Spectral Theory of Orthogonal Polynomials

Periodic and Ergodic Spectral Problems

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Lecture 3: Three Kinds of Polynomial Asymptotics

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- Lecture 2: Szegő Theorem for OPUC
- Lecture 3: Three Kinds of Polynomial Asymptotics
- Lecture 4: Potential Theory
- Lecture 5: Isospectral Tori
- Lecture 6: Fuchsian Groups
- Lecture 7: Chebyshev Polynomials, I
- Lecture 8: Chebyshev Polynomials, II

References

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If $f_n(\theta) = \frac{\sin(n+1)\theta}{\sin\theta}$, then $f_{-1} = 0$, $f_0 = 1$, and $f_{n+1} + f_{n-1} = (2\cos\theta) f_n$.

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Thus, by induction, $f_n(\theta)$ is a polynomial in $2\cos\theta$ of degree n, i.e.,

$$f_n(\theta) = p_n(2\cos\theta)$$

where

$$p_{n+1}(x) + p_{n-1}(x) = x p_n(x); \quad p_{-1} = 0, \, p_0 = 1$$

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Since $\sin(k\theta)$ are orthogonal for $\frac{d\theta}{2\pi}$, $f_n(\theta)$ are orthogonal for $\sin^2\theta \frac{d\theta}{2\pi}$ (for normalization on $[0, 2\pi]$).

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But $\theta \mapsto x = 2\cos\theta$ is 2 to 1 from $[0, 2\pi]$ to $[-2, 2]$, so we want to look at $2\sin^2\theta\frac{d\theta}{2\pi}$ on $[0, \pi]$.

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$$d\mu(x) = \frac{1}{2\pi}\sqrt{4 - x^2}\,dx$$

is the orthogonality measure for this problem.

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If $x \notin [-2, 2]$ ($x \in \mathbb{C}$), $e^{\pm i\theta}$ have different rates of growth so one dominates for $\sin(n + 1)\theta/ \sin\theta$ for n large.

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$$\frac{|p_n(x)|}{\left|\frac{x}{2} + \sqrt{1 - \left(\frac{x}{2}\right)^2}\right|^n} \to 1$$

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One question we'll answer in the next lecture is where $\frac{x}{2} + \sqrt{1 - \left(\frac{x}{2}\right)^2}$ comes from.

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One theme theme in these lectures will be to explore when these conditions hold for OPUC/OPRL close to the "free" case ($\alpha_n \equiv 0$ for OPUC; $a_n \equiv 1, b_n \equiv 0$ for OPRL).

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We'll look at this asymptotics away from $\operatorname{supp}(d\mu)$ because on $\operatorname{supp}(d\mu)$, the asymptotics are typically unusual (decay rather than growth for isolated points in $\operatorname{supp}(d\mu)$; oscillation on the a.c. part of $d\mu$.)

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OPUC Transfer Matrices

We begin by looking at all solutions of the difference equations that describe recursion. In some sense, they are both second order, so there is a 2×2 "update" matrix that takes data at $n = 0$ to data at $n = m$.

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For OPUC, we saw that $A(z;\alpha_n)\binom{\varphi_n}{\varphi_n^*} = \binom{\varphi_{n+1}}{\varphi_{n+1}^*}$

$$A(z;\alpha) = \rho^{-1}\begin{pmatrix} z & -\bar{\alpha} \\ -\alpha z & 1 \end{pmatrix}$$

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Notice that $\det A(z;\alpha) = z$, so for $z \neq 0$, $z \in \mathbb{C}$, we have A invertible and for $z \in \partial\mathbb{D}$,

$$\|A^{-1}\| = \|A\|$$

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$$T_n(z;\alpha_{n-1},\dots,\alpha_0) = A(z;\alpha_{n-1})\,A(z;\alpha_{n-2})\cdots A(z;\alpha_0)$$

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shows that

$$\psi_n\bigl(z;\{\alpha_j\}_{j=0}^{n-1}\bigr) = \varphi_n\bigl(z;\{-\alpha_j\}_{j=0}^{n-1}\bigr)$$

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Theorem *If*

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2. A much stronger result is known (Baxter's Theorem): $\sum_{j=0}^{\infty}|\alpha_j(d\mu)| < \infty \Leftrightarrow \sum_{j=0}^{\infty}|c_j(d\mu)| < \infty + (d\mu = w(\theta)\frac{d\theta}{2\pi}$, w continuous with $\inf w > 0$.)

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but $\|A^{-1}\| = \|A\|$ for $|z| = 1$ and $|\varphi| = |\varphi^*|$

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Thus, by Bernstein–Szegő, we get the desired result.

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The difference equation can be rewritten (we take $a_0 = 1$)

$$\begin{pmatrix} u_{n+1} \\ a_n u_n \end{pmatrix} = A(z; a_n, b_n) \begin{pmatrix} u_n \\ a_{n-1}\, u_{n-1} \end{pmatrix};$$

$$A(z; a, b) = \frac{1}{a}\begin{pmatrix} z-b & -1 \\ a^2 & 0 \end{pmatrix}$$

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As for OPUC, we define

$T_n(z; \{a_j, b_j\}_{j=1}^n) = A(z; a_n, b_n) \cdots A(z : a_1, b_1)$ so

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By induction, q_n is a degree $n-1$ polynomial with $q_0(z) = 0$ and, for $n = 1, \dots,$

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Using the Carmona Simon formula, one can prove an L^1 perturbation result for OPRL.

Szegő Asymptotics for OPUC

For OPUC, the condition for $d\mu = f(\theta)\frac{d\theta}{2\pi} + d\mu_s$

$$\int \log f(\theta)\frac{d\theta}{2\pi} > -\infty$$

is called the Szegő condition.

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is called the Szegő condition. When it holds, we define the Szegő function, $D(z)$, on $\mathbb{D}$ by

$$D(z) = \exp\biggl(\int \frac{e^{i\theta} + z}{e^{i\theta} - z} \log\bigl(f(\theta)\bigr) \frac{d\theta}{4\pi}\biggr)$$

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Lemma. *If the Szegő condition holds, $D \in H^2(\mathbb{D})$, indeed,*

$$\sup_{0\le r<1}\int |D(re^{i\theta})|^2\frac{d\theta}{2\pi} \le 1$$

and, with $D(e^{i\theta}) \equiv \lim_{r\uparrow 1} D(re^{i\theta})$,

$$|D(e^{i\theta})|^2 = f(\theta)$$

Proof. Let $f_\varepsilon(\theta) = \min\bigl(f(\theta), \varepsilon^{-1}\bigr)$. Then $\log\bigl(f_\varepsilon(\theta)\bigr)$ is bounded above by $\log(\varepsilon^{-1})$, so

$$\mathrm{Re}\biggl(\int \frac{e^{i\theta}+z}{e^{i\theta}-z}\log\bigl(f_\varepsilon(\theta)\bigr)\,\frac{d\theta}{4\pi}\biggr) \le \frac{1}{2}\log(\varepsilon^{-1})$$

so $|D_\varepsilon| \le \varepsilon^{-1/2}$.

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so $|D_\varepsilon| \le \varepsilon^{-1/2}$. Thus, D_ε lies in H^∞ and has boundary values

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so $|D_\varepsilon| \le \varepsilon^{-1/2}$. Thus, D_ε lies in H^∞ and has boundary values

$$|D_\varepsilon(e^{i\theta})|^2 = f_\varepsilon(\theta)$$

Therefore, $D_\varepsilon \in H^2$ and

$$\sup_{0\le r<1}\int |D_\varepsilon(re^{i\theta})|^2\frac{d\theta}{2\pi} = \int |D_\varepsilon(e^{i\theta})|^2\frac{d\theta}{2\pi} \le 1$$

Szegő Asymptotics for OPUC

Proof. Let $f_\varepsilon(\theta) = \min\bigl(f(\theta), \varepsilon^{-1}\bigr)$. Then $\log\bigl(f_\varepsilon(\theta)\bigr)$ is bounded above by $\log(\varepsilon^{-1})$, so

$$\operatorname{Re}\biggl(\int \frac{e^{i\theta}+z}{e^{i\theta}-z}\log\bigl(f_\varepsilon(\theta)\bigr)\frac{d\theta}{4\pi}\biggr) \le \frac{1}{2}\log(\varepsilon^{-1})$$

so $|D_\varepsilon| \le \varepsilon^{-1/2}$. Thus, D_ε lies in H^∞ and has boundary values

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Taking $\varepsilon \downarrow 0$, we see that $D \in H^2$ and the rest follows.

Szegő Asymptotics for OPUC

We have the following beautiful calculation of Szegő:

$$\int |\varphi_n^*(e^{i\theta})\, D(e^{i\theta}) - 1|^2 \frac{d\theta}{2\pi} + \int |\varphi_n^*(e^{i\theta})|^2 d\mu_s = 2\Big(1 - \prod_{j=n}^{\infty} \rho_j\Big)$$

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For

$$\text{LHS} = \int \frac{d\theta}{2\pi} + \int |\varphi_n^*(e^{i\theta})|^2 d\mu - 2\,\mathrm{Re} \int D(e^{i\theta}) \varphi_n^*(e^{i\theta}) \frac{d\theta}{2\pi}$$

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Thus, uniformly in $|z| \ge r^{-1} > 1$,

$$z^{-n}\varphi_n(z) \to \left[\overline{D\left(\frac{1}{z}\right)}\right]^{-1}$$

which is Szegő asymptotics for φ_n.

(Extended) Shohat–Nevai Theorem

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$\prod_{n=1}^N a_n$, $\sum_{n=1}^N (a_n - 1)$, $\sum_{n=1}^N b_n$ *all have limits*

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3. Recall Lieb–Thirring (proven for Jacobi by Hundertmark–Simon)

$$\sum(|E_n^\pm| - 2)^p \leq \sum(|a_n - 1|^{p+\frac{1}{2}} + |b_n|^{p+\frac{1}{2}}) \text{ for } p \geq \tfrac{1}{2}.$$

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Thus $J - J_0 \in \ell_1$ (trace class), equivalent to $\sum |a_n - 1| + |b_n| < \infty$ implies both Blaschke condition and $\lim\,(a_1 \cdots a_n)$ exists

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We refer you to [SzTh], Section 3.8 for the proof of the extended Shohat–Nevai Theorem.The idea is to use the use suitable sum rules. In this form it is due to Killip-Simon and Simon-Zlatos. This form of the eigenvalue condition is independently due to Peherstorfer-Yuditskii.

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Theorem (Damanik–Simon [Inv. Math **165** (2006), 1–50]). *Let the Jacobi parameters obey*

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Then, for all $z \in \mathbb{D} \setminus \{0\}$ *with* $z + z^{-1} \notin \sigma(J)$, $\lim_{n\to\infty} z^n\, p_n(z+z^{-1})$ *exists uniformly on compacts and is non-zero.*

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Conversely, if that limit exists uniformly and is non-zero for $\{z \mid |z| = r\}$ *for all* $r \in (0, \varepsilon)$, *then (a)–(c) hold.*

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Corollary (Peherstorfer-Yuditskii [Proc. AMS **129** (2001) 3213–3220]). *If* $\sum_{n,\pm}(|E_n^{\pm}| - 2)^{\frac{1}{2}} < \infty$ *and Szegő condition holds, then* $\lim z^n \, p_n(z + z^{-1})$ *exists, etc.*

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For such examples, the Szegő condition fails, but you still get Szegő asymptotics ! This came as a surprise to many. Of course, if an ℓ^2 condition holds, then the sum is finite for $p=\frac{3}{2}$ by Killip–Simon.

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$$z^n\, p_n(z + \frac{1}{z}) = \frac{1}{a_1 \cdots a_n}\big(1 + z(\sum_{j=1}^{n} b_j) + O(z^2)\big)$$

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The first two coefficients give convergence of $\prod_1^n a_j$ and $\sum_1^n b_j$ by the above and the third coefficient yields the conditional convergence of $\sum_1^n (a_j - 1)^2 + b_j^2$ but since the sum of positive numbers, conditional convergence implies absolute convergence.

Killip–Simon Theorem

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Theorem. Let $d\mu(x) = f(x)\,dx + d\mu_s$ with Jacobi parameters $\{a_n, b_n\}_{n=1}^{\infty}$. Then

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if and only if

(i) (Blumental–Weyl) $\sigma_{\text{ess}}(J) = \operatorname{ess\,supp}(d\mu) = [-2, 2]$, i.e., $\operatorname{supp}(d\mu)$ is a set of pure points whose only possible limit points are ± 2: $E_1^- < E_2^- < \ldots < -2$; $2 < \ldots < E_2^+ < E_1^+$.

(ii) (Lieb–Thirring) $\sum_{\pm, j} (|E_j^{\pm}| - 2)^{3/2} < \infty$.

(iii) (Quasi-Szegő) $\int (x^2 - 4)^{1/2} \log\big(f(x)\big)\,dx < \infty$.

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For Schrödinger operators in 1D (and so on half line), Lieb–Thirring proved (initially for $p > \frac{1}{2}$, $p = \frac{1}{2}$ is Weidl and then Hundertmark–Lieb–Thomas)

$$\sum_{E_j, \pm} |E_j^{\pm}|^p \le C_p \int_0^\infty |V(x)|^{p+\frac{1}{2}}$$

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Hundertmark–Simon (Killip–Simon for $p = \frac{3}{2}$)

$$\sum \bigl(|E_j^\pm| - 2\bigr)^p \le \widetilde{C}_p \sum_{n=0}^{\infty} |a_j - 1|^{p+\frac{1}{2}} + |b_j|^{p+\frac{1}{2}}$$

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Quasi-Sezgő because power is $+\frac{1}{2}$, not $-\frac{1}{2}$ of Szegő condition.

Ratio Asymptotics

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OR $\exists\, a \in (0,1]$ *and* $\lambda \in \partial\mathbb{D}$ *so* $\lim_{n\to\infty} |\alpha_n| = a$, $\lim_{n\to\infty} \bar{\alpha}_{n+1}\, \alpha_n = a^2\, \lambda$

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Szegő's Asymptotic Theorem for OPUC says $\Phi_n^*(z) \to D(0)D(z)^{-1}$ as $n \to \infty$ so $\Phi_{n+1}^*/\Phi_n^* \to 1$. We state without proof

Krushchev's Theorem (see [OPUC2], Section 9.5).

$\Phi_{n+1}^*(z)/\Phi_n^*(z)$ *converges uniformly on each* $\{z \mid |z| < 1 - \varepsilon\}$ *if and only if either*

For $\ell = 1, 2, \dots$, $\lim_{n\to\infty} \alpha_{n+\ell}\, \alpha_n = 0$; *limit is then 1.*

OR $\exists\, a \in (0,1]$ *and* $\lambda \in \partial\mathbb{D}$ *so* $\lim_{n\to\infty} |\alpha_n| = a$, $\lim_{n\to\infty} \bar{\alpha}_{n+1}\, \alpha_n = a^2\, \lambda$

and then limit $\frac{1}{2}\left[(1+\lambda z) + \sqrt{(1-z\lambda)^2 + 4a^2\,\lambda z}\right]$.

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and the limit is

$$\frac{1}{2}\left[(z-b)+\sqrt{(z-b)^2-4a^2}\right] \quad \text{(root with } \sqrt{} = z \text{ near } \infty\text{)}$$

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Moral is ratio and Szegő asymptotics unusual. Expect oscillations.