = -u_0 h/h_0$, then

$$-\frac{1}{u_0}h_T + \frac{3}{2h_0}hh_X + \frac{1}{6}h_0^3 h_{XXX} = 0\,,$$

the familiar form of the KdV found in textbooks, noting that $u_0 = \pm\sqrt{gh_0}$ (since $\mathscr{B}'(u_0) = 0$).

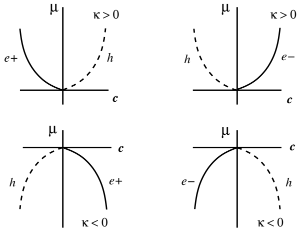
$$\mathscr{K} = \frac{2\mathscr{A}_u}{6}\omega_{kkk} = \Delta S = \frac{h_0^3}{3}\,.$$

Dynamical systems interpretation of $\mathscr{K}$

Write the universal KdV as a first order system

$$\begin{aligned} -\mathscr{A}_k q_T - \mu_X &= 0 \\ \mathscr{A}_k \phi_T - p_X &= \mu - \tfrac{1}{2}\mathscr{B}_{kk} q^2 \\ \phi_X &= q \\ q_X &= sp \quad s = -\mathscr{K} \,, \end{aligned}$$

Steady RE: $\phi(X) = cX + \phi_0$. There are 4 classes depending on the sign of s and $\kappa = \mathscr{B}_{kk}$. $e^{\pm}$ are the Krein signatures of the "incoming" periodic solution.



Emergence of KdV based on modulation of RE

Consider a PDE with Lagrangian

$$\mathscr{L}(Z) = \int\int L(Z_t, Z_x, Z)\,\mathrm{d}x\mathrm{d}t\,,$$

with Euler-Lagrange equation

$$\frac{\partial}{\partial t}\left(\frac{\delta L}{\delta Z_t}\right) + \frac{\partial}{\partial x}\left(\frac{\delta L}{\delta Z_x}\right) - \frac{\delta L}{\delta Z} = 0\,.$$

Symmetry, Noether, CLAW, relative equilibrium

$$Z(x,t) = \widehat{Z}(\theta,k)\,, \quad \theta = kx + \theta_0\,,$$

associated with symmetry and a conservation law

$$A_t + B_x = 0\,.$$

Modulate: $Z(x,t) = \widehat{Z}(\theta + \varepsilon\psi, k + \varepsilon^2 q) + \varepsilon^3 W(\theta, X, T, \varepsilon)$

Modulate and expand in powers of ε

Modulate the family of RE: $\widehat{Z}(\theta, k)$

$$Z(x,t) = \widehat{Z}(\theta + \varepsilon\psi, k + \varepsilon^2 q) + \varepsilon^3 W(\theta, X, T, \varepsilon)\,,$$

with $\psi(X, T, \varepsilon)$, $q(X, T, \varepsilon)$ and scaling

$$T = \varepsilon^3 t \quad \text{and} \quad X = \varepsilon x\,.$$

With $W = W_1 + \varepsilon W_1 + \varepsilon^2 W_2 + \cdots$, the ε^n terms give

$$\begin{aligned} \varepsilon^2 : &\quad q = \psi_X \\ \varepsilon^3 : &\quad \text{equation for } W_1 \text{ solvable iff} \quad \mathscr{B}'(k) = 0 \\ \varepsilon^4 : &\quad \text{gives equation for } W_2\,, \end{aligned}$$

Fifth order terms

and at ε^5,

$$\mathbf{L}W_3 = \left(\mathbf{M}\widehat{Z}_k + \mathbf{J}\zeta\right) q_T + \left(\mathbf{J}\widehat{Z}_{kk} + \mathbf{J}(\widehat{\xi}_3)_\theta - D^3 S(\widehat{Z}^o)(\widehat{Z}_k, \widehat{\xi}_3)\right) q q_X + \mathbf{J}\widehat{\xi}_4 q_{XXX} \,.$$

Solvability gives (after a few pages!)

$$\boxed{2\mathscr{A}'(k) q_T + \mathscr{B}''(k) q q_X + \mathscr{K} q_{XXX} = 0}$$

Symmetry and conservation laws

Lagrangian + symmetry + Noether's Theorem $\Rightarrow$ CLAW

Consider a classical finite-dimensional Hamiltonian system

$$\mathbf{J}Z_t = \nabla H(Z)\,.$$

Suppose it has a one-parameter symmetry with action and generator

$$G_\theta Z \quad \text{and} \quad \frac{d}{d\theta}G_\theta Z\Big|_{\theta=0} = \widehat{Z}_\theta\,.$$

Then invariance of H: $H(G_\theta Z) = H(Z)$ gives $A_t = 0$ with

$$A(Z) = \tfrac{1}{2}\langle \mathbf{J}\widehat{Z}_\theta, Z\rangle \quad \Rightarrow \quad \nabla A(Z) = \mathbf{J}\widehat{Z}_\theta\,.$$

Hence

$$\mathscr{A}_c = \langle \nabla A(\widehat{Z}), \widehat{Z}_c\rangle = \langle \mathbf{J}\widehat{Z}_\theta, \widehat{Z}_c\rangle\,.$$

Lagrangian → Hamiltonian → Multisymplectic

Start with a Lagrangian formulation

$$\mathscr{L}(U) = \int\int L(U_t, U_x, U)\,\mathrm{d}x\mathrm{d}t\,,$$

Legendre transform $V = \delta L/\delta U_t$, giving a Hamiltonian formulation

$$\mathscr{L}(W) = \int\int \left[\tfrac{1}{2}\langle \mathbf{M}W_t, W\rangle - H(W_x, W)\right]\mathrm{d}x\mathrm{d}t\,,$$

Legendre transform again $P = \delta L/\delta W_x$, giving a multisymplectic Hamiltonian formulation

$$\mathscr{L}(Z) = \int\int \left[\tfrac{1}{2}\langle \mathbf{M}Z_t, Z\rangle + \tfrac{1}{2}\langle \mathbf{J}Z_x, Z\rangle - S(Z)\right]\mathrm{d}x\mathrm{d}t\,,$$

two symplectic structures and a Hamiltonian function $S(Z)$.

Symmetry and conservation laws

Starting with

$$\mathbf{M}Z_t + \mathbf{J}Z_x = \nabla S(Z)\,, \quad Z \in \mathbb{R}^n\,,$$

with a one-parameter symmetry group

$$G_\theta Z \quad \text{and} \quad \frac{d}{d\theta} G_\theta Z \bigg|_{\theta=0} = \xi Z\,,$$

and associated conservation law

$$A_t + B_x = 0\,,$$

the invariance of $S(Z)$ in the multisymplectic setting gives

$$\nabla A(Z) = \mathbf{M}\frac{d}{d\theta} G_\theta Z \bigg|_{\theta=0} \quad \text{and} \quad \nabla B(Z) = \mathbf{J}\frac{d}{d\theta} G_\theta Z \bigg|_{\theta=0}$$

Moving frame

Moving frame: $x \mapsto x - ct$,

$$\mathbf{M}Z_t + \big(\mathbf{J} - c\mathbf{M}\big)Z_x = \nabla S(Z)\,, \quad Z \in \mathbb{R}^n\,.$$

Let $\widehat{\mathbf{J}}(c) = \mathbf{J} - c\mathbf{M}$

$$\mathbf{M}Z_t + \widehat{\mathbf{J}}(c)Z_x = \nabla S(Z)\,, \quad Z \in \mathbb{R}^n\,.$$

and proceed as before.

$$\text{Criticality:} \quad \mathscr{B}_k - c\mathscr{A}_k = 0\,,$$

and emergent KdV is replaced by

$$2\mathscr{A}_k q_T + (\mathscr{B}_{kk} - c\mathscr{A}_{kk})qq_X + \mathscr{K}(c)q_{XXX} = 0$$

with $\mathscr{A}(k,c)$ and $\mathscr{B}(k,c)$.

Classical multiple scales vs modulation

Given a basic state, represented by $\widehat{Z}(\theta, k)$, with $\theta = kx + \theta_0$, a multiple scales perturbation would be

$$Z(x,t) = \widehat{Z}(\theta, k) + \varepsilon^d \widetilde{W}(\theta, X, T, \varepsilon)\,,$$

with slow space and time scales $T = \varepsilon^\alpha t, X = \varepsilon^\beta x$.
Include modulation of the basic state

$$Z(x,t) = \widehat{Z}(\theta + \varepsilon^a \psi, k + \varepsilon^b q) + \varepsilon^d W(\theta, X, T, \varepsilon)\,,$$

with slow space and time scales $T = \varepsilon^\alpha t, X = \varepsilon^\beta x$.

They are equivalent ... but the second formulation encodes info about basic state.

History

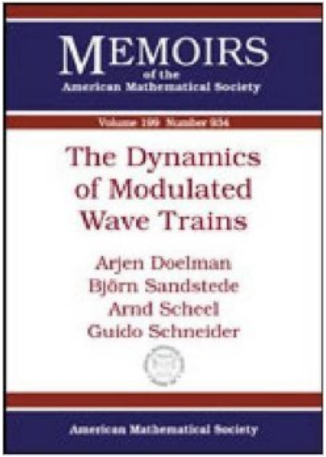
- homoclinic bifurcation from RE – clearly related to steady KdV $\alpha'' q q_T - s q_{TTT}$ with T replaced by X.
- Must be a generalization to KDV

$$[?] q_T + \kappa q q_X + \mathscr{K} q_{XXX} = 0$$

  where κ is some curvature.
- ad hoc approach: nf transforms in space then in time - works: coefficient is related to CLAW density★
- Not completely satisfactory approach ....
- Roger Grimshaw: "... can you do it with a solvabity condition?"

★ TJB [2012] PRSLA, Emergence of DSWs

History 2



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History 3: Doelman et al (2009)

Start with reaction diffusion system

$$\mathbf{U}_t = \mathbf{D}\mathbf{U}_{xx} + \mathbf{f}(\mathbf{U}) , \quad \mathbf{U} \in \mathbb{R}^n$$

with period solution $\widehat{\mathbf{U}}(\theta, k)$, $\theta = kx - \omega t$.
Modulate

$$\mathbf{U}(x,t) = \widehat{\mathbf{U}}(\theta + \phi, k + \varepsilon q) + \varepsilon^2 W ,$$

with $X = \varepsilon x$, $T = \varepsilon^2 t$. Expansion and solvability lead to

$$q_T + a q_X = \nu q_{XX} \quad \text{(Burger's equation)}$$

Kivshar (1990): use modulation for defocussing NLS to KdV

$$\Psi(x,t) = (\Psi_0 + \varepsilon^2 q(X, T, \varepsilon))\mathrm{e}^{\mathrm{i}(kx + \varepsilon\phi(X,T,\varepsilon))}$$

with $T = \varepsilon^3 t$.

History 4: other influences

- Hall & Hewitt (1998) modulation of shear flows in Navier-Stokes leading to Burger's equation coupled to mean flow
- Whitham (1965) modulation of periodic travelling waves leading to conservation of wave action
- Kivshar (1990) modulation of plane waves
- Grimshaw (2012) Madelung transformation, solvability condition

Combine: (a) generalize scaling ansatz ; (b) association with symmetry & RE; (c) Lag/Ham/MSS setting; (d) use of geometry of RE and conservation laws; (e) curvature & coefficients.