Risk Sharing and Contagion in Networks

Antonio Cabrales    Piero Gottardi    Fernando Vega Redondo

UCL, EUI, Bocconi

Systemic Risk Workshop, Cambridge, August 2014
Aim to investigate the effects of different forms of interconnections among (financial) firms, in particular for the capacity of the system to withstand shocks.

Will determine the optimal pattern of connections and study whether this can be sustained in equilibrium.

Key trade-off: More interconnections
⇒ higher levels of insurance, but also
⇒ higher risk of contagion (large shocks can generate widespread default in the system)
Introduction: Main Features of the Model

As in various other papers, also at this workshop:

- Network with nodes = financial firms:
  
  • Link (direct or indirect) among two firms: they are in a situation of mutual (direct or indirect) exposure

  • Degree of exposure of a firm to another firm: depends on number of linkages the firm has and the distance between them.

- When a random shock hits the income of a firm:
  
  • all firms directly or indirectly linked to the firm hit must bear a part of the shock, increasing in their exposure to the firm
  
  • firms whose assets' income falls below their liabilities must default, and this is costly
Tractable specification, allows to compare performance of network structures that differ in two main dimensions:

i. Degree of *segmentation* (number of disjoint components in which system is divided).

ii. *Relative density* of connections (extent to which firms are directly or indirectly linked among them)

- Address two main questions:
  1. How does the optimal network structure vary with the stochastic structure of shocks?
  2. What is the relationship between optimality and incentives of individual firms (stability)?
Introduction: Main Results

- Trade-off insurance/contagion clearly emerges:

  ia) Optimal segmentation maximal when shock distribution exhibits fat tails, minimal with thin tails;

  ib) Intermediate levels of segmentation optimal with sufficient probability mass both on large and small shocks

  ii) Maximal connectivity within a component not always optimal: sparser structures (rings) optimal for some distribution of shocks (low density preferred to more segmentation as mechanism to limit contagion).

  iii) With heterogeneous firms: optimality requires perfect assortativity (homogeneous components).
Introduction: Main Results

Potential conflict Equilibrium/Optimality:
Equilibrium typically requires asymmetry in size of components, not optimal.
Two main strands:

1. Detailed microfoundation of linkages, simple network and shock structures:
   Allen and Gale (2000), Freixas et al. (2000), Allen et al. (2011), ...

2. Random (large) networks, simple contagion mechanics
   Nier et al. (2007), Leitner (2005), Blume et al. (2011), ...

More recently: Elliott et al. (2013), Acemoglu et al. (2013), Glasserman and Young (2013)

Optimality and pairwise stability: Bramoullé Kranton (2007)
The Environment

- $N$ financial firms, ex ante identical (for now)

- At any point in time each firm has a - risky - investment opportunity, to be financed with riskless deposits/bonds.

- Gross return on investment is a random variable:
  - with prob. $1 - q$, normal return: $R$
  - with prob. $q$, a (random) shock hits: $R - \tilde{L}$, with cdf $\Phi(L)$.

- If unable to make due payment to investors, firm defaults (high cost)

- Risk neutral investors: require expected gross payment $r$

  $\Rightarrow$ interest rate $M = \frac{r}{1-\varphi}$ with $\varphi$ prob. of default (endogenous).
Project is viable, but if firm relies on its own resources, can’t repay depositors and hence must default if a shock (of any size) occurs.

Each firm can establish linkages with one or more other firms:

Return on firms’ assets becomes *linear function* of return on own and partners’ projects:

\[ \mathbf{V} = \mathbf{AR} \]

where the matrix \( \mathbf{A} = [a_{i,j}] \) describes the pattern of exposure in the system.

*Linkages:*

- *direct*: exchange of a fraction of its assets with (the same amount of) assets of one or more other firms.

- *indirect*: formed through repeated rounds of exchanges (*securitization*) with some firms.
Note: Default of a firm has no direct implication for solvency of other firms. Mutual exposure/risk of contagion comes from cross ownership of firms’ assets (not mutual lending) \( \implies \) linearity (w.r.t. shocks) of the map determining value of firms’ assets

- Shocks are rare, at most one firm is hit by a shock.

Thus risk sharing is beneficial, allowing full insurance against small shocks. But diversification exposes firm to risk of contagion when a big shock hits.
**Question:** What is the pattern of linkages that allows to maximize welfare (that is, minimize the probability of default of a firm)?

\[
\min_A \varphi(A; \Phi(.)) = q \pi_b \Pr \{ \alpha(R - L_b) + (1 - \alpha)R < M \} + q \pi_b \sum_{j \neq i}^N \Pr \{ R - M < a_{ij} L_b \}
\]

s.t. \( M = r(1 - \varphi) \)
- In the situation considered, autarky never optimal.

- Compare network structures involving the same ’externalization’ of risk: $a_{ii} = \alpha$ for all $i$

Different network structures have then different implications only for the ability of a firm to survive when indirectly hit by a large $b$ shock (shock hits another firm to whom firm is connected).
Network Structures: segmentation

Will consider structures that differ in two main dimensions:

1. **Degree of segmentation**:
   number of disjoint components
   (component $i$ has $K_i + 1$ firms, directly or indirectly linked among them).

   - $K_i = 1$: maximal segmentation, provides maximal insulation from shocks, but also maximal exposure to shocks hitting a firm to whom the firm is connected.
   - $K = N - 1$: fully connected system
     Minimal exposure to shocks hitting connected firms. But risk of generalized default in the system (when a large shocks $L$ hits).
2. **Density of internal connections:**
   fraction of direct vs. indirect linkages within each component.
   Focus on two polar cases:

   i) *Completely connected components*: only direct linkages

   In a component of size $K + 1$ each firm ends up with a fraction $\alpha$ of its original assets and $(1 - \alpha)/K$ of assets of each of the other firms.
ii) *Minimally connected components (rings)*:

- Each firm is directly linked with two other firms.

- Exposure of a firm $i$ to any other firm $j$ ($a_{ij}$) decreases with distance between $i$ and $j$. 
Proposition 1. Let $L$ be Pareto distributed on $[1, \infty)$ ($\Phi(L) = 1 - 1/L^\gamma$).

- The optimal degree of segmentation both for the ring and the complete structures is
  i. maximal ($K^* = 1$) if $0 < \gamma < 1$
  ii. minimal ($K^* = N - 1$) if $\gamma > 1$

- Complete dominate ring structures for all $\gamma$.

When distribution of shocks exhibits fat tails, defaults minimized by minimizing linkages.
With thin tails, optimal to have a single connected component.
Proposition 2. Let $\Phi(L)$ be a mixture of two Pareto distributions with $\gamma > 1$ and $\gamma' < 1$, with weights $p$ and $1 - p$. Then for an open set of values of $p$, the optimal pattern of segmentation for the complete structure is symmetric, with components of intermediate size $1 < K^* < N - 1$.

- For ring structures numerical analysis yields similar results, though optimal size of components is larger (compensate lower density of connections with larger size).
- Complete structure still better than ring.
Proposition 3. Let $\Phi(L)$ be a mixture of a Pareto distribution with \(\gamma \in (1, 2)\), with weight $p$, and a Dirac distribution with all mass on $\bar{L} = 2(N - 1) + 1$.

Then if $N$ is large enough and

\[
\frac{(1 - p)}{p} < (\gamma - 1) \left(\frac{1}{2(N - 1)}\right)^\gamma,
\]

the optimal financial structure is a single component ring network.

- Pareto distribution has thin tails $\Rightarrow$ single component optimal in complete structure.

- But with low probability $(1 - p)$ big shock $\bar{L}$ hits, causing default of everybody in single complete component, while with sparser connections (ring) some firms survive.
Stability and optimality

Is optimality consistent with firms’ incentives to establish linkages?

- *Coalition Proof Equilibrium (CPE)*: no subset of firms can improve (lower their default probability) by deleting or adding linkages.

**Proposition 4.** Let $\Phi(L)$ be a mixture of two Pareto distributions with $\gamma > 1$ and $\gamma' < 1$ such that the optimal structure is completely connected with $1 < K^* < N - 1$ (as in Prop. 2). Generically the optimal structure is not supportable as a CPE.

- Optimality requires all components of same size.
- CPE requires all but one component to have individually optimal size $\hat{K} \in \arg\min D_c(K)$, minimizing expected defaults in the component (generically $\neq K^*$), and one, strictly smaller component.
Stability and optimality (II)

More specifically:

- A CPE structure:
  - cannot have any complete component with $K > \hat{K}$: a subset of its members would want to delete some links (get smaller).
  - cannot have more than one component with $K < \hat{K}$: a subset of firms in one of these components would benefit by deleting links and joining another component.

- At the CPE structure, no firms in components of size $\hat{K}$ want to deviate. Firms in smaller component cannot deviate (won’t be accepted).
firms now differ in the distribution of the $b$ shock, $F_l$, $l = 1, \ldots, n$.

What is the optimal composition of components? Should matching be assortative or not? Focus here on completely connected structures.

**Proposition 5.** The optimal segmentation structure has homogeneous components, where firms are all of the same type, and size

$\hat{K}_{N_l} \in \arg \min D_{F_l} (K)$, for $l = 1, \ldots, n$.

key feature: expected number of indirect defaults in a component only depends on the type $l$ of the firm directly hit and the size of the component, not on the types of other firms in component.
Firms can be of different 'sizes': there are also larger firms, of size $\beta$ (assets and liabilities multiplied by $\beta$, size affects the probability of being hit by a shock - $\beta q$ - not their distribution)

A large firm exchanges assets with a small firm in proportion $1/\beta$ to 1.

**Proposition 6.** The optimal segmentation structure has homogeneous components, where firms are all of the same size.
Conclusions

- Have considered a stylized model to study trade-off of forming linkages: benefits of risk sharing vs. costs of contagion.
- If shocks are typically large (small), the optimal configuration exhibits maximal (minimal) segmentation in complete components.
- For richer shock patterns, the optimal configuration may involve intermediate levels of segmentation and/or sparse connectivity.
- If firms are asymmetric in size or shock distribution, assortativity in size/type.
- Social optimality and individual incentives are typically in conflict: in equilibrium we have both too large and too small components.