Spectral Theory of Orthogonal Polynomials

Periodic and Ergodic Spectral Problems

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Lecture 5: Isospectral Tori

- Lecture 1: Introduction and Overview
- Lecture 2: Szegő Theorem for OPUC
- Lecture 3: Three Kinds of Polynomial Asymptotics
- Lecture 4: Potential Theory
- Lecture 5: Isospectral Tori
- Lecture 6: Fuchsian Groups
- Lecture 7: Chebyshev Polynomials, I
- Lecture 8: Chebyshev Polynomials, II

References

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[OPUC2] B. Simon, *Orthogonal Polynomials on the Unit Circle, Part 2: Spectral Theory*, AMS Colloquium Series, **54.2**, American Mathematical Society, Providence, RI, 2005.

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For $z \in \mathbb{C}$ fixed, we are interested in solutions $\{u_n\}_{n=0}^{\infty}$ of

$$a_n u_{n+1} + b_n u_n + a_{n-1} u_{n-1} = z u_n$$

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Conversely, if $\widetilde{u}$ solves this, the unique u with $u_{n+p} = \lambda u_n$ and $\widetilde{u} = \{u_n\}_{n=1}^{\infty}$ is a Floquet solution.

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If $\lambda \neq \pm 1$, then $\bar{\lambda} \neq \lambda$. If u is a Floquet solution for λ, since z is real, $\bar{u}$ is a Floquet solution for $\bar{\lambda}$ so there is a unique solution for that z.

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In terms of the OP's for $\{a_n, b_n\}_{n=1}^{\infty}$,

$$T_p(z) = \begin{pmatrix} p_p(z) & -q_p(z) \\ a_p p_{p-1}(z) & -a_p q_{p-1}(z) \end{pmatrix}$$

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$$\Delta(z) = \mathrm{Tr}\big(T_p(z)\big) = p_p(z) - a_p q_{p-1}(z)$$

is a (real) polynomial of degree exactly p.

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An important consequence of the fact that $\Delta(z) \in (-2, 2)$ implies all z's are real is $\Delta^{-1}\bigl[(-2,2)\bigr] \subset \mathbb{R}$.

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A basic fact of analytic functions is that if $f(z)$ is real (i.e., $f(\bar{z}) = \overline{f(z)}$), $x_0 \in \mathbb{R}$ with $f'(x_0) = 0$, there are non-real z's near x_0 with $f(z)$ real and near $f(x_0)$.

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If $\beta_{j-1} = \alpha_j$, there is one less point where $\Delta(x) = (-1)^{p-j-1} 2$, but $\Delta'(\alpha_j) = 0$ since $\Delta - (-1)^{p-j-1} 2$ has the same sign on both sides of α_j. It follows that

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and in that case, Δ'' *is not zero at that point.*

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If $\beta_j = \alpha_{j+1}$, we say gap j is closed.

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Each of the gaps where $\Delta(x) \geq 2$ has two periodic solutions—either two at $\beta_j = \alpha_{j+1}$ or one each at β_j and α_{j+1} so there are p periodic Floquet solutions, as there must be from the J^{per} analysis.

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One can write out an explicit spectral representation with Floquet solutions with $z \in (\alpha_j, \beta_j)$ as continuum eigenfunctions.

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The answer will lie in potential theory.

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Theorem. $\gamma(z)$ *as given above is the potential theorists' Green's function and periodic Jacobi parameters are associated to regular measures (in the Stahl–Totik sense).*

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Thus, the equilibrium measure is

$$f_{\mathfrak{e}}(z) = \frac{1}{p\pi}\frac{|\Delta'(x)|}{\sqrt{4-\Delta^2(x)}} = \frac{1}{p\pi}\left|\frac{d}{dx}\arccos\left(\frac{\Delta(x)}{2}\right)\right|$$

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This is also a density of zeros way of understanding why the above $f_{\mathfrak{e}}$ is the DOS. For the periodic eigenfunctions with a box of size kp are the Floquet solutions with $\lambda = e^{2\pi ij/k}$, $j = 0, 1, 2, \ldots, k-1$.

Weyl Solutions

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If $\inf a_n > 0$, constancy of the Wronskian shows this is the unique ℓ^2-solution.

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which upon iterations yields the continued fraction of Jacobi, Markov and Stieltjes:

$$m(z) = \cfrac{1}{b_1 - z - \cfrac{a_1^2}{b_2 - z - \cfrac{a_2^2}{b_3 - z - \dots}}}$$

Quadratic Equations

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By a direct calculation, the two discriminants are related by

$$\beta^2 - 4\alpha\gamma = \sqrt{\Delta(z)^2 - 4}$$

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Those points on the surface which "project down" to a gap, are a circle - two intervals glued together at the ends.

Dirichlet Data

There is thus a map from the m-function to the set of points on the product over the gaps of the points on the surface that project down to that gap, i.e. onto a ℓ-dimensional torus.

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The possible m-functions are thus precisely the Herglotz functions (i.e. analytic functions in the upper half plane with positive imaginary part) that have an meromorphic continuation to the Riemann surface of $\sqrt{\Delta(z)^2 - 4}$ that have minimal degree and which are normalized to look like $-/z$ near ∞. Christiansen, Simon and Zinchenko call these *minimal Herglotz functions*.

Finite Gap Sets

Now consider a general compact subset $\mathfrak{e} \subset \mathbb{R}$ which has $\ell + 1$ components so its complement in $\mathbb{R}$ has ℓ gaps.

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The same method which we didn't describe that constructs minimal Herglotz functions in the periodic case lets us do the same for the Reimann surface of

$$\sqrt{\prod_{j=1}^{\ell+1} (z - \alpha_j)(z - \beta_j)}$$

Almost Periodic Isospectral Torus

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Theorem Every finite gap set, $\mathfrak{e}$ has an isospectral torus of almost periodic Jacobi matrices associated to it. These are all periodic with period p if and only if each band has harmonic measure $\frac{j}{p}$ for $j \in \{1, ..., p-1\}$.