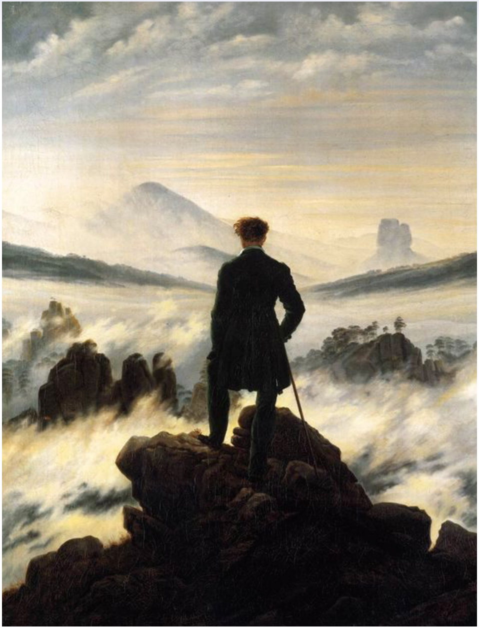

Mathias and Set Theory

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- 1965, BA at Trinity College, Cambridge University
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Années de pèlerinage:

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$\omega \longrightarrow (\omega)^\omega_2$ and Mathias Reals

For a set X of ordinals, $[X]^\gamma = \{y \subseteq X \mid y \text{ has order type } \gamma\}$, and

$$\beta \longrightarrow (\alpha)^\gamma_\delta$$

asserts that for any $f\colon [\beta]^\gamma \to \delta$, there is an $H \in [\beta]^\alpha$ *homogeneous* for f, i.e. $|f``[H]^\gamma| \leq 1$.

Ramsey's Theorem: For $0 < r, k < \omega$,

$$\omega \longrightarrow (\omega)^r_k\,,$$

Erdős: The "far-reaching generalization"

$$\omega \longrightarrow (\omega)^\omega_2,$$

"the strong partition property for ω" in what follows.

$Y \subseteq [\omega]^\omega$ is *Ramsey iff* $\exists x \in [\omega]^\omega([x]^\omega \subseteq Y \text{ or } [x]^\omega \subseteq [\omega]^\omega - Y)$.

So, $\omega \longrightarrow (\omega)^\omega_2$ *iff every* $Y \subseteq \omega$ is Ramsey.

Galvin-Prikry (1967): If Y is Borel, then Y Ramsey.

Silver (1968): If Y is analytic, then Y is Ramsey.

Solovay: In the Levy collapse extension $V[G]$,
(*) if Y is a set of reals definable from a real r, then there is a formula $\varphi(\cdot, \cdot)$ such that: $x \in Y$ *iff* $V[r][x] \models \varphi[r, x]$.

Mathias forcing has as conditions $\langle s, A\rangle$, where $s \subseteq \omega$ is finite, $A \subseteq \omega$ is infinite, and $\max(s) < \min(A)$, ordered by:
$\langle t, B\rangle \leq \langle s, A\rangle$ *iff* s is an initial segment of t and $B \cup (t - s) \subseteq A$.
(a) For any condition $\langle s, A\rangle$ and formula ψ of the forcing language, there is an infinite $B \subseteq A$ such that $\langle s, B\rangle$ decides ψ, i.e. s need not be extended to decide formulas, and
(b) If x is a Mathias real over a model M and $y \subseteq x$ is infinite, then y is a Mathias real over M.

$\omega \longrightarrow (\omega)^{\omega}_2$ in Solovay's inner model:
For a set Y of reals definable from a real r as in (*), by (a) there is in $V[r]$ a Mathias condition $\langle \emptyset, A\rangle$ that decides $\varphi(r, c)$, where c is the canonical name for a Mathias real. There is surely in $V[G]$ a real $x \subseteq A$ Mathias generic over $V[r]$, and by (b) x confirms that Y is Ramsey.

An ultrafilter U over ω is *Ramsey iff* for any $f : [\omega]^2 \longrightarrow 2$ there is an $H \in U$ homogeneous for f.

For a filter F, *F-Mathias forcing* is Mathias forcing with the additional proviso that conditions $\langle s, A\rangle$ are to satisfy $A \in F$.

Mathias [11]: a real x Mathias over a ground model V generates a Ramsey ultrafilter F on $\mathcal{P}(\omega) \cap V$ given by $F = \{X \in \mathcal{P}(\omega) \cap V \mid x - X \text{ is finite}\}$, and generically adjoining x to V is *equivalent* to first generically adjoining the corresponding F, without adjoining any reals, and then doing F-Mathias forcing over $V[F]$ to adjoin x.

A set A of infinite subsets of ω is a *happy family iff* $\mathcal{P}(\omega) - A$ is an ideal, and: whenever $X_i \in A$ with $X_{i+1} \subseteq X_i$ for $i \in \omega$ there is a $Y \in A$ which diagonalizes the X_i's, i.e. its increasing enumeration f satisfies $f(i+1) \in X_i$ for every $i \in \omega$. A happy family is just the sort through which one can force a Ramsey ultrafilter without adjoining any reals.

Varia

An filter F over ω is a *p-point iff* whenever $X_i \in F$ for $i \in \omega$, there is a $Y \in F$ such that $Y - X_i$ is finite for every $i \in \omega$.

The *Rudin-Keisler ordering* $\leq_{\mathrm{RK}}$ is generally defined by: For filters $F, G \subseteq \mathcal{P}(I)$, $F \leq_{\mathrm{RK}} G$ *iff* there is an $f: I \to I$ such that $F = f_*(G)$, where $f_*(G) = \{X \subseteq I \mid f^{-1}(X) \in G\}$.

$Fr = \{X \subseteq \omega \mid \omega - X \text{ is finite}\}$—the *Fréchet filter*. A filter over ω is *feeble*, being as far from being an ultrafilter as can be, *iff* there is a finite-to-one $f: \omega \to \omega$ such that $f_*(F) = Fr$.

Rudin: CH implies that there are p-points.

Mathias [4]: CH implies that there is an ultrafilter U over ω with no p-points below it in the RK-ordering.

Mathias [8]: For filters F over ω extending Fr, F is feeble *iff* a corresponding $P^F \subseteq \mathcal{P}(\omega)$ is Ramsey. Hence, $\omega \to (\omega)^\omega_2$ implies that every filter over ω extending Fr is feeble!

A filter over ω is *coherent iff* it extends Fr, is a p-point, and is not feeble.

Mathias [13]: if $0^\sharp$ does not exist or $2^{\aleph_0} \leq \aleph_{\omega+1}$, then there are coherent filters.

The elegant proof depended on a covering property for families of sets, with Jensen's recent Covering Theorem for L providing the ballast with $0^\sharp$.

Open Question: In ZFC, are there any coherent filters?

κ has the *strong partition property iff*

$$\kappa \longrightarrow (\kappa)^{\kappa}_{\alpha} \text{ for every } \alpha < \kappa.$$

Kechris-Woodin: In $L(\mathbb{R})$, AD iff Θ is a limit of cardinals satisfying the strong partition property.

Henle-Mathias [15]: A remarkably strong continuity for functions $[\kappa]^{\kappa} \to [\kappa]^{\kappa}$ and results about the Rudin-Keisler ordering in this context.

Henle-Mathias-Woodin [18]: Assuming $\omega \to (\omega)^{\omega}_2$, the Hausdorff extension not only adjoins no new real but no new sets of ordinals at all. So e.g. if AD, then in the Hausdorff extension of $L(\mathbb{R})$, Θ is still the limit of cardinals κ having the strong partition property *and* there is a Ramsey ultrafilter over ω. In particular, AD fails, and so $V = L(\mathbb{R})$ is necessary for the Kechris-Woodin characterization.

Mathias [17]:

An ordinal is *unsound iff* it has subsets A_n for $n \in \omega$ such that *un*countably many ordinals are realized as ordertypes of sets of form $\bigcup\{A_n \mid n \in a\}$ for some $a \subseteq \omega$.

Woodin had asked whether there is an unsound ordinal, eventually showed that AD implies that there is one less than ω_2.

While the issue remains unsettled in ZF, Mathias [17] showed:
(a) If ω_1 is regular, then every ordinal less than $\omega_1^{\omega+2}$ is sound, and
(b) if moreover $\aleph_1 \leq 2^{\aleph_0}$, i.e. there is a uncountable, well-orderable set of reals, then $\omega_1^{\omega+2}$ is exactly the least unsound ordinal.

The proofs proceed through an intricate analysis of indecomposable ordinals and a generalization of the well-known Milnor-Rado paradox.

Logic and terror (1978) [21]: $A \vee \neg A$ and $A \wedge \neg A$.

The Ignorance of Bourbaki (1986) [23]: Ignored Gödel's work on in completeness and the set theory that they fixed on was inadequate.

The Mac Lane – Mathias controversy: *What is Mac Lane missing?* (1992) [22], on Mac Lane's ZBQC, which is ZFC without Replacement and only Δ_0 Separation. *Is Mathias an ontologist?* (1992). *Strong statements of analysis* (2000) [26]. *Contrary statements about mathematics* (2000).

Mac Lane Set Theory

Zermelo set theory (Z): Extensionality, Empty Set, Pairing, Union, Power Set, Foundation, Infinity, and Separation.

Kripke-Platek set theory (KP): Extensionality, Empty Set, Pairing, Union, Foundation for Π_1 Classes (at Mathias' insistence), Δ_0 Separation, and Δ_0 Collection.

(Collection: $\forall x \in a \exists y \varphi \longrightarrow \exists b \forall y (\exists x (x \in a \wedge \varphi \longrightarrow y \in b))$.

Slim models of set theory (1996) [27]: In Z + KP, supertransitive inner models can be recursively defined that exhibit evident failures of Replacement, e.g. V_ω is not a set.

The strength of Mac Lane set theory (2001) [28]:

- Mathias carries out von Neumann's classical construction, with minimal hypotheses, of the inner model of well-founded sets, to effect the relative consistency of Foundation as well as Transitive Containment, viz. that every set is a member of a transitive set.
  MAC: ZBQC + Transitive Containment
  M: MAC – Choice.
- Axiom H: $\forall u \exists t (\cup t \subseteq t \wedge \forall z (\cup z \subseteq z \wedge |z| \leq |u| \longrightarrow z \subseteq t)$
  Mathias approximates having Mostowski collapses, i.e. transitizations of well-founded, extensional relations, and makes isomorphic identifications to get the relative consistency of having Axiom H, e.g. Con(MAC) implies Con(MAC + H). Drawing out the centrality of Axiom H, Mathias shows that over a minimal set theory H is equivalent to actually having Mostowski collapses, and with a minimal Skolem hull argument, that over MAC, H subsumes KP, being equivalent to Σ_1 Separation together with Δ_0 Collection.

- Mathias simulates Gödel's recursive set-by-set generation along well-orderings, takes transitizations, and makes identifications as before for adjoining H, to define L. Working out condensation, he then gets

  $$\text{Con(M)} \longrightarrow \text{Con(M + KP} + V = L).$$

  Mathias' approach notably provides the first explicit proof of Con(Z) $\longrightarrow$ Con(Z + AC), once announced by Gödel.
- Mathias layers, with the mediation of L, the region between M + KP and Z + KP in terms of Σ_n Separation. He shows that if $V = L$, Σ_n Separation implies the consistency of Σ_{n-1} Separation, and Σ_n Separation implies the same in the sense of L.

- Mathias develops a theory subsuming M + KP first isolated for "power-admissible sets" by Harvey Friedman. The germ is to incorporate $\forall x \subseteq y$ and $\exists x \subseteq y$ as part of bounded quantification. Working through a subtle syntactical analysis, Mathias develops normal forms and situates $\Delta_0^{\mathcal{P}}$ Separation.

  $\mathrm{KP}^{\mathcal{P}}$: M + KP + $\Pi_1^{\mathcal{P}}$ Foundation + $\Delta_0^{\mathcal{P}}$ Collection.

  Working in the Gandy Basis Theorem, standard parts of admissible sets, and forcing over ill-founded models, Mathias is able to establish the remarkable and surprising result that, unlike for KP,

  $$\mathrm{KP}^{\mathcal{P}} + V = L \text{ proves the consistency of } \mathrm{KP}^{\mathcal{P}},$$

  and delimitative results, e.g. even $\mathrm{KP}^{\mathcal{P}}$ + AC + "every cardinal has a successor" does not prove H (nor therefore Σ_1 Separation).

- Mathias attends, lastly, to systems type-theoretic in spirit, working the theme that MAC, with its Power Set and Δ_0 Separation, is latently in this direction. Mathias shows that e.g. in MAC "strong stratifiable Σ_1 Collection" is provable, a narrow bridge to Quine's NF. Also, Mathias provides the first explicit proof of a result implicit in Kemeny's 1949 thesis, that MAC is equiconsistent with the simple theory of types (together with the Axiom of Infinity).

A note on the schemes of replacement and collection (2007) [35]:

(Repcoll) $\forall y \in u \exists ! z \varphi(y, z) \longrightarrow \exists w \forall y \in u(\varphi(y, z) \rightarrow z \in \omega)$.

Loosely speaking, any class function restricted to a set has range included in a set. Replacement is with the "$\rightarrow$" replaced by "$\leftrightarrow$" and Collection is with the uniqueness "!" deleted, and hence Repcoll is weaker than either.

The weak theory M + Repcoll fully implies ZF.

(a) Infinity can be effaced from both sides. (b) Transitive Containment can be effaced from the left side.

Mac Lane had envisioned his foundational set theory as sufficient for mathematics, inclusive of category theory, and the Mac Lane–Mathias controversy had turned on the possible necessity of strong, even large cardinal, hypotheses. In recent affirmations, the very strong large cardinal principle, Vopenka's Principle, has been pressed to display categorical consequences, e.g. that all reflective classes in locally presentable categories are small-orthogonality classes. In collaborative work with Joan Bagaria, Carles Casacuberta and Jiří Rosický, Mathias [36][40] (a) sharpened this result by reducing the hypothesis to having a proper class of supercompact cardinals yet still drawing a substantial conclusion; (b) got categorical equivalents to Vopenka's Principle; and (c) showed as a consequence that "the existence of cohomological localizations of simplicial sets, a long-standing open problem in algebraic topology, is implied by the existence of arbitrarily large supercompact cardinals".

Dynamics

For χ be a Polish space (complete, separable metric space) and $f\colon \chi \to \chi$ a continuous function, define a relation $\curvearrowright_f$ on χ by:

$x \curvearrowright_f y$ *iff* $\exists$ an increasing $\alpha\colon \omega \to \omega$ with $\lim_{n\to\infty} f^{\alpha(n)}(x) = y$.

$$\omega_f(x) = \{y \mid x \curvearrowright_f y\} \text{ and}$$
$$\Gamma_f(X) = \bigcup\{\omega_f(x) \mid x \in X\},$$

both being $\curvearrowright_f$-closed as $\curvearrowright_f$ is a transitive relation. For $a \in \chi$,

$$A^0(a,f) = \omega_f(a)\,,$$
$$A^{\beta+1}(a,f) = \Gamma_f(A^\beta(a,f))\,, \text{and}$$
$$A^\lambda(a,f) = \bigcap\nolimits_{\nu<\lambda} A^\nu(a,f) \text{ for limit } \lambda.$$

Then $A^0(a,f) \supseteq A^1(a,f) \supseteq A^2(a,f) \supseteq \ldots$ again by the transitivity of $\curvearrowright_f$. Let $\theta(a,f)$ by the least ordinal θ such that $A^{\theta+1}(a,f) = A^\theta(a,f)$, and let $A(a,f) = A^{\theta(a,f)}(a,f)$. The thrust of Mathias' work is to investigate the closure ordinal $\theta(a,f)$ as providing the dynamic sense of $\curvearrowright_f$.

Mathias [29]: $\theta(a, f) \leq \omega_1$, *with the inequality being strict when* $A(a, f)$ *is Borel.*

He associated to each $x \in \omega_f(a)$ a tree of $\curvearrowright_f$-descending finite sequences, so that $x \notin A(a, f)$ *iff* the tree is well-founded. Then he adapted to $\curvearrowright_f$ the Kunen proof of the Kunen-Martin Theorem on bounding ranks of well-founded trees. Notably, Mathias' argument works for any transitive relation in place of $\curvearrowright_f$, and so it can be seen as a nice appeal to well-foundedness in the study of transitivity.

Particular to $\curvearrowright_f$ and dynamics, Mathias established a striking result about recurrent points, i.e. points b such that $b \curvearrowright_f b$. With an intricate metric construction of a recurrent point, he showed that $y \in A(a, f)$ *iff* for some z, $a \curvearrowright_f z \curvearrowright_f z \curvearrowright_f y)$, so that in particular there are recurrent points in $\omega_f(a)$ exactly when $A(a, f) = \emptyset$.

More particular still with χ being Baire space, ${}^{\omega}\omega$, Mathias showed that if $s\colon {}^{\omega}\omega \to {}^{\omega}\omega$ is the shift function given by $s(g)(n) = g(n+1)$, then for each $\zeta < \omega$ there is an $a \in {}^{\omega}\omega$ such that $\theta(a, s) = \zeta$, a "long delay". For this, he carefully embedded countable well-founded trees into the graph of $\curvearrowright_f$.

Whether $\theta(a, f)$ can be ω_1 for any χ, a, and f remains an open question. Adapting his embedding apparatus to ill-founded tress and using a Cantor-Bendixson analysis on the hyperarithmetic hierarchy, Mathias provided an effective answer. There is a recursive $a \in {}^{\omega}\omega$ such that $\theta(a, s) = \omega_1^{\mathrm{CK}}$, the first non-recursive ordinal.

Weaker Set Theories

75-page *Weak systems of Gandy, Jensen, and Devlin* [34]:

Definitive analysis of set theories weaker than Kripke-Platek (KP) for lack of full Δ_0 Collection.

- ReS: KP without Δ_0 Collection.
- DB: ReS augmented with having Cartesian products.
- GJ: DB augmented with Rudimentary Replacement: for Δ_0 formulas ϕ,

$$\forall x \exists w \forall v \in x \exists t \in w \forall u (u \in t \longleftrightarrow u \in x \wedge \phi(u, v)) .$$

- Further augmentations with restricted versions of Δ_0 Collection are formulated, getting closer full KP.

Distinctive contribution: Add an axiom S asserting for all x the existence of $S(x) = \{y \mid y \subseteq x \text{ is finite}\}$. He allowed $\forall y \in S(x)$ and $\exists y \in S(x)$ as part of bounded quantification and set up a corresponding hierarchy of Σ^S_n formulas. With that, he augmented the various systems with Infinity and S, and established corresponding Separation, Collection.

Mathias detailed the flaws in Devlin's *Constructibility*, especially the inadequacy of his basic set theory for formulating the satisfaction predicate. GJ + Infinity does work, as it axiomatizes the rudimentary functions. Mathias shows that what also works for a parsimonious development of constructibility is Devlin's system (DB + Infinity + (full) Foundation) as augmented by S, as well as a particularly enticing subsystem MV:

$$\text{DB} + \text{Infinity} + \forall a \forall k \in \omega([a]^k \in V).$$

57-page *Rudimentary recursion, gentle functions and provident sets* [41], with Nathan Bowler: Worked to a weakest theory that will support a smooth, recognizable theory of forcing.

- Rudimentary recursive functions, given by $F(x) = G(F \restriction x)$ where G is rudimentary.
- Gentle functions, functions $H \circ F$ where H is rudimentary and F is rudimentary recursive. The composition of gentle functions *is* gentle.
- p-rudimentary recursive functions, given by $F(x) = G(p, F \restriction x)$ with p as a parameter in the recursion.
- Forcing is to be done over provident sets. A set is *provident iff* it is non-empty, transitive, closed under pairing and for all $x, p \in A$ and p-rudimentary recursive F, $F(x) \in A$.

The Jensen rudimentary functions are nine, and can be but together into one set formation function T such that for transitive u, $\bigcup_{n\in\omega} T^n(u)$ is rudimentarily closed. With this, given a transitive set c, define c_ν and P^c_ν by simultaneous recursion:

$$\begin{aligned} c_0 &= \emptyset\,, & c_{\nu+1} &= c \cap \{x \mid x \subseteq c_\nu\}\,, & c_\lambda &= \textstyle\bigcup_{\nu<\lambda} c_\nu \\ P^c_0 &= \emptyset\,, & P^c_{\nu+1} &= T(P^c_\nu) \cup c_{\nu+1} \cup \{c_\nu\}\,, & P^c_\lambda &= \textstyle\bigcup_{\nu<\lambda} P^c_\nu\,. \end{aligned}$$

- For transitive c and indecomposable ordinal θ, P^c_θ is provident.
- The *provident closure* of any M: $\mathrm{Prov}(M) = \bigcup\{P^c_\theta \mid c$ is the transitive closure of some finite subset of $M\}$, where θ is the least indecomposable ordinal not less than the set-theoretic rank of M. So if M is already provident, $\mathrm{Prov}(M) = M$ exhibits a canonical ramification.
- A finite set of axioms Prov warranting the recursion so that the transitive models of Prov are exactly the provident sets. J_ν is provident *iff* $\omega\nu$ is indecomposable, so that $J_\omega = \mathrm{HF}$ is provident, the only provident set not satisfying Infinity, and so are $J_{\omega^2}, J_{\omega^3}\,\ldots$.

49-page *Provident sets and rudimentary set forcing* [42]: Carries out a parsimonious development of forcing. The Shoenfield-Kunen approach is taken, carefully tailored to be effected in Prov + Infinity.

- If M is provident, $P \in M$ is a forcing partial order, and G is P-generic over M, then $M[G]$ is provident, with $M[G] = \mathrm{Prov}(M \cup \{G\})$.
- The various axioms of set theory, like Power Set, persist from M to $M[G]$.

[42] is a veritable paean to formalism and forcing, one that exhibits an intricate melding of axiomatics and technique in set theory. As such, it together with [41] is Mathias' arguably most impressive accomplishment in axiomatics.

Bourbaki

Bourbaki in their 1954-1957 Théorie des Ensembles adapted for purposes of quantification the Hilbert ε-operator, which for each formula φ introduces a term $\varepsilon x \varphi$ replicating the entire formula. Hilbert had $\varphi(t) \rightarrow \varphi(\varepsilon x \varphi)$ for terms t, and defined the quantifiers by $\exists x \varphi \longleftrightarrow \varphi(\varepsilon x \varphi)$ and $\forall x \varphi \longleftrightarrow \varphi(\varepsilon x \neg \varphi)$. Mathias [30] underlined the cumulating complications in Bourbaki's rendition by showing that Bourbaki's definition of the cardinal number 1, itself awkward, when written out in his formalism would have length 4,523,659,424,929 !

Bourbaki's 1949 system has Extensionality, Separation, Power Set, ordered pair as primitive with axioms to match, and Cartesian products. From Separation and Power set, one gets singletons $\{x\}$. Mathias [37] showed, bluntly, that there is a model of Bourbaki's system in which the Axiom of (unordered) Pairs fails!

109-page *Hilbert, Bourbaki and the scorning of logic* [39]:

(a) Hilbert in 1922 proposed an alternative treatment of first-order logic using his ε-operator,
(b) which, despite its many unsatisfactory aspects, was adopted by Bourbaki
(c) and by Godement for his classic *Cours d'Algèbre*, though leading him to express distrust of logic.
(d) It is this distrust, intensified to a phobia by the vehemence of Dieudonné's writings,
(e) and fostered by, for example, the errors and obscurities of a well-known undergraduate text,
(f) that has, it is suggested, led to the exclusion of logic from the CAPES examination—"tout exposé de logique formelle est exclu".
(g) Centralist rigidity has preserved the underlying confusion and consequently flawed teaching;
(h) the recovered will start when mathematicians adopt a post-Gödelian treatment of logic.