# Invariance of IDS under Darboux transformation and its application

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Random and Other Ergodic Problems
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# Motivation

The KdV equation is defined by

$$\partial_t q = \partial_x^3 q - 6q\partial_x q, \qquad q\left(0,x\right) = q\left(x\right).$$

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1. Solvability of the KdV equation starting from initial functions as general as possible, especially stationaly ergodic functions.
2. Study the asymptotic behavior of the solutions.

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- Then, $L^{q(t)}$ and $L^{q(0)}$ are **unitarily equivalent** by $U(t)$, that is

$$L^{q(t)} = U(t)^{-1}L^{q(0)}U(t).$$

# Weyl-functions (m-functions)

- 1D Schrödinger op. on $\mathbb{R}$: $L = L^q = -d^2/dx^2 + q$ for $q \in \mathcal{Q}$ :

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- For $\forall\lambda \in \mathbb{C} \setminus \mathbb{R}$, $\exists 1 f_\pm = f_\pm(x, \lambda, q)$ satisfying

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- Green function: For $x \geq y$

$$\left(L^q - \lambda\right)^{-1}(x, y) = g_\lambda(x, y, q) = -\frac{f_+(x, \lambda, q) f_-(y, \lambda, q)}{m_+(\lambda, q) + m_-(\lambda, q)}$$

# Reflectionless property

## Definition (reflectionless set)

$$\Sigma_{re}(q) = \left\{ \xi \in \mathbb{R}; \;\; m_+(\xi + i0, q) = -\overline{m_-(\xi + i0, q)} \right\}$$

- $\Sigma_{re}(q) \subset \Sigma_{ac}(q)$ and

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- A Rybkin (2008): Let $q(t, x)$ be a solution to the KdV equation

$$\partial_t q = \partial_x^3 q - 6q\partial_x q.$$

  Then, we have

$$\Sigma_{re}(q(0)) = \Sigma_{re}(q(t)).$$

# Construction of KdV flow

- For $\lambda_0 < \lambda_1$ set

$$\Omega_{\lambda_0,\lambda_1} = \{q \in \mathcal{Q};\ \Sigma(q) \subset [\lambda_0, \infty),\ \ \Sigma_{re}(q) \supset [\lambda_1, \infty)\}.$$

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- Marchenko-Lundina: $\Omega_{\lambda_0,\lambda_1} \ni q$ is holomorphic on

$$\left\{z \in \mathbb{C};\ \ |\mathrm{Im}\, z| < \sqrt{\gamma}^{-1}\right\} \qquad (\gamma = \lambda_1 - \lambda_0)$$

with uniform bound

$$|q(z) - \lambda_1| \le 2\gamma \left(1 - \sqrt{\gamma}\, |\mathrm{Im}\, z|\right)^{-2} \Longrightarrow \Omega_{\lambda_0,\lambda_1} \text{ compact.}$$

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## Theorem

*$\exists$A smooth flow $\{K(g)\}_{g\in\Gamma}$ on $\mathcal{W}\left(\supset \Omega_{-1,0}\right)$ s.t. $\{K(g)\}_{g\in\Gamma_{\text{real}}}$ is a flow on $\Omega_{-1,0}$ and*

$$\begin{aligned} \text{if } g_t(z) &= e^{-tz} \Longrightarrow \left(K(g_t)q\right)(x) = \left(S_t q\right)(x) = q(x+t) \ \ \textit{shift} \\ \text{if } g_t(z) &= e^{-4tz^3} \Longrightarrow \left(K(g_t)q\right)(x) \ \ \textit{solves the KdV equation} \\ &\ \ \vdots \end{aligned}$$

# Darboux transformation 1

- For $\zeta \in C$ such that $|\zeta| > 1$

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$$\Gamma_{\mathrm{real}} \ni \left(r_{\zeta_1} r_{\overline{\zeta_1}}\right)\left(r_{\zeta_2} r_{\overline{\zeta_2}}\right)\cdots\left(r_{\zeta_m} r_{\overline{\zeta_m}}\right), \quad (\zeta_i \text{ are non-real}).$$

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- For instance, choosing non-real $\zeta$ such that $t = \zeta^{-1} + \overline{\zeta}^{-1}$

$$\begin{cases} e^{-tz} = \lim_{n\to\infty}\left(r_{n\zeta}(z)\, r_{n\overline{\zeta}}(z)\right)^n \\ e^{-tz^3} = \lim_{n\to\infty}\left(\prod_j r_{n^{1/3}\omega_j}(z)\, r_{n^{1/3}\overline{\omega_j}}(z)\right)^n \end{cases}$$

where $\{\omega_j\}_{j=1,2,3}$ are distinct solutions to $z^3 = \zeta$.

# Darboux transformation 2

- For $\zeta \in \mathbb{C}$ s.t. $|\zeta| > 1$ and $q \in \Omega_{-1,0}$

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- If $\zeta \in \mathbb{R}$, then the Weyl function of $K\left(r_\zeta\right) q$ is

$$m_+\left(z, K\left(r_\zeta\right) q\right) = -\frac{z+\zeta^2}{m_+(z,q) - m_+\left(-\zeta^2, q\right)} - m_+\left(-\zeta^2, q\right).$$

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- For $q \in \mathcal{Q}$, $\lambda_0 \leq \inf \mathrm{sp} L^q$ Darboux defined a transformation of $q$

$$\left(\Delta_{\lambda_0} q\right)(x) = q(x) - 2\partial_x^2 \log f_+\left(x, \lambda_0, q\right).$$

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which shows $K\left(r_{\zeta}\right) q = \Delta_{-\zeta^{2}} q$.

# Darboux transformation 3

- Since for $q \in \mathcal{Q}$, $f_+(x, \lambda, q) = \exp\left(\int_0^x m_+(\lambda, S_t q)\, dt\right) \Longrightarrow$

$$\begin{aligned} \left(\Delta_{\lambda_0} q\right)(x) &= q(x) - 2\partial_x^2 \log f_+(x, \lambda_0, q) \\ &= 2\lambda_0 - q(x) + 2m_+(\lambda_0, S_x q)^2 . \end{aligned}$$

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- For $\lambda_0 \in \mathbb{C} \backslash \mathbb{R}$, $\left(\Delta_{\lambda_0} q\right)(x) \in \mathbb{C}$, however $\left(\Delta_{\lambda_0} \Delta_{\overline{\lambda_0}} q\right)(x) \in \mathbb{R}$. Hence

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## Lemma

*$\left\{\widetilde{\Delta}_{\lambda_0}\right\}_{\lambda_0 \in \mathbb{C} \backslash \mathbb{R}}$ is commutative, and $\widetilde{\Delta}_{\lambda_0} S_t = S_t \widetilde{\Delta}_{\lambda_0}$. Moreover, $\left(\widetilde{\Delta}_{\lambda_0}\right)^* \mu$ is ergodic if so is $\mu$. ($\mu$: a probability measure on $\mathcal{Q}$.)*

# Invariance of IDS

- Let $\mu$ be a $S_t$-invariant ergodic probability measure on $\mathcal{Q}$. The IDS $N_\mu$ of $\mu$ is defined by

$$\langle g_\lambda (0,0,q) \rangle_\mu \equiv \int_{\mathcal{Q}} g_\lambda (0,0,q) \, \mu (dq) = \int_{\mathbb{R}} \frac{1}{\xi - \lambda} dN_\mu (\xi)$$

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**Theorem**

*For any $\lambda_0 \in \mathbb{C}\backslash\mathbb{R}$ we have*

$$N_{\left(\widetilde{\Delta}_{\lambda_0}\right)^* \mu}(\xi) = N_\mu(\xi).$$

# Proof

- Johnson-Moser's method: For functions $f, g, h$ on $\mathcal{Q}$

$$f(S_x q) - g(S_x q) = \partial_x h(S_x q) \Longrightarrow \langle f(q) \rangle_\mu = \langle g(q) \rangle_\mu .$$

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- The Ricatti equation for $m_+\left(\lambda, S_x q\right)$

$$\partial_x m_+\left(\lambda, S_x q\right) = \frac{\partial_x f_+\left(x, \lambda, q\right)}{f_+\left(x, \lambda, q\right)} = q\left(x\right) - \lambda - m_+\left(\lambda, S_x q\right)^2 .$$

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$$\begin{aligned} & \partial_x \log\left(m_+(\lambda, S_x q) - m_+(\lambda_0, S_x q)\right) \\ = & \frac{\partial_x\left(m_+(\lambda, S_x q) - m_+(\lambda_0, S_x q)\right)}{m_+(\lambda, S_x q) - m_+(\lambda_0, S_x q)} \\ = & \frac{\lambda_0 - \lambda}{m_+(\lambda, S_x q) - m_+(\lambda_0, S_x q)} - m_+(\lambda_0, S_x q) - m_+(\lambda, S_x q) \end{aligned}$$

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- $\left\langle -\frac{\lambda - \lambda_0}{m_+(\lambda, q) - m_+(\lambda_0, q)} - m_+(\lambda_0, q)\right\rangle_\mu = \langle m_+(\lambda, q)\rangle_\mu$

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- Therefore, for any $\lambda_0 \in \mathbb{C}\backslash \mathrm{sp} L^q$

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- On the other hand, Johnson-Moser showed

$$\frac{d}{d\lambda} \left\langle m_+ \left(\lambda, q\right)\right\rangle_\mu = \int_{\mathbb{R}} \frac{1}{\xi - \lambda} dN_\mu \left(\xi\right) .$$

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- Consequently

$$\int_{\mathbb{R}} \frac{1}{\xi - \lambda} dN_{\left( \widetilde{\Delta}_{\lambda_0} \right)^* \mu} \left( \xi \right) = \int_{\mathbb{R}} \frac{1}{\xi - \lambda} dN_\mu \left( \xi \right) .$$

# Application to KdV

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**Theorem**

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  1. $K(g)\,q$ is constructed by iteration of Darboux transformation.

# Application to KdV

- On $\Omega_{-1,0}$ we have constructed a KdV flow $\{K(g)\}_{g\in\Gamma_{\text{real}}}$.

**Theorem**

*Let $g \in \Gamma_{\text{real}}$. Then, for any ergodic probability measure $\mu$ on $\Omega_{-1,0}$, $K(g)^{*}\mu$ is shift invariant and ergodic, and*

$$N_{K(g)^{*}\mu}(\xi) = N_{\mu}(\xi).$$

- This theorem is a consequence of
  1. $K(g)\,q$ is constructed by iteration of Darboux transformation.
  2. For each Darboux transformation the IDS remains as it was.

# Open problem

- Recall that on $\Omega_{\lambda_0,\lambda_1}$ the Weyl function $m_+ (\lambda, q(t))$ for a solution $q(t) = q(t, \cdot)$ to the KdV equation is obtained by

$$m_+ (\lambda, q(t)) = \lim_{n\to\infty} m_+ \left( \lambda, \left( \prod_j \widetilde{\Delta}_{n^{1/3}\omega_j} \right)^n q \right),$$

where $t = \zeta^{-1} + \overline{\zeta}^{-1}$ and

$\{\omega_j\}_{j=1,2,3}$ are distinct solutions to $z^3 = \zeta$.

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$\{\omega_j\}_{j=1,2,3}$ are distinct solutions to $z^3 = \zeta$.

- For $\lambda_0 \in \mathbb{C}\backslash\mathbb{R}$ and a Herglotz function $m$, define Darboux transformation by

$$\begin{cases} (\Delta_{\lambda_0} m)(z) = -\dfrac{z-\lambda_0}{m(z)-m(\lambda_0)} - m(\lambda_0) \\ \left(\widetilde{\Delta}_{\lambda_0} m\right)(z) = \left(\Delta_{\overline{\lambda_0}} \Delta_{\lambda_0} m\right)(z). \end{cases}$$

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- In the course of the proof the invariance of IDS under the Darboux transformation might play an important role.