Ramsey theory in topological dynamics

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The Role of the Higher Infinite in Mathematics and Other Disciplines
INI, Cambridge
December 14, 2015

The author is supported by the grant FAPESP 2013/14458-9.

Ramsey everywhere

Ramsey-type phenomena appear accross mathematics - in set theory, model theory, finite combinatorics, geometry, analysis, topological dynamics, ergodic theory, additive number theory, ...

Invisible structures

"Is there a meaningful classification/clusterization of Ramsey phenomena based on deep invisible structures inherent in them?" [1]

[1] Misha Gromov, *Number of Questions*.

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- For every colouring of k-element suborders of an N-element linear order by r colours, there is an m-element suborder whose further k-element suborders have the same colour.

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- For every colouring of k-element suborders of an N-element linear order by r colours, there is an m-element suborder whose further k-element suborders have the same colour.
- For every colouring of k-dimensional faces of an N-dimensional simplex δ^N by r colours, there is an m-dimensional face δ^m of δ^N whose k-dimensional faces all have the same colour.

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RAMSEY CLASSES

- finite linear orders (Ramsey)
- finite linearly ordered graphs (Nešetřil and Rödl)
- finite linearly ordered metric spaces (Nešetřil)
- finite Boolean algebras (Graham and Rothschild)

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Then $\mathcal{K}$ satisfies the amalgamation property:

For $A, B, C \in \mathcal{A}$ and embeddings $i : A \longrightarrow B$ and $j : A \longrightarrow C$ there are $D \in \mathcal{K}$ and embeddings $k : B \longrightarrow D$ and $l : C \longrightarrow D$ such that $k \circ i = l \circ j$.

High symmetry

Groups of homeomorphisms

Ramsey $\iff$ thick

Ramsey $\iff$ extremely amenable

Disjoint syndetic sets

Lots of syndetic sets

Finite Ramsey degree

Eamples

Finite Ramsey degree $\Longleftrightarrow$ metrizable UMF

Finite Ramsey degree $\iff$ precompact Ramsey expansion

Uncountable case

Spheres

Continuous objects

Projective Fraïssé

Lelek fan

Lelek fan II

The pseudo-arc

Approximate Ramsey property

Gurarij spaces $\mathbb{G}$

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KUBIŚ-SOLECKI; HENSON
Simple proof - metric Fraïssé theory.

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Katětov construction

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Theorem (BLLM)

$\mathrm{Iso}_l(\mathbb{G})$ *is extremely amenable.*

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Theorem (Nešetřil)

Linearly ordered finite metric spaces satisfy the (exact) Ramsey property.

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FACT

$T : \{0,1\}^{\mathbb{Z}} \longrightarrow \{0,1\}^{\mathbb{Z}}$ the shift $\Rightarrow$ T-invariant probability measures form P

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Theorem (B-LA-M)

$M(AH(P)) \cong \widehat{AH(P)/AH_p}(P) \cong P$

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What is the universal minimal flow of $\mathrm{Homeo}(\mathcal{Q})$?

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Theorem (BLLM)

$M(\mathrm{Aut}(\mathcal{Q})) = \{-1,1\}^{\mathbb{N}} \times LO(\mathbb{N})$.

THANK YOU

OBRIGADA