Spectral Theory of Orthogonal Polynomials

Periodic and Ergodic Spectral Problems

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Lecture 1: Introduction and Overview

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- Lecture 2: Szegő Theorem for OPUC
- Lecture 3: Three Kinds of Polynomial Asymptotics
- Lecture 4: Potential Theory
- Lecture 5: Isospectral Tori
- Lecture 6: Fuchsian Groups
- Lecture 7: Chebyshev Polynomials, I
- Lecture 8: Chebyshev Polynomials, II

References

[OPUC] B. Simon, *Orthogonal Polynomials on the Unit Circle, Part 1: Classical Theory*, AMS Colloquium Series **54.1**, American Mathematical Society, Providence, RI, 2005.

[OPUC2] B. Simon, *Orthogonal Polynomials on the Unit Circle, Part 2: Spectral Theory*, AMS Colloquium Series, **54.2**, American Mathematical Society, Providence, RI, 2005.

[SzThm] B. Simon, *Szegő's Theorem and Its Descendants: Spectral Theory for L^2 Perturbations of Orthogonal Polynomials*, M. B. Porter Lectures, Princeton University Press, Princeton, NJ, 2011.

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- Isospectral manifold for the harmonic oscillator

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For example, the domain of definition of the harmonic oscillator isospectral "manifold" is unknown. It is not even known if it is connected!

OPs

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OPs also enter in many application—both specific polynomials and the general theory.
Indeed, my own interest came from studying discrete Schrödinger operators on $\ell^2(\mathbb{Z})$

$$\big(Hu\big)_n = u_{n+1} + u_{n-1} + V u_n$$

and the realization that when restricted to $\mathbb{Z}_+$, one had a special case of OPRL.

OPRL basics

μ will be a probability measure on $\mathbb{R}$. We'll always suppose that μ has bounded support $[a, b]$ which is not a finite set of points. (We then say that μ is non-trivial.) This implies that $1, x, x^2, \ldots$ are independent since $\int |P(x)|^2 \, d\mu = 0 \Rightarrow \mu$ is supported on the zeroes of P.

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Apply Gram Schmidt to $1, x, \ldots$ and get monic polynomials

$$P_j(x) = x^j + \alpha_{j,1} x^{j-1} + \ldots$$

and orthonormal (ON) polynomials

$$p_j = P_j / \|P_j\|$$

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One difference from the case of $\mathbb{R}$, the linear combination of $\{x^j\}_{j=0}^{\infty}$ are dense in $L^2(\mathbb{R}, d\mu)$ by Weierstrass. This may or may not be true if $\operatorname{supp}(d\mu) \not\subset \mathbb{R}$.

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If $d\mu = d\theta/2\pi$ on $\partial\mathbb{D}$, the span of $\{z^j\}_{j=0}^{\infty}$ is not dense in L^2 (but is only H^2). Perhaps, surprisingly, we'll see later that there are measures $d\mu$ on $\partial\mathbb{D}$ for which they are dense (e.g., μ purely singular).

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More significantly, the argument we'll give for our recursion relation fails if $\operatorname{supp}(d\mu) \not\subset \mathbb{R}$.

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Since P_k is monic and $\{P_j\}_{j=0}^{k+1}$ span polynomials of degree at most $k+1$, we have

$$xP_k = P_{k+1} + \sum_{j=0}^{k} B_{k,j}\, P_j$$

Clearly

$$B_{k,j} = \langle P_j, xP_k\rangle / \|P_j\|^2$$

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If $j = k-1$, $\langle P_{k-1}, xP_k \rangle = \langle xP_{k-1}, P_k \rangle = \|P_k\|^2$.

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Thus $(P_{-1} \equiv 0)$; $\{a_j\}_{j=1}^{\infty}$, $\{b_j\}_{j=1}^{\infty}$: *Jacobi recursion*

$$xP_N = P_{N+1} + b_{N+1}P_N + a_N^2 P_{N-1}$$

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This, in turn, implies $p_n = P_n/a_1 \dots a_n$ obeys

$$xp_n = a_{n+1}p_{n+1} + b_{n+1}p_n + a_n p_{n-1}$$

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In the orthonormal basis $\{p_n\}_{n=0}^{\infty}$, multiplication by x has the matrix

$$J = \begin{pmatrix} b_1 & a_1 & 0 & 0 & \dots \\ a_1 & b_2 & a_2 & 0 & \dots \\ 0 & a_2 & b_3 & a_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

called a *Jacobi matrix*.

Favard's Theorem

Since

$$b_n = \int x p_{n-1}^2(x)\, d\mu, \quad a_n = \int x p_{n-1}(x) p_n(x)\, d\mu$$

$\operatorname{supp}(\mu) \subset [-R, R] \Rightarrow |b_n| \le R,\ |a_n| \le R.$

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Conversely, if $\sup_n \big(|a_n| + |b_n|\big) = \alpha < \infty$, J is a bounded matrix of norm at most 3α. In that case, the spectral theorem implies there is a measure $d\mu$ so that

$$\langle (1, 0, \ldots)^t, J^\ell (1, 0, \ldots)^t \rangle = \int x^\ell d\mu(x)$$

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If one uses Gram-Schmidt to orthonormalize $\{J^\ell (1, 0, \dots)^t\}_{\ell=0}^\infty$, one finds μ has Jacobi matrix exactly given by J.

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Favard's Theorem. *There is a one–one correspondence between bounded Jacobi parameters*

$$\{a_n, b_n\}_{n=1}^{\infty} \in \big[(0,\infty) \times \mathbb{R}\big]^{\infty}$$

and non-trivial probability measures, μ, of bounded support via:

$$\mu \Rightarrow \{a_n, b_n\} \quad \textit{(OP recursion)}$$

$$\{a_n, b_n\} \Rightarrow \mu \quad \textit{(Spectral Theorem)}$$

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There are also results for μ's with unbounded support so long as $\int x^n \, d\mu < \infty$. In this case, $\{a_n, b_n\} \Rightarrow \mu$ may not be unique because J may not be essentially self-adjoint on vectors of finite support.

OPUC basics

Let $d\mu$ be a non-trivial probability measure on $\partial\mathbb{D}$. As in the OPRL case, we use Gram-Schmidt to define monic OPs, $\Phi_n(z)$ and ON OP's $\varphi_n(z)$.

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In the OPRL case, $P_{n+1} - xP_n \perp \{1, x_1, \dots, x_{n-2}\}$.

OPUC basics

In the OPUC case, $\Phi_{n+1} - z\Phi_n \perp \{z, \ldots, z^n\}$, since

$$\langle z\Phi, z^j \rangle = \langle \Phi, z^{j-1} \rangle$$

if $j \geq 1$.

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In the OPUC case, $\Phi_{n+1} - z\Phi_n \perp \{z, \ldots, z^n\}$, since

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if $j \geq 1$.

In the OPRL case, we used deg $P = n$ and $P \perp \{1, x, \ldots, x^{n-2}\} \Rightarrow P = c_1 P_n + c_2 P_{n-1}$.

In the OPUC case, we want to characterize deg $P = n$, $P \perp \{z, z^2, \ldots, z^n\}$.

OPUC basics

Define * on degree n polynomials to themselves by

$$Q^*(z) = z^n \, \overline{Q\left(\frac{1}{\bar{z}}\right)}$$

(bad but standard notation!) or

$$Q(z) = \sum_{j=0}^{n} c_j z^j \Rightarrow Q^*(z) = \sum_{j=0}^{n} \bar{c}_{n-j} \, z^j$$

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Then, * is unitary and so for deg $Q = n$

$$Q \perp \{1, \ldots, z^{n-1}\} \Leftrightarrow Q = c\,\Phi_n$$

is equivalent to

$$Q \perp \{z, \ldots, z^n\} \Leftrightarrow Q = c\,\Phi_n^*$$

Szegő recursion and Verblunsky coefficients

Thus, we see, there are parameters $\{\alpha_n\}_{n=0}^{\infty}$ (called Verblunsky coefficients) so that

$$\Phi_{n+1}(z) = z\Phi_n - \overline{\alpha}_n \Phi_n^*(z)$$

This is the Szegő Recursion (History: Szegő and Geronimus in 1939; Verblunsky in 1935–36)

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The strange looking $-\bar{\alpha}_n$ rather than say $+\alpha_n$ is to have the α_n be the Schur parameter of the Schur function of μ (Geronimus); also the Verblunsky parameterization then agrees with α_n. These are discussed in [OPUC1].

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Φ_n monic $\Rightarrow$ constant term in Φ_n^* is $1 \Rightarrow \Phi_n^*(0) = 1$.

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This plus $\Phi_{n+1} = z\Phi_n - \bar{\alpha}_n \Phi_n^*(z)$ implies

$$-\overline{\Phi_{n+1}(0)} = \alpha_n$$

i.e., Φ_n determines α_{n-1}.

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$$\|\Phi_{n+1}\|^2 = \rho_n^2 \|\Phi_n\|^2; \quad \rho_n^2 = 1 - |\alpha_n|^2$$

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which implies $|\alpha_n| < 1$ (i.e., $\alpha_n \in \mathbb{D}$) and

$$\|\Phi_n\| = \rho_{n-1} \cdots \rho_0$$

Szegő recursion and Verblunsky coefficients

$$\begin{pmatrix} \varphi_{n+1} \\ \varphi_{n+1}^* \end{pmatrix} = A_n(z) \begin{pmatrix} \varphi_n \\ \varphi_n^* \end{pmatrix} x; \quad A_n = \rho_n^{-1} \begin{pmatrix} z & -\bar{\alpha}_n \\ -\alpha_n\, z & 1 \end{pmatrix}$$

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$$z\Phi_n = \rho_n^{-2}\left[\Phi_{n+1} + \bar{\alpha}_n \Phi_{n+1}^*\right]$$

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Corollary. *All zeros of $\Phi_n^*(z)$ lie in $\mathbb{C} \setminus \overline{\mathbb{D}}$.*

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As β varies in $\mathbb{D}$, all zeros of $\Phi_{n+1}^{(\beta)}$ are trapped in $\mathbb{D}$. QED.

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Then $d\mu_n$ *has Verblunsky coefficients*

$$\alpha_j(d\mu_n) = \begin{cases} \alpha_j^{(0)} & j = 0, \dots, n-1 \\ 0 & j \geq n \end{cases}$$

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The first step of the proof is to show that

$$k, \ell, n \text{ with } k < n + \ell \Rightarrow \int_{z=e^{i\theta}} \bar{z}^k \, z^\ell \, \varphi_n(z) d\mu_n(\theta) = 0$$

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is zero since $\left[\varphi_n^*(z)\right]^{-1}$ is analytic on a neighborhood of $\overline{\mathbb{D}}$ and $\ell + n - k - 1 \geq 0$.

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As we saw, Φ_n determines $\{\alpha_j\}_{j=0}^{n-1}$ and Φ_j by inverse Szegő recursion and $-\overline{\Phi_{j+1}(0)} = \alpha_j$. This shows that

$$\varphi_j(z; d\mu) = \begin{cases} \varphi_j(x) & j = 0, \dots, n \\ z^{j-n} \varphi_n(z) & j = n, n+1, \dots \end{cases}$$

implying the claimed result.

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Given $\{\alpha_j\}_{j=0}^{\infty} \subset \mathbb{D}^{\infty}$, we can form $d\mu_n$ as above. Via $\int \Phi_j(e^{i\theta})d\mu(e^{i\theta}) = 0$, $\{\Phi_j\}_{j=0}^{n}$ determines $\{\int z^j d\mu\}_{j=0}^{n}$ inductively (actually they determine more moments).

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We have thus proven

Verblunsky's Theorem. $\mu \to \{\alpha_j(\mu)\}_{j=0}^\infty$ sets up a 1–1 correspondence between non-trivial probability measures on $\partial\mathbb{D}$ and $\mathbb{D}^\infty$.

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In 2001, Cantero, Moral and Velázquez had the lovely idea of orthonormalizing $\{1, z, z^{-1}, z^2, z^{-2}, \dots\}$ which always produces a complete set. Remarkably, they found that this ON basis can be expressed in terms of suitable $z^\ell \varphi_k$ and $z^\ell \varphi_k^*$.

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One strange aspect of this formula is that for $\ell > 2n-1$, the moments of $d\nu_n$ are infinite!

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Remark. A much stronger result is known (see e.g., Simon [Proc AMS **124** (1996), 3361-3369]); I can be any set and c can be x-dependent.