Rado's Conjecture, Strong Chang's Conjecture, Tree Properties and Two Cardinal Square Principles

Víctor Torres-Pérez

Vienna University of Technology

The Role of the Higher Infinite in Mathematics and Other Disciplines
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Rado's Conjecture (RC)

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A family of intervals of a linearly ordered set is the union of countably many disjoint subfamilies (σ-disjoint) if and only if every subfamily of size $\aleph_1$ is σ-disjoint.

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However MA_{ω_1} implies the negation of RC.

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Rado's Conjecture implies the presaturation of the nonstationary ideal on ω_1.

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For every regular cardinal $\kappa \geq \omega_2$, there are arbitrary large λ such that for every countable $M \prec H_\lambda$ and for every $a \in [\kappa]^{\omega_1}$, there is a countable $M^* \prec H_\lambda$ and $b \in M^* \cap [\kappa]^{\omega_1}$ such that $M^* \supseteq M$ and $M^* \cap \omega_1 = M \cap \omega_1$.

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A regular cardinal κ has the *tree property* and we denote it by $\mathrm{TP}(\kappa)$, if every tree T of height κ, with levels of size less than κ has a cofinal branch.

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What about trees of height ω_2 and levels of size ω_1?

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A natural question is if under RC, the negation of the Continuum Hypothesis is enough to imply there are no $\aleph_2$-Aronzajn trees at all, i.e. if $\mathrm{TP}(\omega_2)$ holds.

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$\mathrm{CC}^* + \neg\mathrm{CH} \to \mathrm{TP}(\omega_2)$.

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We call $\mathscr{F} = \bigcup_{a \in [\kappa]^{<\lambda}} \mathscr{F}_a$ a (κ, λ)-tree, and $\mathscr{F}_a$ the level a of $\mathscr{F}$ for $a \in [\kappa]^{<\lambda}$.

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Observe that in general, $\leq_{\mathscr{F}}$ is not a tree order.

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We say that λ has the Strong Tree Property if every (κ, λ)-tree has a cofinal branch for every $\kappa \geq \lambda$.

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3. If $C \in \mathscr{C}_\beta$ and if α is a limit point of C, then $C \cap \alpha \in \mathscr{C}_\alpha$.

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Under CC^*, $\square_{\omega_1}^{\omega_1}$ *and* CH *are equivalent.*

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(3) there is no closed unbounded set $C \subseteq \theta$ such that for every $\gamma \in \mathrm{Lim}(C)$, $C \cap \gamma = C_\gamma$.

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(3) if $C \in \mathcal{C}_\delta$ and if $\gamma < \delta$ is a limit point of C then $C \cap \gamma$ belongs to $\mathcal{C}_\gamma$.

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Thus the existence of ascent θ-paths of subsets of θ of other small cardinalities is a natural weakening of this notion.
In fact let us say that a given $\square_{<\lambda}(\theta)$-sequence $\mathcal{C}_\delta$ $(\delta < \theta)$ is κ-*trivial* whenever it admits an ascent θ-path consisting of nonempty sets of cardinalities at most κ.

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Theorem (Todorcevic-T., 2014)

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Theorem (Todorcevic-T., 2014)

Assume RC. *Let* θ *be a regular cardinal* $\geq \omega_2$ *with the property that for every* $\delta < \theta$, *the set* $[\delta]^\omega$ *contains a closed and unbounded subset of size* $< \theta$. *Then every* $\square_{<\mathfrak{b}}(\theta)$*-sequence* $\langle \mathscr{C}_\delta : \delta < \theta \rangle$ *has an ascent* θ*-path of countable subsets of* θ.

Theorem (T.-Wu, 2015)

Assume CC^*. *Then every* $\square_{<\omega_1}(\theta)$*-sequence is 1-trivial for every regular cardinal* $\theta \geq \omega_2$.

Thanks!