SCHRAMM–LOEWNER EVOLUTION

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Contents

1. Three discrete models

The aim of this mini-course is to present the theory of *Stochastic Loewner Evolution* or *Schramm–Loewner Evolution* (SLE for short), which is a one-parameter family of probability distributions on curves in the plane that are, under reasonable hypotheses, the only possible scaling limits of many critical discrete models from statistical physics. SLE stands at the intersection of probability and complex analysis, and we first describe a few discrete models to give a feel of where the intuition behind the introduction of SLE comes from.

None of these models will be treated in any depth, because this would have to take too much time; much more detail on such discrete models and their convergence to SLE will be given in the mini-course of Clément Hongler.

1.1. **Simple random walk.** Let $(X_n)_{n\geqslant 0}$ be a simple random walk on the lattice $\mathbb{Z}^2$, started from the origin. It is well known that it can be asymptotically rescaled into a continuous process, which is two-dimensional Brownian motion, in the following sense: the process $(n^{-1/2}X_{\lfloor nt\rfloor})_{t\geqslant 0}$ converges in distribution to Brownian motion

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$(B_t)_{t\geqslant 0}$ in $\mathbb{R}^2$, started from 0 as well, as $n \to \infty$. This is known as a *scaling limit*, and can be related to the convergence of planar maps to the Brownian map in the mini-course of Grégory Miermont, or that of discrete Gaussian free field to continuous GFF in that of Jason Miller. We will see a few others very shortly.

One particular feature of 2d Brownian motion is its *conformal invariance*, which we now spend some time defining; along the way, we will introduce some of the notions from complex analysis that we will require in the next hours as well. From now on we will always identify the complex plane $\mathbb{C}$ and the Euclidean plane $\mathbb{R}^2$.

A *domain* is a simply connected, open, strict subset of the complex plane. For ease of notation, we will always assume a little bit of topological regularity on domains, namely that their boundaries are locally connected (such as *e.g.* if they are the interior of a Jordan loop). If Ω_1 and Ω_2 are two domains, a map $\Phi : \Omega_1 \to \Omega_2$ is *conformal* if it is bijective and holomorphic (*i.e.* differentiable in its complex variable). The *Riemann mapping theorem* states the existence of such a conformal map between any two domains — see further for a more precise statement.

Fix such a conformal map $\Phi : \Omega_1 \to \Omega_2$, and let $z_1 \in \Omega_2$ and $z_2 := \Phi(z_1) \in \Omega_2$. We can define two random compact subsets $\overline{\Omega_2}$:

- First, let (B_t) be Brownian motion started from z_1, and let τ be its hitting time of $\partial\Omega_1$; define K_1 as the image of the path of B up to time τ, under the map Φ:
$$K_1 := \{\Phi(B_t) : 0 \leqslant t \leqslant \tau\}.$$
(The map Φ extends to $\partial\Omega_1$ if the boundary of Ω_2 is locally connected, which we are assuming.)
- Second, let (W_s) be Brownian motion started from z_2, and let σ be its hitting time of $\partial\Omega_2$; define K_2 as the path of W up to time σ:
$$K_2 := \{W_s : 0 \leqslant s \leqslant \sigma\}.$$

Theorem 1 (*Lévy*)

The sets K_1 and K_2 have the same distribution.

This statement is known as the *conformal invariance* of Brownian motion. Its proof, the details of which are a nice exercise in stochastic calculus for the reader, actually shows a much stronger result, at the level of processes instead of supports. Let $Y_t := \Phi(B_t)$ be defined up to time τ. One can apply Itô's formula to compute it, and use the Cauchy-Riemann equations coming from the holomorphicity of Φ to obtain first that the drift term vanishes, and second that the diffusion term is given by a scalar multiplication:
$$\mathrm{d}Y_t = \Phi'(B_t)\mathrm{d}B_t.$$
This implies that (Y_t) can be seen as a time-change of (W_s), from which the theorem follows.

As it turns out, the geometry of the set K_2 is quite interesting. To give one statement that is representative of what one can prove using SLE, first define $\tilde{K}_2$ to be the *hull* of K_2; namely, the union of K_2 and the set of all the points in Ω_2 that are separated from $\partial\Omega_2$ by K_2. In other words, one goes from K_2 to $\tilde{K}_2$ by "filling up its the holes". Clearly the hull of Brownian motion is as conformally invariant as Brownian motion itself.

Theorem 2 (*Lawler–Schramm–Werner*)

The Hausdorff dimension of $\partial\tilde{K}_2$ is almost surely equal to $4/3$.

1.2. **Percolation.** The second model I would like to discuss is *planar percolation*, more precisely critical site-percolation on the triangular lattice. Consider a discretization T_δ of the upper-half plane by a triangular lattice of mesh $\delta > 0$, aligned in such a way that $\pm\delta/2 \in T_\delta$; color each vertex of T_δ either black or white, independently of the others, with equal probability $1/2$. Equivalently, color each hexagonal face of the dual lattice white or black with probability $1/2$. This is a critical model, in the sense that all the connected monochromatic components are almost surely finite, but the probability that two points are in the same component decays only polynomially in the distance — while for any other choice of the coloring parameter, one of the colors will have an infinite component while the connection probabilities decay exponentially for the other.

One can look at interfaces between colored regions in a realization of percolation. Those form a family of discrete, non-intersecting, finite loops, so defining a scaling limit will be difficult. [This would also have been a problem for the SRW but there the limit was explicit.] To remedy it, one way is to introduce *boundary conditions*, and a common choice here is to force all the vertices along $\mathbb{R}_+$ (resp. $\mathbb{R}_-$) to be black (resp. white). This forces the existence of exactly one infinite interface γ_δ within the upper-half plane, started at the origin, having black vertices on its right and white vertices on its left. This interface is also known as the *exploration process* of percolation.

The counterpart to the invariance principle for SRW is the determination of what happens in this setup as $\delta \to 0$. The following is one of the most celebrated recent results in probability; we state it in slightly vague terms, a more precise statement will appear a bit later:

Theorem 3 (*Smirnov*)

As $\delta \to 0$, γ_δ *converges in distribution to a random continuous curve* γ *in* $\overline{\mathbb{H}}$.

As we will see later, the limit is in fact the SLE(6) process, which so far we haven't defined yet. As an extension, the exploration process can be defined in any domain Ω whose boundary is partitioned into two intervals $\partial_w\Omega$ and $\partial_b\Omega$, in a natural way: discretize Ω by taking the intersection with the triangular lattice of mesh δ, and force all the sites along $\partial_w\Omega$ (resp. $\partial_b\Omega$) to be white (resp. black). As before this creates a collection of loops, plus one interface γ_δ^Ω joining the two special points on the boundary of Ω where the boundary condition changes.

Again, as $\delta \to 0$, the discrete exploration process converges in distribution to a random continuous curve γ^Ω; the scaling limit of percolation, like planar Brownian motion, turns out to be conformally invariant as well:

Theorem 4 (***Smirnov***)

Let Ω_1 and Ω_2 be two domains, whose boundaries are each partitioned as above into a white and a black interval; let $\Phi : \Omega_1 \to \Omega_2$ be a conformal map sending $\partial_b \Omega_1$ to $\partial_b \Omega_2$ (and $\partial_w \Omega_1$ to $\partial_w \Omega_2$). Then, up to parametrization,

$$\Phi(\gamma^{\Omega_1}) \stackrel{\text{(d)}}{=} \gamma^{\Omega_2}.$$

The exploration process has one more feature which the random walk didn't have. Fix a domain Ω with bi-colored boundary, $\delta > 0$ and $n \in \mathbb{Z}_+$ (small enough, in a sense that will soon be evident). Let $\gamma_\delta[0, n]$ denote the first n steps of the exploration process, and let Ω_n be the domain $\Omega \setminus \gamma_\delta[0, n]$; color the boundary of Ω_n like that of Ω where they match, and black (resp. white) along the right (resp. left) side of $\gamma_\delta[0, n]$.

Conditionally on $\gamma_\delta[0, n]$, the model we see in Ω_n is still exactly percolation, because the states of the vertices away from the exploration path are independent of each other and of the path itself. In other words: the distribution of the future of the exploration process after n steps, given the first n steps, is the same as that of an exploration process in the domain remaining after removing those n steps from the initial domain. [Here, n needs to be small enough to make the event that γ reaches the other boundary point before time n impossible.]

This is known as the *domain Markov property*; a similar statement holds for the Gaussian free field and appeared in the course of Jason Miller. Schramm's fundamental insight is that, together with conformal invariance, it will give enough information on the possible scaling limit of the exploration process to characterize it up to the value of one real parameter; we will go into this shortly.

1.3. **Self-avoiding walk (if time allows).** A third model I would like to discuss is supported on simple discrete paths, which makes it topologically simpler to set up. Let S_n be the set of all self-avoiding paths of n steps in $\mathbb{Z}^2$, starting at 0 and remaining in $\mathbb{Z} \times \mathbb{Z}_+$. The *connective constant* of $\mathbb{Z}^2$ is

$$\mu := \lim_{n\to\infty} |S_n|^{1/n}.$$

Again we would like to define a scaling limit of an element of S_n chosen uniformly at random, and the same obstacle as for percolation is present here, the elements of S_n are bounded so letting the lattice mesh go to 0 might shrink everything to a point.

For $0 < k < n$, let $P_{k,n}$ be the distribution of the first k steps of a uniform element of S_n. It is a measure on S_k, which is certainly *not* uniform. Nevertheless, it is a result by Kesten that $P_{k,n}$ has a limit P_k as $n \to \infty$; and the measures P_k are consistent, in the sense that the distribution of the first k steps of a walk sampled according to P_{k+1} is exactly P_k. Hence there is a probability measure P on the collection of infinite self-avoiding walks, having the P_k as marginals.

Since P is defined on infinite objects, now it makes sense to ask the question of existence of a scaling limit: sample γ according to P, rescale it by a positive factor $\delta > 0$ and let δ go to 0.

Conjecture 1

As $\delta \to 0$, $\delta\gamma$ converges in distribution to a continuous random curve — namely, SLE(8/3).

The measure P can also be defined in other domains, and it also has the domain Markov property, because the conditional distribution of the last $n-k$ steps of a uniform self-avoiding walk of length n, given ist first k steps, is uniform in the possible continuations. It has another property which makes it stand on its own, known as the *restriction property*: namely, if $\Omega \subset \Omega'$ are two unbounded domains contained in the upper-half plane, and agreeing in a neighborhood of the origin, the conditional distribution of infinite SAW in Ω', conditioned to remain in Ω, is the same as the distribution of infinite SAW in Ω — again the finite version of this states that the restriction of a uniform measure is uniform, which is obvious.

2. Definition of SLE

Schramm's insight was to realize that, under mild (and reasonable) assumptions in addition to conformal invariance, the only possible scaling limits have to be distributed as one of a one-parameter family of measures on curves, which he named Stochastic Loewner Evolutions (they are now universally known as {Schramm–Loewner Evolutions}). The aim of this section is to define these random curves and give a few of their fundamental properties.

2.1. Reminder on conformal maps.

- Riemann mapping theorem
- Uniqueness / normalization
- Special case of the disk, conformal radius
- Special case of the upper-half plane, capacity
- Two computations, $(z+a) \circ (z+b)$, $(z+a/z) \circ (z+b/z)$
- Two examples, $z+1/z$ and $\sqrt{z^2+1}$

2.2. **Loewner chains in the upper-half plane.** Let $\mathbb{H}$ denote the open upper half-plane, and let $\gamma : [0,\infty) \to \overline{\mathbb{H}}$ be a continuous, non-self-crossing curve starting from 0 and such that $|\gamma_t| \to \infty$ as $t \to \infty$. Let $H_t := \mathbb{H} \setminus \gamma_{[0,t]}$ be the complement of the path up to time t: it is a simply connected domain, and therefore Riemann's mapping theorem can be applied to show that there exists a conformal map

$$g_t : H_t \to \mathbb{H},$$

uniquely determined if one imposes the *hydrodynamic normalization*:

$$g_t(z) = z + \frac{a(t)}{z} + \mathcal{O}_\infty(z^{-2}).$$

In particular $g_0(z) = z$.

One can easily check that $a(t) > 0$ for all $t > 0$; it is known as the *half-plane capacity* of $\gamma_{[0,t]}$. In fact, it is continuous and strictly increasing, which can be

shown from positivity by composition:
$$\left[z+\frac{a_1}{z}+\mathcal{O}_\infty(z^{-2})\right]\circ\left[z+\frac{a_2}{z}+\mathcal{O}_\infty(z^{-2})\right]=\left[z+\frac{a_1+a_2}{z}+\mathcal{O}_\infty(z^{-2})\right].$$
This implies that a can be used as a time parameter: up to reparametrization, one can assume, and we will from now on, that $a(t)=2t$ for all $t>0$.

Remark 1. *One thing that is* not *always true is that $a(t)\to\infty$ with t, and it is a nice exercise to construct a curve γ with bounded half-plane capacity.*

With this choice of normalization, and assuming that the capacity does go to $+\infty$, one can describe (g_t) as the flow of a differential equation:

Theorem 5 (*Loewner*)

There exists a continuous function $\beta:[0,\infty)\to\mathbb{R}$ such that, for every $t\geqslant 0$ and every $z\in H_t$,
$$\partial_t g_t(z)=\frac{2}{g_t(z)-\beta_t}.$$
This is called the Loewner equation, *and we will refer to β as the* driving function.

What this theorem really means is that one can encode the geometry of a two-dimensional curve (up to parametrization) using a real-valued function, which is a much simpler object. To make this precise, one simply has to notice that the whole construction can be done in reverse: from β one can compute (g_t) by solving the Loewner equation, and from there recover the curve γ.

Remark 2. *Again, this is not to say that any continuous function β can be plugged in the Loewner equation to produce a curve — which is wrong. It is however true if β is Hölder of exponent $1/2$ and small enough norm, which actually produces a simple curve, but the proof of this is quite subtle.*

2.3. **Chordal SLE.** Consider, as before, critical site-percolation on the triangular lattice of mesh $\delta>0$ in the upper half-plane, with boundary conditions white to the right of the origin and black to the left, and let γ be its scaling limit as $\delta\to 0$; assume that this limit exists and exhibits conformal invariance (in addition to the domain Markov property inherited from the discrete model). The curve γ satisfies the hypotheses above, so it can be described as a Loewner chain — of course the driving function β will then be random as well.

We now aim to identify this β, but first we introduce two definition in a more formal way than before.

Definition 1 (*Conformal invariance*)

A conformally invariant family of random curves *is a collection $(\mu_{\Omega,a,b})$ of random measures, indexed by triples (Ω,a,b) where Ω is a domain and $a,b\in\partial\Omega$, such that $\mu_{\Omega,a,b}$ is supported on non-self-crossing curves going from a to b in $\overline{\Omega}$, and safisfying the* conformal invariance property*: for every triple (Ω,a,b) and any conformal map defined from Ω,*
$$\Phi_*\mu_{\Omega,a,b}=\mu_{\Phi(\Omega),\Phi(a),\Phi(b)}.$$

Implicitly, we identify two curves that are conjugated by an increasing time-change; in what follows, we fix a conformally invariant family $(\mu_{\Omega,a,b})$.

Remark 3. *It is interesting to look at the case $\Omega = \mathbb{H}$, $a = 0$ and $b = \infty$. One conformal map in this case is $z \mapsto \lambda z$ for a given $\lambda > 0$, and this means that conformal invariance implies scale invariance as a particular case.*

Definition 2 (*Domain Markov property*)

If Ω is a domain, $a, b \in \partial\Omega$ and γ is distributed according to $\mu_{\Omega,a,b}$, and if τ is a stopping time for γ, let Ω_τ be the connected component of $\Omega \setminus \gamma_{[0,\tau]}$ having b on its boundary, and $\mu^{(\tau)}_{\Omega,a,b}$ be the law of the curve γ after time τ. The family $(\mu_{\Omega,a,b})$ satisfies the domain Markov property *if*

$$E(\mu^{(\tau)}_{\Omega,a,b} | \gamma_{[0,\tau]}) = \mu_{\Omega_\tau,\gamma_\tau,b}.$$

The main theorem in this section is the following:

Theorem 6 (*Schramm*)

If $(\mu_{\Omega,a,b})$ is a conformally invariant family of random curves satisfying the domain Markov property, then there exists $\kappa \in \mathbb{R}_+$ such that $\mu_{\mathbb{H},0,\infty}$ can be obtained by solving the Loewner equation with $(\beta_t) = (\sqrt{\kappa} B_t)$, where (B_t) is a standard Brownian motion on $\mathbb{R}$. The corresponding random curve is then known as (chordal) SLE(κ).

Proof (sketch). Consider $\mu_{\mathbb{H},0,\infty}$ and represent it using Loewner's theorem into a family (g_t) of conformal maps; fix $t > 0$, and recall that H_t is the domain of g_t, which is playing the same role as Ω_t in the definition of the domain Markov property. g_t maps it back to $\mathbb{H}$, but it maps γ_t to β_t rather than 0, so it is natural to define

$$h_t(z) := g_t(z) - \beta_t.$$

By the domain Markov property and the conformal invariance, conditionally on $\gamma_{[0,t]}$, the image of γ after time t under the map h_t is distributed the same way as γ. In particular, we get an equality in distribution:

$$h_{t+s} \overset{(d)}{=} \tilde{h}_s \circ h_t$$

where (as well as below) the tilde means "an independent copy of". Looking at the expansion at infinity leads to a similar relation at the level of the driving function, namely

$$\beta_{t+s} \overset{(d)}{=} \beta_t + \tilde{\beta}_s$$

from which the fact that β is a Brownian motion follows. □

2.4. **Radial SLE (here in the notes, but later in the course).** We just say a few words here about the case of radial Loewner chains, since not much needs to be changed from the chordal setup. Here, we are given a continuous, Jordan curve γ in the unit disk $\mathbb{D}$, satisfying $\gamma_0 = 1$, $\gamma_t \neq 0$ for all $t > 0$ and $\gamma_t \to 0$ as $t \to 0$. In other words, the reference domain is not the upper half-plane with two marked boundary points, but the unit disk with one marked boundary point and one marked interior point.

Let D_t be the complement of $\gamma_{[0,t]}$ in the unit disk; notice that 0 is in the interior of D_t, so there exists a conformal map g_t from D_t onto $\mathbb{D}$ fixing 0; this map is unique if one requires in addition that $g_t'(0) \in \mathbb{R}_+$, which we will do from now on.

The natural parametrization of the curve still needs to be additive under composition of conformal maps; here, the only choice (up to a multiplicative constant) is the logarithm of $g_t'(0)$: up to reparametrization, we can ensure that for every $t > 0$, $g_t'(0) = e^t$. With this choice, we have the following:

Theorem 7 (***Loewner***)

There exists a continuous function $\theta : [0, \infty) \to \mathbb{R}$ such that, for every $t \geqslant 0$ and every $z \in D_t$,

$$\partial_t g_t(z) = \frac{e^{i\theta_t} + g_t(z)}{e^{i\theta_t} - g_t(z)} g_t(z).$$

This is known as the radial Loewner equation.

Everything we just saw in the chordal case extends to the radial case. In particular, the notions of conformally invariant family of random curves and domain Markov property have natural counterparts for domains with one marked point on the boundary and one in the interior, and essentially the same argument as before shows that a family of random curves having both conformal invariance and the domain Markov property has to be in a one-paramenter family defined as follows:

Definition 3 (***Schramm***)

Radial SLE(κ) *is the solution of the radial Loewner equation with driving function $(\theta_t) = (\sqrt{\kappa} B_t)$, where (B_t) is a standard Brownian motion in $\mathbb{R}$.*

The local behavior of this equation around the singularity at $z = e^{i\theta_t}$ involves a numerator of norm 2; it is the same 2 as in Loewner's equation in the upper half-plane, in the sense that the local behavior of the solution for the same value of κ will then be the same on both sides (this is known as the *chordal-radial equivalence*).

2.5. **What's next.** The plan for the rest of the lecture is then to work our way backwards from there, namely we will follow the following steps:

(1) go from (β_t) to (g_t), (H_t) and (K_t), which is easy;
(2) from there, reconstruct (γ_t), which is *very* difficult;
(3) use stochastic calculus to study properties of the curve; and
(4) possibly use what we obtained to get discrete statements.

3. Some properties of SLE

3.1. **Geometry.** The first, very non-trivial question arising about SLE is whether it actually fits the above derivation, which more specifically means whether the curve γ exists. In the case of regular enough driving functions, namely Hölder with exponent 1/2 and small enough norm, Marshall and Rohde proved that it does, but it is possible to construct counterexamples even in the Hölder-1/2 class.

It turns out that SLE does actually fit the description, up to one notable change. One can always solve Loewner's equation to obtain $g_t : \mathbb{H} \setminus K_t \to \mathbb{H}$ where K_t is the (relatively compact) set of points in the upper half-plane from which the solution blows up before time t. Then:

Theorem 8 (*Rohde-Schramm, Lawler-Schramm-Werner*)

For every $\kappa > 0$, SLE(κ is generated by a curve, in the following sense: there exists a (random) continuous curve γ in the closure of the upper half-plane $\overline{\mathbb{H}}$, called the SLE trace, *such that, for every $t > 0$, $\mathbb{H} \setminus K_t$ and $\mathbb{H} \setminus \gamma_{[0,t]}$ have the same unbounded connected component (in other words, K_t is the hull of $\gamma_{[0,t]}$.*

Given the existence of the trace, it is natural to ask whether K itself is a curve or not. Whether this happens depends on the value of κ:

Theorem 9 (*Rohde-Schramm*)

The topology of the SLE trace undergoes two transitions:

- *If $\kappa \leqslant 4$, then γ is almost surely a simple curve, and besides $\gamma_t \in \mathbb{H}$ for every $t > 0$;*
- *If $4 < \kappa < 8$, then γ does have double points, and $\gamma_{[0,t]} \subsetneq K_t$;*
- *If $8 \leqslant \kappa$, then γ is almost surely a space-filling curve,* i.e. $\gamma_{[0,\infty)} = \overline{\mathbb{H}}$.

Proof. The proof of this theorem involves the first use of SLE in computations. Let us start with the transition across $\kappa = 4$. Let $x > 0$, and trace the evolution of x under the (chordal) SLE flow by defining

$$Y_t^x := g_t(x) - \beta_t.$$

From Loewner's equation, one gets

$$\mathrm{d}Y_t^x = \partial_t g_t(x)\,\mathrm{d}t - \mathrm{d}\beta_t = \frac{2}{g_t(x) - \beta_t}\,\mathrm{d}t - \mathrm{d}\beta_t = \frac{2\,\mathrm{d}t}{Y_t^x} - \sqrt{\kappa}\,\mathrm{d}B_t.$$

Up to a linear time change, this is exactly a *Bessel process of dimension* $1 + 4/\kappa$. In particular, it will hit the origin (meaning that x is swallowed by the curve in finite time) if and only it the dimension of the process is less than 2, if and only if $\kappa > 4$.

One way to see that without knowing about Bessel process is to consider the process $Z_t^x = (Y_t^x)^{2-d}$ where $d = 1 + 4/\kappa$ is the dimension (or $Z_t^x = \log Y_t^x$ if $d = 2$, meaning $\kappa = 4$). Then, one has

$$\mathrm{d}Z_t^x = (2-d)(Y_t^x)^{1-d}\mathrm{d}Y_t^x + \frac{(2-d)(1-d)}{2}(Y_t^x)^{-d}\kappa\mathrm{d}t$$

and replacing $\mathrm{d}Y_t^x$ and d by their values, one gets (for $\kappa \neq 4$)

$$\mathrm{d}Z_t^x = 2\left(\frac{2}{\sqrt{\kappa}} - \frac{\sqrt{\kappa}}{2}\right)(Y_t^x)^{-4/\kappa}\,\mathrm{d}B_t.$$

So Z_t^x is a local martingale, and whenever Y_t^x reaches 0 it reaches either 0 or $+\infty$ according to the value of κ.

The transition across $\kappa = 8$ is a bit more problematic, and involves estimates for the probability of hitting a ball inside the domain, proving that this probability is equal to 1 if and only if $\kappa \geqslant 8$; we leave that as an exercise, which can be skipped on first reading. □

3.2. **Probability.** We now turn to uses of SLE in computing the probabilities of various events which are of particular interest in the framework of scaling limits of discrete models; we focus on two kinds of estimates, for crossings and for arm events.

3.2.1. *Crossings: Cardy's formula for SLE.* The initial motivation behind the definition of SLE was the conformal invariance of some scaling limits; here we looked particularly at percolation through Cardy's formula. We still have to identify the value of κ though, and crossing probabilities are a natural way to do it: one can compute them in terms of κ and match the result with Smirnov's theorem.

Theorem 10

Let $\kappa > 4$, $a < 0 < c$, and let $E_{a,c}$ be the event that the chordal SLE(κ) trace visits $[c, +\infty)$ before $(-\infty, a]$. Then,

$$P[E_{a,c}] = F\left(\frac{-a}{c-a}\right) \quad \text{where} \quad F(x) = \frac{1}{Z_\kappa}\int_0^x \frac{\mathrm{d}u}{u^{4/\kappa}(1-u)^{4/\kappa}}$$

and Z_κ is chosen so that $F(1) = 1$.

Proof. The proof follows essentially the same lines as that of the phase transition at $\kappa = 4$, but it is rather instructive, so we still give a very rough outline here for the benefit of the serious reader willing to do the computation. Let $A_t := g_t(a)$, $C_t = g_t(c)$ and

$$Z_t := \frac{\beta_t - A_t}{C_t - A_t}.$$

From Itô's formula, it is straightforward to obtain

$$\mathrm{d}Z_t = \frac{\sqrt{\kappa}\,\mathrm{d}B_t}{C_t - A_t} + \frac{2\,\mathrm{d}t}{(C_t - A_t)^2}\left(\frac{1}{Z_t} - \frac{1}{1 - Z_t}\right),$$

which after the time-change $\mathrm{d}s = \mathrm{d}t/(C_t - A_t)^2$, $\tilde{Z}_s = Z_t$ leads to

$$\mathrm{d}\tilde{Z}_s = \sqrt{\kappa}\,\mathrm{d}\tilde{B}_s + 2\left(\frac{1}{\tilde{Z}_s} - \frac{1}{1 - \tilde{Z}_s}\right).$$

Finding F now amounts to writing that the drift term of $F(Z_{\tilde{s}})$ should vanish, thus leading to the following differential equation:

$$\frac{\kappa}{4}F''(x) + \left(\frac{1}{x} - \frac{1}{1-x}\right)F'(x) = 0.$$

Proceeding from this is left as an exercise. □

Proposition 1

The crossing probability obtained above in the case $\kappa = 6$ matches with Cardy's formula for critical percolation on the triangular lattice; equivalently, in that case the map F in the statement of the previous theorem extends conformally to a map from $\mathbb{H}$ to the equilateral triangle with vertices $\{0, 1, e^{\pi i/3}\}$.

Proof (sketch). One only has to show the existence of the conformal extension of F, and to identify its image. The existence is clear because the term $\varphi(u) := u^{-4/\kappa}(1-u)^{-4/\kappa}$ has a unique holomorphic extension to $\mathbb{H}$ (which is simply connected and does not contain the singularities 0 and 1). The argument of φ is then constant on each of the intervals $(-\infty, 0)$, $(0, 1)$ and $(1, +\infty)$ of the real line, with the value respectively $-2\pi/3$, 0 and $2\pi/3$, which shows that the image is an equilateral triangle. The rest of the proof follows directly from the statement of Smirnov's theorem, and we refer to Clément's course for reference. □

Remark 4. *The extension of F to $\mathbb{H}$, mapping it to a triangle is an example of what is known as a* Schwarz-Christoffel transformation*, and in the most general case the conformal map between $\mathbb{H}$ and any simply connected polygon in $\mathbb{C}$ can be obtained using a similar integral formula, with exponents in the denominator coresponding to the angles of the polygon. The particular cases given by the previous theorem for other special values of κ (namely $16/3$ and 8) are interesting in relation to discrete models (respectively the FK-Ising model and the uniform spanning tree); drawing the corresponding pictures is left as an exercise.*

3.2.2. *Critical exponents.* We now turn to radial in the disk. For the remaining of the section, fix $\kappa > 4$ (as the estimates we are going to consider would be trivial in the case $\kappa \leqslant 4$). We will compute the *one-arm* and *two-arm exponents* of radial SLE(κ); the names should become clear as soon as one sees the SLE trace as an exploration process.

For $\varepsilon > 0$, let τ_ε be the first time the radial SLE(κ) trace visits the circle of radius ε around 0. Besides, let T be the first time t when K_t contains the whole unit circle. In addition, let U be the first time t when $\gamma_{[0,t]}$ contains both a clockwise and a counterclockwise loop separating 0 from the unit circle; note that almost surely $0 < T < U < \infty$.

Theorem 11 (*Lawler–Schramm–Werner*)

As $\varepsilon \to 0$, the one-arm probability *scales like*

$$P\left[\tau_\varepsilon < U\right] = \varepsilon^{\lambda_\kappa^{(1)} + o(1)} \quad \textit{with} \quad \lambda_\kappa^{(1)} = \frac{\kappa^2 - 16}{32\kappa};$$

the two-arm probability *(or* non-disconnection probability*) behaves like*

$$P\left[\tau_\varepsilon < T\right] = \varepsilon^{\lambda_k^{(2)} + o(1)} \quad \textit{with} \quad \lambda_k^{(2)} = \frac{\kappa - 4}{8}.$$

The case $\kappa = 6$ is of particular interest for us because of its ties to critical percolation: from these SLE estimates, one gets that the one-arm exponent of 2D percolation is $5/48$ and that the two-arm exponent is equal to $1/4$.

Proof (draftish). The overall idea is the same as that of the proof of Cardy's formula, *i.e.* to derive a PDE from Loewner's equation, to identify boundary conditions, and to exhibit a positive eigenfunction.

We first consider the case of the two-arm event. Let $x \in (0, 2\pi)$ and let $\mathcal{H}(x,t)$ be the event that radial SLE(κ) starting from 1 does not disconnect e^{ix} from 0 before time t. We will show that there exists $c > 0$ such that

$$e^{-\lambda_\kappa^{(2)} t} \left(\sin \frac{x}{2}\right)^{b_\kappa} \leqslant \mathbb{P}\big[\mathcal{H}(x,t)\big] \leqslant c e^{-\lambda_\kappa^{(2)} t} \left(\sin \frac{x}{2}\right)^{b_\kappa}.$$

Indeed, one can define a continuous, real valued process Y_t^x such that $Y_0^x = x$ and $g_t(e^{ix}) = \zeta_t \exp(i(\theta_t + Y_t^x))$ for every $t < \tau(e^{ix})$ (where $\tau(z)$ is the first time when the SLE disconnects z from 0); it plays the same role as the similarly named process in the proof of the crossing probability. Applying Itô's formula, one gets

$$\mathrm{d}Y_t^x = \sqrt{\kappa}\, \mathrm{d}B_t + \cot \frac{Y_t^x}{2}\, \mathrm{d}t.$$

Notice that $\tau(e^{ix}) = \inf\{t > 0 : Y_t^x \in \{0, 2\pi\}\}$, so $\mathbb{P}[\mathcal{H}(x,t)] = \mathbb{P}[\tau(e^{ix}) > t]$. Assuming that $f(x,t) := \mathbb{P}\big[\mathcal{H}(x,t)\big]$ is smooth on $(0,2\pi) \times [0,\infty)$ (the general theory of diffusion processes guarantees that), another application of Itô's formula shows that

$$\frac{\kappa}{2} f'' + \cot \frac{x}{2} f' = \partial_t f,$$

and besides $\lim_{x\to 0^+} f(x,t) = \lim_{x \to 2\pi^-} f(x,t) = 0$ and $f(x,0) = 1$ for all $x \in (0,2\pi)$. All that remains is to find a positive eigenfunction for the differential operator in the left-hand side with the appropriate boundary conditions, and the corresponding eigenvalue will be $-\lambda_\kappa^{(2)}$; the eigenfunction turns out to be precisely $\sin(x/2)^{b_\kappa}$ with $b_\kappa = 1 - 4/\kappa$.

The case of the one-arm exponent is exactly similar, the only difference being in the boundary condition: indeed the SLE curve is allowed to close loops in one direction, meaning that instead of being killed on both extremities of $(0, 2\pi)$, the process (Y_t^x) is instead killed at 0 and reflected at 2π. The corresponding boundary conditions are Dirichlet at 0 and Neumann at 2π, and the positive eigenfunction is then $\sin(x/4)^{b_\kappa}$ (with the same exponent as above) and the corresponding eigenvalue is precisely $-\lambda_\kappa^{(1)}$. □

4. Further topics not covered during the course