Default Cascades in Financial Networks

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joint with Rama Cont and Andreea Minca

Systemic Risk : Models and Mechanisms - 29 August 2014
Network structures of banking systems

Figure: Austria: scale-free structure (Boss et al. 2004), Switzerland: sparse and centralized structure (Müller 2006).
Fundamental Questions

- How does the default of a bank affect its counterparties, counterparties of counterparties, ... (domino effect) ?
- Can the default of one or few institutions generate a macro-cascade / large-scale instability of financial network ?
- How do the answers to the above depend on network structure ?

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- How does the default of a bank affect its counterparties, counterparties of counterparties, ... (domino effect)?
- Can the default of one or few institutions generate a macro-cascade / large-scale instability of financial network?
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Outline

1 Network Model of Default Contagion

2 Asymptotic Analysis and Limit Theorems

3 Stress Tests

4 Conclusion
1. Network Model of Default Contagion

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3. Stress Tests

4. Conclusion
Financial Network

Can be modeled as a **weighted directed graph** \((V, e, c)\) on the vertex set \(V = \{1, \ldots, n\}\):

- \(n\) vertices represent financial market participants;
- \(e(i, j)\) is the **exposure** of \(i\) to \(j\);
- \(c(i)\) is the **capital buffer** of institution \(i\) which absorbs market losses.

Suppose a loss \(\varepsilon\) in the assets of institution \(i\): 
\[
c(i) \rightarrow (c(i) - \varepsilon)_+. \]

**Solvency condition**: 
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c(i) > 0. \]
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## Balance Sheet

<table>
<thead>
<tr>
<th>Assets</th>
<th>Liabilities</th>
</tr>
</thead>
<tbody>
<tr>
<td>Interbank assets $\sum_j e(i,j)$</td>
<td>Interbank liabilities $\sum_j e(j,i)$</td>
</tr>
<tr>
<td>Deposits $D(i)$</td>
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</tr>
<tr>
<td>Other assets $x(i)$</td>
<td>Equity $c(i)$</td>
</tr>
</tbody>
</table>

**TABLE:** Stylized balance sheet of bank $i$.

### Balance sheet equation:

$$c(i) = x(i) + \sum_{j \neq i} e(i,j) - \sum_{j \neq i} e(j,i) - D(i).$$
Insolvency Cascades

The set of initially insolvent institutions is

$$\mathcal{D}_0(e, c) = \{ i \in V \mid c(i) \leq 0 \}.$$ 

The default of a market participant $j$ affects its counterparties:
- Creditors lose a fraction $(1 - R)$ of their exposure;
- This leads to default of $i$ if

$$c(i) < (1 - R)e(i, j).$$

$R \approx 0$ in the short term (liquidation takes time) and Insolvency occurs if

$$\text{Loss}(i) > c(i).$$
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Example
Example
Example
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Contagion lasts 3 rounds.

- **Fundamental defaults**: \{a\}.
- **Default cluster**: \{a, b, c, d\}.
- **Final number of defaults**: \(N_{\text{def}}(e, c) = 4\).
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Observation

Real financial networks are large (thousands of nodes).

The Brazilian interbank network

The structure of interbank network

- **Number of nodes** $n \approx 2500$
- **Heterogeneity in number of debtors/creditors**
- **Heterogeneous exposures sizes**

Source: Cont et al. (2010)
Degree sequences

- **Financial system**: $(e_n, c_n)$ with the vertex set $[n] := \{1, \ldots, n\}$;
- The **out-degree** of node $i$ is given by its number of debtors

$$d_n^+(i) = \#\{j \mid e_n(i, j) > 0\}$$

and its **in-degree** is given by its number of creditors

$$d_n^-(i) = \#\{j \mid e_n(j, i) > 0\};$$

- The **empirical distribution of the degrees**:

$$\mu_n(j, k) := \frac{1}{n} \#\{i : d_n^+(i) = j, d_n^-(i) = k\}.$$ 

- The **final fraction of defaults**:

$$\alpha_n(e_n, c_n) = \frac{|N_{def}(e_n, c_n)|}{n}.$$
Random financial network

We introduce a random network ensemble of which the network \((e_n, c_n)\) may be considered as a typical sample.

\(G_n(e_n)\) : The set of all weighted directed graphs with degree sequence \(d_n^+, d_n^-\) s.t. for all \(i\), the set of exposures is given by the non-zero elements of line \(i\) in the exposure matrix \(e_n\).

We define \(E_n\) as a random financial network uniformly distributed on \(G_n(e_n)\).

We endow the nodes in \(E_n\) with the capital buffers \(c_n\).

\[
\#\{j \in [n], E_n(j, i) > 0\} = d_n^-(j), \\
\#\{j \in [n], E_n(i, j) > 0\} = d_n^+(i), \\
\{E_n(i, j), E_n(i, j) \neq 0\} = \{e_n(i, j), e_n(i, j) \neq 0\} \quad \mathbb{P} - \text{a.s.,}
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for all \(i = 1, \ldots, n\).
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\]

for all \(i = 1, \ldots, n\).
The random network $E_n$ has the same distribution as the random multigraph given by configuration model conditional on it being simple.
Default threshold

$\Theta(i, \tau)$: measures how many counterparty defaults $i$ can tolerate before it becomes insolvent, if its counterparties default in the order specified by $\tau$:

$$\Theta(i, \tau) := \min\{k \geq 0, c(i) < \sum_{j=1}^{k} (1 - R) e(i, \tau(j))\}.$$ 

We let

$$p_n(j, k, \theta) := \frac{\#\{(i, \tau) \mid d_n^+(i) = j, d_n^-(i) = k, \Theta(i, \tau) = \theta\}}{n \mu_n(j, k) j!}.$$
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Natural assumptions

There exists a

1. probability distribution \( \mu \) with average \( \lambda \in (0, \infty) \) s.t.

\[
\mu_n(j, k) \to \mu(j, k) \text{ as } n \to \infty;
\]

2. function \( p \) s.t.

\[
p_n(j, k, \theta) \to p(j, k, \theta), \text{ as } n \to \infty;
\]

for all \( j, k, \theta \in \mathbb{N} \).

**Ex:** exposures and capitals i.i.d. or exchangeable arrays.
The asymptotic size of contagion

We can completely describe the asymptotic behaviour of contagion:

**Theorem (A., Cont, Minca)**

There exists $l : [0,1] \rightarrow [0,1]$ s.t. if $\pi^*$ is the smallest fixed point of $l$, we have

1. If $\pi^* = 1$, then asymptotically all nodes default during the cascades;
2. If $\pi^* < 1$ and furthermore $\pi^*$ is a stable fixed point of $l$, then

$$
\alpha_n(E_n, c_n) \xrightarrow{p} J(\pi^*) := \sum_{j,k} \mu(j,k) \sum_{\theta=0}^j p(j,k,\theta) P(\text{Bin}(j, \pi^*) \geq \theta).
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$$
l(\pi) := \sum_{j,k} \frac{k \mu(j,k)}{\lambda} \sum_{\theta=0}^j p(j,k,\theta) P(\text{Bin}(j, \pi) \geq \theta).
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$$\alpha_n(E_n, c_n) \xrightarrow{p} J(\pi^*):= \sum_{j,k} \mu(j, k) \sum_{\theta=0}^{j} p(j, k, \theta) \mathbb{P}(\text{Bin}(j, \pi^*) \geq \theta).$$

$$I(\pi) := \sum_{j,k} \frac{k \mu(j, k)}{\lambda} \sum_{\theta=0}^{j} p(j, k, \theta) \mathbb{P}(\text{Bin}(j, \pi) \geq \theta).$$
Resilience condition

Corollary

If

\[ 1 - \sum_{j,k} jk \frac{\mu(j,k)}{\lambda} p(j, k, 1) > 0, \]

then w.h.p. (with probability \( \to 1 \) as \( n \to \infty \)), the default of a finite set of nodes cannot trigger the default of a positive fraction of the financial network.

We say \( i \to j \) is a contagious link if the default of \( j \) generates the default of \( i \).

\( p(j, k, 1) \) : proportion of contagious exposures belonging to nodes with degree \((j, k)\).
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Amplification of default by contagion

Assume that the resilience condition is satisfied:

- When a fraction $\varepsilon$ of all nodes represent fundamental defaults, i.e. $p(j, k, 0) = \varepsilon$ for all $j, k$:

$$\alpha_n(E_n, c_n) \xrightarrow{p} \varepsilon \left(1 + \frac{\sum_{j,k} j \mu(j, k) p(j, k, 1)}{1 - \sum_{j,k} \frac{\mu(j, k)jk}{\lambda} p(j, k, 1)}\right) + o(\varepsilon).$$

- When $p(d^+, d^-, 0) = \varepsilon$ and $p(j, k, 0) = 0$ for all $(j, k) \neq (d^+, d^-)$:

$$\alpha_n(E_n, c_n) \xrightarrow{p} \varepsilon \mu(d^+, d^-) \left(1 + \frac{d^-}{\lambda} \frac{\sum_{j,k} \frac{\mu(j, k)jk}{\lambda} p(j, k, 1)}{1 - \sum_{j,k} \frac{\mu(j, k)jk}{\lambda} p(j, k, 1)}\right) + o(\varepsilon).$$
The skeleton of contagious links

Theorem (A., Cont, Minca)

If the resilience condition fails:

\[ 1 - \sum_{j,k} jk \frac{\mu(j,k)}{\lambda} p(j,k,1) < 0, \]

then w.h.p. there exists a strongly connected set of nodes representing a positive fraction of the financial system s.t. any node belonging to this set can trigger the default of all nodes in the set.

- A decentralized recipe for regulating systemic risk;
- No need to monitor/know the entire network of counterparty exposures but simply the skeleton of contagious links.
The skeleton of contagious links

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Related to bootstrap percolation

**Bootstrap percolation process** with **activation threshold** an integer $\theta \geq 2$:

- $A_0 \subseteq V$: Initially infected vertices, selected deterministically or randomly;
- If an uninfected vertex has at least $\theta$ of its neighbours infected, then it also becomes infected and remains so forever.

$A_f$ denotes the final infected set.

Bootstrap Percolation in Power-Law Random Graphs

In power-law random graphs with parameter \(2 < \beta < 3\) and maximum degree \(d_{\text{max}} = \Theta(n^\zeta)\):

\[
a_c(n) = n^{\frac{\theta(1-\zeta)+\zeta(\beta-1)-1}{\theta}} = o(n).
\]

**Theorem (A., Fountoulakis 2014)**

If \(|\mathcal{A}_0| \ll a_c(n)\), then w.h.p. \(\mathcal{A}_f = \mathcal{A}_0\); If \(|\mathcal{A}_0| \gg a_c(n)\), then there exists \(\varepsilon > 0\) such that w.h.p. \(|\mathcal{A}_f| > \varepsilon n\).
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These analytical results may be used for stress-test the resilience of a banking system, without the need for large scale simulation.

**Stress scenario**: apply a common macro-shock $Z$, measured in % loss in asset value, to all balance sheets in network;

Analytical result allow to compute fraction of defaults as function of $Z$;

Network remains resilient (nomacro-cascade) as long as

$$\sum_{j,k} \frac{\mu(j,k)}{\lambda} p_Z(j, k, 1) < 1 \iff Z < Z^*.$$
Numerical results

**Figure:** (a) The distribution of in-degree has a Pareto tail with exponent 2.19, (b) The distribution of the out-degree has a Pareto tail with exponent 1.98, (c) The distribution of the exposures (tail-exponent 2.61).
The finite sample

In a finite network the resilience condition becomes

\[
\frac{1}{m_n} \sum_i d_n^-(i) q_n^{(Z)}(i) < 1,
\]

with \( m_n \) the total number of links in the network and \( q_n^{(Z)}(i) \) : the number of ’contagious’ links of bank \( i \).
Phase transition

Figure: Function $I$ for increasing magnitude of the macroeconomic shock $Z$.
Phase transition
Amplification of the number of defaults in a Scale-Free Network.
The impact of heterogeneity

Amplification of the number of defaults in a Scale-Free Network (in and out-degree of the scale-free network are Pareto distributed with tail coefficients 2.19 and 1.98 respectively, the exposures are Pareto distributed with tail coefficient 2.61), the same network with equal weights and an Erdös Rényi Network with equal exposures $n = 10000$. 

Hamed Amini (EPFL)

Default Cascades

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Conclusions

- We have established asymptotic results linking the cascading behavior of a financial network to the topology of network.
- These results hold for a model flexible enough to accommodate interpretations as insolvency cascade of illiquidity cascade.
- The regulator can efficiently contain insolvency contagion by focusing on fragile nodes, especially those with high connectivity and over-exposed.
- In particular, higher capital requirements could be imposed on them to reduce their number of contagious links.
THANK YOU!

References

