Recent Results and Open Problems on Many-Body Localization

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Random and Other Ergodic Problems
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What is Many-Body Localization?

Recall: One-Body Localization:

- Anderson model for single 'electron' in disordered background (here 1D):

$$(Au)(n) = -u(n+1) - u(n-1) + \nu_n u(n), \quad u \in \ell^2(\mathbb{Z}),$$

  $(\nu_n)_{n\in\mathbb{Z}}$ i.i.d. random variables, sufficiently smooth distribution
- Localization (of eigencorrelators, in particular dynamical):

$$\mathbb{E}\left(\sup_{|g|\leq 1} |(g(A))_{jk}|\right) \leq Ce^{-\mu|j-k|}$$

- Main simplification of the underlying physics:

  No interaction between electrons!

Beyond the Anderson Model:

- Effect of interactions? (on a many-body electron gas)
- Does the many-body system stay localized? (for small interaction strength, low electron density, . . .)
- Neglected in physics for almost 50 years!
- Now very active (Quantum Information Theory)

What is many-body localization?

First (and rough) attempt at an answer:

System should have properties **similar to a non-interacting many-body system** of the form

$$H_N = \sum_{j=1}^{N} h_j \quad \text{on } \mathcal{H}_N = \bigotimes_{j=1}^{N} \mathcal{H}, \;\; N >> 1$$

Here

$$h_j = I \otimes \ldots \otimes I \otimes h \otimes I \otimes \ldots \otimes I \quad \text{(on } j\text{-th site)}$$

$$h = \text{one-particle Hamiltonian on } \mathcal{H}$$

Eigenstates: $\varphi = \varphi_1 \otimes \ldots \otimes \varphi_N$, Dynamics: $e^{-itH_N} = \bigotimes_{j=1}^{N} e^{-ith}$

Generally accepted manifestations of Many-Body Localization:

- Eigenstates close to product states, e.g. **exponential decay of correlations** for
  - ground state without assuming ground state gap ($T = 0$)
  - thermal states ($0 < T < \infty$)
  - general excited eigenstates ($T = \infty$)
- **Area laws for entanglement** of ground/excited/thermal states

More manifestations of MBL:

- **No many-body transport** (say in form of a zero-velocity Lieb-Robinson bound)
- Slow growth of **dynamical entanglement**
- **“Fock space localization”** (Gornyi et al 2005, Basko et al 2006) ?

Best available models for rigorous work (so far):

- Disordered harmonic oscillator systems (Nachtergaele/Sims/St. 2012/13)
- XY spin chain in random field (Hamza/Sims/St. 2012, Abdul-Rahman/Nachtergaele/Sims/St., Sims/Warzel 2015+)

These are simple *toy models*: Both can be reduced to study of an

Effective One-Particle Hamiltonian.

Thus results on localization for non-interacting systems (Anderson model and related) can be applied, sometimes after suitable extension.

Anisotropic XY chain in random field:

Particles = Spins (qubits), one-particle Hilbert space: $\mathbb{C}^2$

N-particle Hilbert space: $\mathcal{H} = \bigotimes_{j=1}^{N} \mathbb{C}^2 \quad (N >> 1)$

Hamiltonian:

$$H_N = -\sum_{j=1}^{N-1} \left[(1+\gamma)\sigma_j^X \sigma_{j+1}^X + (1-\gamma)\sigma_j^Y \sigma_{j+1}^Y \right] - \sum_{j=1}^{N} \nu_j \sigma_j^Z$$

Will mostly consider *isotropic* XY model: $\gamma = 0$

$\sum_{j=1}^{N} \nu_j \sigma_j^Z$ models transversal exterior magnetic field

$(\nu_j)_{j=1}^{\infty}$ i.i.d. random, "nice" distribution

Jordan-Wigner transform:

- $c_j := \sigma_1^Z \dots \sigma_{j-1}^Z a_j, \quad a_j = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}_j$
- Satisfy CAR:

$$\{c_j, c_k^*\} = \delta_{jk} I, \quad \{c_j, c_k\} = \{c_j^*, c_k^*\} = 0$$

- $H_N = \mathcal{C}^* M_N \mathcal{C}$, where $\mathcal{C} := (c_1, \dots, c_N, c_1^*, \dots, c_N^*)^t$
- Effective 1-particle Hamiltonian: $M_N = \begin{pmatrix} A_N & B_N \\ -B_N & -A_N \end{pmatrix}$

$$A_N = \begin{pmatrix} -\nu_1 & 1 & & \\ 1 & \ddots & \ddots & \\ & \ddots & \ddots & 1 \\ & & 1 & -\nu_N \end{pmatrix}, \quad B_N = \begin{pmatrix} 0 & \gamma & & \\ -\gamma & \ddots & \ddots & \\ & \ddots & \ddots & \gamma \\ & & -\gamma & 0 \end{pmatrix}$$

Rigorous results

Theorem (Hamza/Sims/St. 2012)

Let M_N be dynamically (one-particle) localized, i.e. there exist $\mu > 0$ and $C < \infty$ such that

$$\mathbb{E}\left(\sup_t |(e^{-iM_N t})_{jk}|\right) \le Ce^{-\mu|j-k|}$$

for all N, j and k.

Then H_N satisfies a **zero-velocity Lieb-Robinson bound** *after disorder averaging: There exist $C' < \infty$ and $\mu' > 0$ such that*

$$\mathbb{E}\left(\sup_t \|[e^{itH_N} A_j e^{-itH_N}, B_k]\|\right) \le C'\|A\|\|B\|e^{-\mu'|j-k|}$$

for all N, j, k and $A, B \in \mathbb{C}^{2\times 2}$.

Ground state correlations:

Theorem (Hamza/Sims/St. 2012)

Let $\gamma = 0$ (isotropic XY) and ψ_0 the normalized ground state of H_N (almost surely non-degenerate). Then there exist $C < \infty$ and $\eta > 0$ such that

$$\mathbb{E}\left(|\langle \psi_0, A_j B_k \psi_0 \rangle - \langle \psi_0, A_j \psi_0 \rangle \langle \psi_0, B_k \psi_0 \rangle|\right) \leq C \|A\| \|B\| N \, e^{-\eta |j-k|}$$

for all $N \in \mathbb{N}$, $1 \leq j, k \leq N$, $A, B \in \mathbb{C}^{2 \times 2}$.

Better results (including excited states):

Work in preparation by Sims and Warzel

Bipartite entanglement

Chain: $\Lambda = \{1, \dots, N\}$
Subchain: $\Lambda^A = \{r, \dots, r+\ell-1\} \subset \Lambda$, Length ℓ

$\mathcal{H}_N = \mathcal{H}^A \otimes \mathcal{H}^B$, where

$$\mathcal{H}^A = \bigotimes_{j \in \Lambda^A} \mathbb{C}^2, \quad \mathcal{H}^B = \bigotimes_{j = \Lambda \setminus \Lambda^A} \mathbb{C}^2$$

Pure state in $\mathcal{H}_N$: $\varphi \in \mathcal{H}$, $\|\varphi\| = 1$, $\rho_\varphi = |\varphi\rangle\langle\varphi|$

Reduction to $\mathcal{H}^A$: $\rho_\varphi^A = \mathrm{Tr}_{\mathcal{H}^B} \rho_\varphi$ (generally mixed)

Bipartite entanglement entropy:

$$\mathcal{E}(\rho_\varphi) := -\mathrm{Tr}\, \rho_\varphi^A \log \rho_\varphi^A$$

Area law uniform in energy

Theorem (Abdul-Rahman/St. 2015)

Suppose that H_N is almost surely simple and

$$\mathbb{E}\Big(\sup_{|g|\le 1} |(g(M_N))_{jk}|\Big) \le \frac{C}{1+|j-k|^\beta}$$

for some $\beta > 2$, uniformly in $N \in \mathbb{N}$ and $1 \le j, k \le N$. Then

$$\mathbb{E}(\sup_\psi \mathcal{E}(\rho_\psi)) \le C' < \infty,$$

uniformly in N, r and ℓ such that $1 \le r \le r+\ell-1 \le N$. Here the supremum is taken over **all** *normalized eigenstates ψ of H_N.*

Remarks:

(i) Why *area law*? Entanglement is proportional to *surface area* 2 of the subsystem $\Lambda^A = \{r, \ldots, r+\ell-1\}$. (By default it can grow like $\log \dim \mathcal{H}^A = \ell =$ *volume* of subsystem.)

(ii) Physical interpretation: All eigenstates satisfy area law, thus H_N is in the *fully many-body localized phase* (e.g. Huse et al 2006+, Bauer/Nayak 2013)

(iii) Related result by Hastings 2007: Area law for ground state entanglement in general 1D gapped spin systems

(iv) Related result by Pastur/Slavin 2014: Area law for expectation of ground state entanglement in disordered quasi-free Fermion systems

Dynamical Entanglement:

Idea:

- $\Lambda^A = \{1, \ldots, \ell\}$, $\Lambda^B = \{\ell + 1, \ldots, N\}$ (for simplicity)
- $\varphi = \varphi^A \otimes \varphi^B$, $\rho = |\varphi\rangle\langle\varphi|$ unentangled in $\mathcal{H}^A \otimes \mathcal{H}^B$:

$$\mathcal{E}(\rho) = 0$$

- $\tau_t(\rho) = e^{-itH_N} \rho \, e^{itH_N}$ becomes entangled at $t \neq 0$:

$$\mathcal{E}(\tau_t(\rho)) \neq 0$$

- In presence of disorder: $\mathcal{E}(\tau_t(\rho))$ should grow slowly in time (or not at all).

Example: Product of local eigenstates

Consider left and right ends of XY chain:

$$H_N^A = -\sum_{j=1}^{\ell-1}\Big[(1+\gamma)\sigma_j^X\sigma_{j+1}^X + (1-\gamma)\sigma_j^Y\sigma_{j+1}^Y)\Big] - \sum_{j=1}^{\ell}\nu_j\sigma_j^Z$$

$$H_N^B = -\sum_{j=\ell+1}^{N-1}\Big[(1+\gamma)\sigma_j^X\sigma_{j+1}^X + (1-\gamma)\sigma_j^Y\sigma_{j+1}^Y)\Big] - \sum_{j=\ell+1}^{N}\nu_j\sigma_j^Z$$

φ^A eigenstate of H_N^A, φ^B eigenstate of H_N^B

$\varphi = \varphi^A \otimes \varphi^B$, $\rho = |\varphi\rangle\langle\varphi|$, $\tau_t(\rho) = e^{-itH_N}\rho\, e^{itH_N}$

Theorem (Abdul-Rahman, Nachtergaele, Sims and St., in preparation)

Suppose that H_N is almost surely simple and

$$\mathbb{E}\big(\sup_{|g|\le 1} |(g(M_N))_{jk}|\big) \le Ce^{-\mu|j-k|}$$

for some $C < \infty$ and $\mu > 0$, uniformly in $N \in \mathbb{N}$ and $1 \le j, k \le N$. Then

$$\mathbb{E}\left(\sup_t \mathcal{E}(\tau_t(\rho))\right) \le C' < \infty$$

uniformly in N and ℓ and in the choice of the eigenstates φ^A and φ^B.

About the proofs

or

Why can we deal with XY?

About the proofs

(1) Jordan-Wigner transform (Lieb/Schultz/Mattis 1961):

$$\mathcal{C} = (c_1, \ldots, c_N, c_1^*, \ldots, c_N^*)^t$$

$$H_N = \mathcal{C}^* M_N \mathcal{C}$$

Thus:

$$\text{Diagonalize } M_N \quad \Longrightarrow \quad \text{Diagonalize } H_N$$

Reduction of dimension to $2N$ from 2^N.

About the proofs

(2) Wick's Rule: Eigenstates $\rho = |\varphi\rangle\langle\varphi|$ of H_N characterized by their *correlation matrices*:

$$\Gamma_\rho^{\mathcal{C}} = \begin{pmatrix} (\mathrm{tr}\, c_j^* c_k \rho)_{jk} & (\mathrm{tr}\, c_j c_k \rho)_{jk} \\ (\mathrm{tr}\, c_j^* c_k^* \rho)_{jk} & (\mathrm{tr}\, c_j c_k^* \rho)_{jk} \end{pmatrix} \in \mathbb{C}^{2N \times 2N}$$

(3) Correlation matrix of ρ = spectral projection of M_N:

$$\Gamma_\rho^{\mathcal{C}} = \chi_B(M_N), \quad B \subset \mathbb{R} \text{ suitable}$$

(ρ ground state: $B = (0, \infty)$)

About the proofs

(4)

$$\mathcal{E}(\rho) = -\operatorname{Tr} \rho^A \log \rho^A = -\operatorname{tr} \Gamma^A \log \Gamma^A$$

where Γ^A is a restriction of $\Gamma_\rho^{\mathcal{C}}$ (Kitaev et al 2003, compare A. Sobolev's talk for free Fermion systems in $\mathbb{R}^d$)

(5) Get area law for $\mathcal{E}(\rho)$ from Anderson localization

$$\mathbb{E}\left((\chi_B(M_N))_{jk}\right) \leq Ce^{-\mu|j-k|}$$

(Pastur/Slavin 2014 for ground state of free Fermion system)

(6) Generalize this to get result for dynamical entanglement.

Summary

- Random XY chain (and harmonic oscillator systems) can serve as toy models to illustrate the physical characteristics of the fully MBL regime.
- Proofs by reducing to an effective one-particle system (a.k.a. 'cheating').
  Physicists say: XY chain is a non-interacting many-body system.
- **Open challenge:** Develop methods which can demonstrate MBL for truly interacting systems such as the Heisenberg model, an electron gas, etc (Imbrie 2014)

Thank you!