Rado’s Conjecture, Strong Chang’s Conjecture, Tree Properties and Two Cardinal Square Principles

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The Role of the Higher Infinite in Mathematics and Other Disciplines
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Rado’s Conjecture (RC)

Definition (Rado’s Conjecture)

A family of intervals of a linearly ordered set is the union of countably many disjoint subfamilies (σ-disjoint) if and only if every subfamily of size ℵ₁ is σ-disjoint.
Rado’s Conjecture (RC)

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A family of intervals of a linearly ordered set is the union of countably many disjoint subfamilies ($\sigma$-disjoint) if and only if every subfamily of size $\aleph_1$ is $\sigma$-disjoint.
Todorˇcevi´c has shown the consistency of this statement relative to the consistency of the existence of a strongly compact cardinal. Moreover it is shown that $\text{RC}$ is consistent with $\text{CH}$ as well as consistent with the negation of $\text{CH}$. However $\text{MA}_{\omega_1}$ implies the negation of $\text{RC}$.
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Some applications of RC

Theorem (Todorˇcevi´c, 1993)
Rado’s Conjecture implies (some examples):
1. $2^{\omega_0} \leq \omega_2$,
2. $\theta^{\omega_0} = \theta$ for all regular $\theta \geq \omega_2$,
3. the Singular Cardinal Hypothesis,
4. $\square_\kappa$ fails for every uncountable cardinal $\kappa$,
5. $\text{CC}^*$, etc.

Theorem (Feng, 1999)
Rado’s Conjecture implies the presaturation of the nonstationary ideal on $\omega_1$.  

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Some applications of RC

Theorem (Todorčević, 1993)

Rado's Conjecture
Some applications of RC
Special Aronszajn trees
The Tree Property
The Strong Tree Property
Weak squares
Ascent paths and square sequences

Rado's Conjecture implies (some examples):
1. $\aleph_0 \leq \omega_2$,
2. $\theta = \aleph_0$ for all regular $\theta \geq \aleph_2$,
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Rado’s Conjecture and special Aronszajn trees
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We recall that an $\kappa^+$-Aronszajn tree is a tree $T$ of height $\kappa^+$ with levels of cardinality $\kappa$, but no chains of length $\kappa^+$. 
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**Theorem (Specker)**

CH implies there is a special $\aleph_2$-Aronszajn tree.
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Strong Chang’s Conjecture

We consider the following strong version of Chang’s Conjecture:

Definition (CC∗):

For every regular cardinal $\kappa \geq \omega^2$, there are arbitrary large $\lambda$ such that for every countable $M \prec H_\lambda$ and for every $a \in [\kappa]^{\omega_1}$, there is a countable $M^* \prec H_\lambda$ and $b \in M^* \cap [\kappa]^{\omega_1}$ such that $M^* \supseteq M$ and $M^* \cap \omega_1 = M \cap \omega_1$. 

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Rado’s Conjecture, Strong Chang’s Conjecture, Tree Properties
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CC∗ is a consequence of RC. Actually, we proved the following:

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The Tree Property (TP)

A regular cardinal $\kappa$ has the tree property and we denote it by $\text{TP}(\kappa)$, if every tree $T$ of height $\kappa$, with levels of size less than $\kappa$ has a cofinal branch.
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- $\text{TP}(\aleph_0)$ holds. (König)
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What about trees of height $\omega_2$ and levels of size $\omega_1$?
The Tree Property for $\omega_2$
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- CH implies there is a special $\aleph_2$-Aronszajn tree. (Specker)
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- PFA implies TP($\omega_2$). (Baumgartner)
The Tree Property for $\omega_2$

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A natural question is if under $\text{RC}$, the negation of the Continuum Hypothesis is enough to imply there are no $\aleph_2$-Aronszajn trees at all, i.e. if $\text{TP}(\omega_2)$ holds.
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Theorem (T.-Wu, 2015)
CC
∗
+
¬
CH
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TP(ω²).

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**Theorem (T.-Wu, 2015)**
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\[ \text{CC}^* + \neg \text{CH} \rightarrow \text{TP}(\omega_2). \]
The Strong Tree Property

Definition (Jech-Weiß)

Let $\kappa > \omega_1$ be a regular cardinal, and $\lambda \geq \kappa$. Suppose we have a collection of sets $\{F_a \in P(2^a) : a \in [\kappa] < \lambda\}$ such that

1. for every $a \in [\kappa] < \lambda$, $|F_a| < \lambda$,
2. for $a, b \in [\kappa] < \lambda$, $a \subseteq b \rightarrow \forall f \in F_b \exists g \in F_a$ such that $f \upharpoonright a = g$.

We call $F = \bigcup_{a \in [\kappa] < \lambda} F_a$ a $\kappa, \lambda$-tree, and $F_a$ the level $a$ of $F$ for $a \in [\kappa] < \lambda$.
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We call $\mathcal{F} = \bigcup_{a \in [\kappa]^{<\lambda}} \mathcal{F}_a$ a $(\kappa, \lambda)$-tree, and $\mathcal{F}_a$ the level $a$ of $\mathcal{F}$ for $a \in [\kappa]^{<\lambda}$. 
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The Strong Tree Property

We furnish $F$ with the following order: for $f, g \in F$, $f \leq F g$ if and only if $g \upharpoonright \text{dom}(f) = f$.

Observe that in general, $\leq F$ is not a tree order.
We furnish $\mathcal{F}$ with the following order: for $f, g \in \mathcal{F}$, $f \leq_{\mathcal{F}} g$ if and only if $g\upharpoonright_{\text{dom}(f)} = f$. 

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Observe that in general, $\leq_{\mathcal{F}}$ is not a tree order.
The Strong Tree Property

Definition
We say that $\lambda$ has the Strong Tree Property if every $(\kappa, \lambda)$-tree has a cofinal branch for every $\kappa \geq \lambda$. 
The Strong Tree Property

A cofinal branch through $\mathcal{F}$ is a function $B : \kappa \to 2$ such that $B \upharpoonright a \in \mathcal{F}$ for every $a \in [\kappa]^{<\lambda}$. 

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Theorem (Weiß)
PFA implies $\aleph_2$ has the Strong Tree Property.

Theorem (Sakai and Velickovic)
$\text{CC}^*$ and $\text{MA}_{\omega_1}(\text{Cohen})$ together imply $\aleph_2$ has the Strong Tree Property.

Theorem (T.-Wu, 2015)
$\text{CC}^*$ and $\neg \text{CH}$ together imply $\aleph_2$ has the Strong Tree Property.

Corollary
RC and $\neg \text{CH}$ together imply $\aleph_2$ has the Strong Tree Property.
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CC* and $\neg$CH together imply $\aleph_2$ has the Strong Tree Property.

Corollary

RC and $\neg$CH together imply $\aleph_2$ has the Strong Tree Property.
Weak squares

We recall the following variation on Jensen's principle $\Box_\kappa$.

**Definition**

For cardinals $\lambda \leq \kappa$, let $\Box_\lambda \kappa$ be the statement that there is a sequence $\langle C_\alpha : \alpha < \kappa^+ \rangle$ such that:

1. $C_\alpha$ is a family of closed subsets of $\alpha$ with at least one unbounded in $\alpha$.
2. $|C_\alpha| \leq \lambda$ and $\text{otp}(C_\alpha) \leq \kappa$ for all $C_\alpha \in C_\alpha$.
3. If $C_\beta \in C_\beta$ and if $\alpha$ is a limit point of $C_\beta$, then $C_\beta \cap \alpha \in C_\alpha$.
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For cardinals $\lambda \leq \kappa$, let $\square^\lambda_\kappa$ be the statement that there is a sequence $\langle C_\alpha : \alpha < \kappa^+ \rangle$ such that:

1. $C_\alpha$ is a family of closed subsets of $\alpha$ with at least one unbounded in $\alpha$.
2. $|C_\alpha| \leq \lambda$ and $\text{otp}(C_\alpha) \leq \kappa$ for all $C_\alpha \in C_\alpha$.
3. If $C_\beta \in C_\beta$ and if $\alpha$ is a limit point of $C_\beta$, then $C_\beta \cap \alpha \in C_\alpha$. 
Weak squares

We recall the following variation on Jensen’s principle □_κ.

Definition
For cardinals \( \lambda \leq \kappa \), let □_κ^\lambda be the statement that there is a sequence \( \langle C_\alpha : \alpha < \kappa^+ \rangle \) such that:

1. \( C_\alpha \) is a family of closed subsets of \( \alpha \) with at least one unbounded in \( \alpha \).
Weak squares

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For cardinals $\lambda \leq \kappa$, let $\Box^\lambda_\kappa$ be the statement that there is a sequence $\langle C_\alpha : \alpha < \kappa^+ \rangle$ such that:

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3. If $C \in C_\beta$ and if $\alpha$ is a limit point of $C$, then $C \cap \alpha \in C_\alpha$. 
Weak squares

κ is a consequence of the cardinal assumption κ = κ and, as it is well-known, □κ is equivalent to the existence of a special κ+ Aronszajn tree. So we get the following corollary:

**Corollary**

Under CC∗, □ω₁ and CH are equivalent.
\( \square^{\kappa}_{\kappa} \) is a consequence of the cardinal assumption \( \kappa^{<\kappa} = \kappa \) and, as it is well-known, \( \square^{\kappa}_{\kappa} \) is equivalent to the existence of a special \( \kappa^+ \)-Aronszajn tree.
Weak squares

\(\square^\kappa_\kappa\) is a consequence of the cardinal assumption \(\kappa^{<\kappa} = \kappa\) and, as it is well-known, \(\square^\kappa_\kappa\) is equivalent to the existence of a special \(\kappa^+\)-Aronszajn tree.
So we get the following corollary:
Weak squares

□\kappa^+ is a consequence of the cardinal assumption \kappa^{<\kappa} = \kappa and, as it is well-known, □\kappa^+ is equivalent to the existence of a special \kappa^+-Aronszajn tree.
So we get the following corollary:

Corollary
$\square_\kappa^\kappa$ is a consequence of the cardinal assumption $\kappa^{<\kappa} = \kappa$ and, as it is well-known, $\square_\kappa^\kappa$ is equivalent to the existence of a special $\kappa^+$-Aronszajn tree.

So we get the following corollary:

**Corollary**

*Under CC*, $\square_{\omega_1}^{\omega_1}$ and CH are equivalent.*
We proved actually the following:

Theorem (Todorcevic-T., 2012)
Assume $\text{RC}$. Then the following holds:

$\neg \Box <\kappa \kappa$ for any uncountable cardinal $\kappa$,
$\neg \Box \omega_1 \omega_1$,
$\neg \Box \kappa \kappa$ for every singular cardinal of cofinality $\omega$. 

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Assume RC. Then the following holds:
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**Theorem (Todorcevic-T., 2012)**

Assume RC. Then the following holds:

- $\neg \Box_{<\kappa}^\kappa$ for any uncountable cardinal $\kappa$,
We proved actually the following:

**Theorem (Todorcevic-T., 2012)**

Assume $\text{RC}$. Then the following holds:

- $\neg \square^\kappa_\kappa$ for any uncountable cardinal $\kappa$,
- $\neg \text{CH}$ implies $\neg \square^\omega_\omega$,
We proved actually the following:

**Theorem (Todorcevic-T., 2012)**

Assume RC. Then the following holds:

- $\neg \Box^<_\kappa$ for any uncountable cardinal $\kappa$,
- $\neg \text{CH}$ implies $\neg \Box^{\omega_1}_{\omega_1}$,
- $\neg \Box^\kappa_\kappa$ for every singular cardinal of cofinality $\omega$. 
We proved actually the following:

**Theorem (Todorcevic-T., 2012)**

Assume RC. Then the following holds:

- $\neg \square^<\kappa$, for any uncountable cardinal $\kappa$,
- $\neg \text{CH}$ implies $\neg \square^\omega_1$,
- $\neg \square^\kappa_\kappa$ for every singular cardinal of cofinality $\omega$. 


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Theorem (Cummings-Magidor, Baumgartner)
Assume MM. Then we have the following:
\[\neg \square < \text{cof}(\kappa) \kappa\]
whenever \(\kappa\) is an uncountable cardinal \(\kappa\),
\[\neg \square \omega_1 \omega_1\] (Baumgartner),
\[\neg \square \kappa \kappa\]
whenever \(\kappa\) is an uncountable cardinal of cofinality \(\omega\),
\[\neg \square \kappa < \kappa\]
whenever \(\kappa\) is an uncountable cardinal of cofinality \(\omega_1\).
Theorem (Cummings-Magidor, Baumgartner)
Theorem (Cummings-Magidor, Baumgartner)

Assume $\text{MM}$. Then we have the following:
Theorem (Cummings-Magidor, Baumgartner)

Assume MM. Then we have the following:

- \( \neg \square_{\kappa}^{\text{cof}(\kappa)} \) whenever \( \kappa \) is an uncountable cardinal \( \kappa \),
Theorem (Cummings-Magidor, Baumgartner)

Assume MM. Then we have the following:

- $\neg \Box_{\kappa}^{<\text{cof}(\kappa)}$ whenever $\kappa$ is an uncountable cardinal $\kappa$,
- $\neg \Box_{\omega_1}^{\omega_1}$ (Baumgartner),

$\Box_{\kappa}$ and $\Box_{\omega_1}$ denote the square principles at $\kappa$ and $\omega_1$, respectively.
Theorem (Cummings-Magidor, Baumgartner)

Assume MM. Then we have the following:

\[ \neg \Box_{<\text{cof}(\kappa)} \kappa \text{ whenever } \kappa \text{ is an uncountable cardinal } \kappa, \]

\[ \neg \Box_{\omega_1} \omega_1 \text{ (Baumgartner),} \]

\[ \neg \Box_{\kappa}^\kappa \text{ whenever } \kappa \text{ is an uncountable cardinal of cofinality } \omega, \]
Theorem (Cummings-Magidor, Baumgartner)

Assume $\text{MM}$. Then we have the following:

1. $\neg \square_{\text{cof}(\kappa)}^{<\kappa}$ whenever $\kappa$ is an uncountable cardinal $\kappa$,
2. $\neg \square_{\omega_1}^\omega$ (Baumgartner),
3. $\neg \square_{\kappa}^\kappa$ whenever $\kappa$ is an uncountable cardinal of cofinality $\omega$,
4. $\neg \square^{<\omega_1}^\kappa$ whenever $\kappa$ is an uncountable cardinal of cofinality $\omega_1$. 
Cummings-Magidor Theorem

If there is a supercompact cardinal, then there is a class forcing extension where

\[ \square^{\text{MM}} \]

and

\[ \square^{\text{cof}(\kappa)} \]

\[ \kappa \]

holds for every cardinal with \( \text{cof}(\kappa) > \omega_1 \),

\[ \square^{\kappa} \]

\[ \kappa \]

holds for every singular cardinal \( \kappa \) of cofinality \( \omega_1 \).
Theorem (Cummings-Magidor)
Theorem (Cummings-Magidor)

*If there is a supercompact cardinal, then there is a class forcing extension where MM holds and*
Theorem (Cummings-Magidor)

If there is a supercompact cardinal, then there is a class forcing extension where MM holds and

\[ \square^\text{cof}(\kappa)_\kappa \text{ holds for every cardinal with } \text{cof}(\kappa) > \omega_1, \]
Theorem (Cummings-Magidor)

If there is a supercompact cardinal, then there is a class forcing extension where MM holds and

- □_{κ}^{\text{cof}(κ)} holds for every cardinal with \text{cof}(κ) > ω_1,
- □_{κ}^{κ} holds for every singular cardinal κ of cofinality ω_1.
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Theorem (Sakai)
Assume CC
\[ \neg \Box < \text{cof}(\kappa) \]
\[ \kappa \]
whenever \( \kappa \) is an uncountable cardinal
\[ \neg \text{CH} \implies \neg \Box \omega_1 \omega_1 \]
\[ \neg \Box \kappa \kappa \]
whenever \( \kappa \) is an uncountable cardinal of cofinality \( \omega \)
\[ \neg \Box < \kappa \kappa \]
whenever \( \kappa \) is an uncountable cardinal of cofinality \( \omega_1 \).
Theorem (Sakai)
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Assume $\text{CC}^*$. Then we have the following:
Theorem (Sakai)

**Assume** $\mathbb{CC}^*$. Then we have the following:

- $\neg \square^\prec_{\text{cof}(\kappa)} \kappa$ whenever $\kappa$ is an uncountable cardinal $\kappa$,
Theorem (Sakai)

Assume $\text{CC}^*$. Then we have the following:

- $\neg \square^<_{\text{cof}(\kappa)}$ whenever $\kappa$ is an uncountable cardinal $\kappa$,
- $\neg \text{CH}$ implies $\neg \square^{\omega_1}_{\omega_1}$,
Theorem (Sakai)

Assume $CC^*$. Then we have the following:

- $\neg \Box^{<\text{cof}(\kappa)}_{\kappa}$ whenever $\kappa$ is an uncountable cardinal $\kappa$,
- $\neg \text{CH}$ implies $\neg \Box^{\omega_1}_{\omega_1}$,
- $\neg \Box^{\kappa}_{\kappa}$ whenever $\kappa$ is an uncountable cardinal of cofinality $\omega$, 

Theorem (Sakai)

Assume $\text{CC}^*$. Then we have the following:

- $\neg \square^<_{\text{cof}(\kappa)}$ whenever $\kappa$ is an uncountable cardinal $\kappa$,
- $\neg \text{CH}$ implies $\neg \square^\omega_{\omega_1}$,
- $\neg \square^\kappa_{\kappa}$ whenever $\kappa$ is an uncountable cardinal of cofinality $\omega$,
- $\neg \square^<_{\kappa}$ whenever $\kappa$ is an uncountable cardinal of cofinality $\omega_1$. 
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Theorem (Sakai)
If there is a supercompact cardinal, then there is a class forcing extension where
\( \square \cof(\kappa) \kappa \) holds for every cardinal with \( \cof(\kappa) > \omega_1 \),
\( \square \kappa \kappa \) holds for every singular cardinal \( \kappa \) of cofinality \( \omega_1 \).
Theorem (Sakai)
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If there is a supercompact cardinal, then there is a class forcing extension where RC holds and
Theorem (Sakai)

If there is a supercompact cardinal, then there is a class forcing extension where RC holds and

\[ \square^\text{cof}(\kappa) \] holds for every cardinal with \( \text{cof}(\kappa) > \omega_1 \),
Theorem (Sakai)

If there is a supercompact cardinal, then there is a class forcing extension where RC holds and

- □⁺_{κ}^{\text{cof}(κ)} holds for every cardinal with \text{cof}(κ) > ω₁,
- □_{κ}^{κ} holds for every singular cardinal κ of cofinality ω₁.
Ascent paths and square sequences

Definition

Fix a regular cardinal $\theta$. The principle $\square(\theta)$ holds if there is a sequence $\langle C_\delta : \delta \in \text{Lim}(\theta) \rangle$ of subsets of $\theta$ such that for every $\delta \in \text{Lim}(\theta)$:

1. $C_\delta$ is a closed and unbounded subset of $\delta$,
2. if $\gamma \in \text{Lim}(C_\delta)$, then $C_\delta \cap \gamma = C_\gamma$,
3. there is no closed unbounded set $C \subseteq \theta$ such that for every $\gamma \in \text{Lim}(C)$, $C \cap \gamma = C_\gamma$. 

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Rado’s Conjecture, Strong Chang’s Conjecture, Tree Properties and Two Cardinal Square Principles
Ascent paths and square sequences

Definition

Fix a regular cardinal $\theta$. The principle $\Box(\theta)$ holds if there is a sequence $\langle C_\delta : \delta \in \text{Lim}(\theta) \rangle$ of subsets of $\theta$ such that for every $\delta \in \text{Lim}(\theta)$:

1. $C_\delta$ is a closed and unbounded subset of $\delta$,
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Ascent paths and square sequences
Remark:
Ascent paths and square sequences

Remark:
\( \Box_{\kappa^+} \) implies \( \Box(\kappa^+) \).
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\(\Box_{\kappa^+} \text{ implies } \Box(\kappa^+).\)

**Theorem (Todorcevic, 1993)**
Remark:
\(\square_{\kappa^+} \implies \square(\kappa^+)\).

**Theorem (Todorcevic, 1993)**

RC implies the negation of \(\square(\theta)\) for every regular cardinal \(\theta \geq \omega_2\).
Remark:
□_{\kappa^+} implies □(\kappa^+).

**Theorem (Todorcevic, 1993)**
RC implies the negation of □(\theta) for every regular cardinal \( \theta \geq \omega_2 \).

**Theorem (Sakai-Velickovic, 2015)**
Remark:
\( \square_{\kappa^+} \) implies \( \square(\kappa^+) \).

**Theorem (Todorcevic, 1993)**

\( \text{RC implies the negation of } \square(\theta) \text{ for every regular cardinal } \theta \geq \omega_2. \)

**Theorem (Sakai-Velickovic, 2015)**

\( \text{CC}^* \text{ implies the negation of } \square(\theta) \text{ for every regular cardinal } \theta \geq \omega_2. \)
Ascent paths and square sequences

Definition

Fix a regular cardinal $\theta$ and some other cardinal $\lambda \leq \theta$. A sequence $C_\delta$ ($\delta \in \text{Lim}(\theta)$) of families of subsets of $\theta$ is said to be a $\square^<\lambda(\theta)$-sequence whenever:

1. $|C_\delta| < \lambda$ for all $\delta \in \text{Lim}(\theta)$,
2. for each $\delta \in \text{Lim}(\theta)$, each $C \in C_\delta$ is a closed and unbounded subset of $\delta$,
3. if $C \in C_\delta$ and if $\gamma < \delta$ is a limit point of $C$ then $C \cap \gamma$ belongs to $C_\gamma$.

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Definition

Fix a regular cardinal $\theta$ and some other cardinal $\lambda \leq \theta$. A sequence $C_\delta (\delta \in \text{Lim}(\theta))$ of families of subsets of $\theta$ is said to be a $\square^{\theta < \lambda}$-sequence whenever:

1. $|C_\delta| < \lambda$ for all $\delta \in \text{Lim}(\theta)$,
2. for each $\delta \in \text{Lim}(\theta)$, each $C \in C_\delta$ is a closed and unbounded subset of $\delta$,
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Ascent paths and square sequences

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1. $|C_\delta| < \lambda$ for all $\delta \in \text{Lim}(\theta)$,
2. for each $\delta \in \text{Lim}(\theta)$, each $C \in C_\delta$ is a closed and unbounded subset of $\delta$,
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Ascent paths and square sequences
Ascent paths and square sequences

Definition

An ascendent path \( \theta \)-sequence \( \square < \lambda \)-sequence \( C \) is a sequence \( A_\xi (\xi < \theta) \) of nonempty subsets of \( \theta \) such that:

1. \( \xi < \eta \) implies \( A_\xi < A_\eta \), i.e., every ordinal of \( A_\xi \) is smaller than every ordinal of \( A_\eta \),
2. \( \xi < \eta \) implies that there exist \( \delta \in A_\eta \) and \( \gamma \in A_\xi \) and \( C \in C_\delta \) such that \( \gamma \) is a limit point of \( C \).
Ascent paths and square sequences

Definition

An ascent $\theta$-path of the $\square_{<\lambda}(\theta)$-sequence $C_\delta$ ($\delta \in \text{Lim}(\theta)$) is a sequence $A_\xi$ ($\xi < \theta$) of nonempty subsets of $\theta$ such that:

1. $\xi < \eta$ implies $A_\xi < A_\eta$, i.e., every ordinal of $A_\xi$ is smaller than every ordinal of $A_\eta$,
2. $\xi < \eta$ implies that there exist $\delta \in A_\eta$ and $\gamma \in A_\xi$ and $C_\gamma \in C_\delta$ such that $\gamma$ is a limit point of $C_\gamma$. 

$\text{V} \acute{\text{i}}\text{c}\text{t}\text{o}r$ $\text{T}\text{o}\text{r}\text{e}$-$\text{P}\text{\'e}r\text{\'e}$

 Raf\text{\'o}$'\text{s} $\text{C}o\text{njecture}$, $\text{S}tr\text{\'o}ng$ $\text{C}h\text{\'a}ng's$ $\text{C}o\text{njecture}$, $\text{T}ree$ $\text{P}r\text{\'o}p\text{\'e}r\text{t}i\text{\'e}$s
Ascent paths and square sequences

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An ascent $\theta$-path of the $\square_{<\lambda}(\theta)$-sequence $C_\delta$ ($\delta \in \text{Lim}(\theta)$) is a sequence $A_\xi$ ($\xi < \theta$) of nonempty subsets of $\theta$ such that:

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Definition

An ascent $\theta$-path of the $\square_{<\lambda}(\theta)$-sequence $C_\delta$ ($\delta \in \text{Lim}(\theta)$) is a sequence $A_\xi$ ($\xi < \theta$) of nonempty subsets of $\theta$ such that:

1. $\xi < \eta$ implies $A_\xi \subset A_\eta$, i.e., every ordinal of $A_\xi$ is smaller than every ordinal of $A_\eta$,

2. $\xi < \eta$ implies that there exist $\delta \in A_\eta$ and $\gamma \in A_\xi$ and $C \in C_\delta$ such that $\gamma$ is a limit point of $C$. 
We recall that the existence of an ascent $\theta$-path of sets of cardinality 1 is what is frequently worded as the triviality of the square sequence. Thus the existence of ascent $\theta$-paths of subsets of $\theta$ of other small cardinalities is a natural weakening of this notion. In fact let us say that a given $\Box < \lambda (\theta)$-sequence $C_\delta (\delta < \theta)$ is $\kappa$-trivial whenever it admits an ascent $\theta$-path consisting of nonempty sets of cardinalities at most $\kappa$. 
Ascent paths and square sequences

We recall that the existence of an ascent $\theta$-path of sets of cardinality 1 is what is frequently worded as the *triviality* of the square sequence.
Ascent paths and square sequences

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Definition

A family $F \subseteq \omega^\omega$ is called unbounded if for every $g \in \omega^\omega$, there is $f \in F$ such that
\[
\{ n \in \omega : f(n) \geq g(n) \}
\]
is infinite.

The bounding number $b$ is the least cardinality of an unbounded family.

Theorem (Todorcevic-T., 2014)

Assume RC. Let $\theta$ be a regular cardinal $\geq \omega_2$ with the property that for every $\delta < \theta$, the set $[\delta]$ contains a closed and unbounded subset of size $< \theta$. Then every $\Box^{< b}(\theta)$-sequence $\langle C_\delta : \delta < \theta \rangle$ has an ascent $\theta$-path of countable subsets of $\theta$. 

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Definition

A family $F \subseteq \omega^\omega$ is called unbounded if for every $g \in \omega^\omega$, there is $f \in F$ such that $\{n \in \omega : f(n) \geq g(n)\}$ is infinite.

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A family $F \subseteq \omega^\omega$ is called *unbounded* if for every $g \in \omega^\omega$, there is $f \in F$ such that $\{n : \in \omega : f(n) \geq g(n)\}$ is infinite.
**Definition**

A family $F \subseteq \omega^\omega$ is called *unbounded* if for every $g \in \omega^\omega$, there is $f \in F$ such that $\{n : \in \omega : f(n) \geq g(n)\}$ is infinite.

The *bounding number* $b$ is the least cardinality of an unbounded family.
Definition

A family $F \subseteq \omega^\omega$ is called \textit{unbounded} if for every $g \in \omega^\omega$, there is $f \in F$ such that $\{ n \in \omega : f(n) \geq g(n) \}$ is infinite.

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Definition

A family \( F \subseteq \omega^\omega \) is called \textit{unbounded} if for every \( g \in \omega^\omega \), there is \( f \in F \) such that \( \{ n : n \in \omega : f(n) \geq g(n) \} \) is infinite.

The \textit{bounding number} \( b \) is the least cardinality of an unbounded family.

Theorem (Todorcevic-T., 2014)

Assume \( RC \). Let \( \theta \) be a regular cardinal \( \geq \omega_2 \) with the property that for every \( \delta < \theta \), the set \([\delta]^{\omega}\) contains a closed and unbounded subset of size \(< \theta \). Then every \( \Box_{<b} (\theta) \)-sequence \( \langle \mathcal{C}_\delta : \delta < \theta \rangle \) has an ascent \( \theta \)-path of countable subsets of \( \theta \).
Theorem (T.-Wu, 2015)

Assume $CC^*$. Then every $\square^{<\omega_1}(\theta)$-sequence is $1$-trivial for every regular cardinal $\theta \geq \omega_2$. 
Thanks!