Approximate Ramsey properties of matrices and finite dimensional normed spaces.

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   - Dual Ramsey Theorem
   - $\mathbb{F} = \mathbb{R}, \mathbb{C}$
   - Grassmannians over $\mathbb{R}, \mathbb{C}$
2. ARP of Finite dimensional Normed spaces
   - Polyhedral spaces
   - Consequences
3. ARP for ℓ_∞^n
   - Borsuk-Ulam Theorem
   - ARP of ℓ_∞^n
4. Extensions

Definition

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4. *Given a vector space* V *over* $\mathbb{F}$*, and* $k \in \mathbb{N}$*, let* $\mathrm{Gr}(k, V)$ *be the collection of all subspaces of* V *of dimension exactly* k*. Combinatorial notation:*

$$\binom{V}{\mathbb{F}^k}.$$

Definition

Given $A \in \mathcal{I}_{n\times k}$, let

$$A = \mathrm{red}(A) \cdot \tau(A)$$

be the unique decomposition of A by the reduced column echelon form of A and a unique invertible matrix $\tau(A) \in \mathrm{GL}_k(\mathbb{F})$.

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Let

$$\mathcal{E}_{n\times k} := \{A \in \mathcal{I}_{n\times k} \,:\, \mathrm{red}(A) = A\}.$$

Theorem (Ramsey degree of full rank matrices)

For every $k, m \in \mathbb{N}$ and every $r \in \mathbb{N}$ there exists n such that for every coloring

$$f : \mathcal{I}_{n\times k}(\mathbb{F}) \to r$$

there exists $R \in \mathcal{E}_{n\times m}$ and $g : \mathrm{GL}_k \to r$ such that

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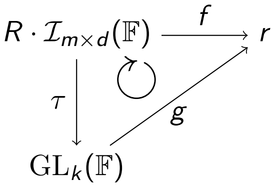

Since every subspace $W \in \mathrm{Gr}(k, \mathbb{F}^n)$ is the image of a matrix $A \in \mathcal{E}_{n\times k}$, the previous theorem gives the following.

Corollary (Graham-Leeb-Rothschild)

For every $k, m \in \mathbb{N}$ and $r \in \mathbb{N}$ there exists n such that for every coloring $c : \mathrm{Gr}(k, \mathbb{F}^n) \to r$ there exists $V \in \mathrm{Gr}(m, \mathbb{F}^n)$ such that c is constant on $\mathrm{Gr}(k, V)$.

G-L-R Theorem is a consequence of the Dual Ramsey Theorem.

Definition

Let $(S, <_S)$ *and* $(T, <_T)$ *be two linearly ordered sets. A surjection* $\theta : S \to T$ *is called a rigid-surjection when* $\min \theta^{-1}(t_0) < \min \theta^{-1}(t_1)$ *for every* $t_0 < t_1$ *in* T. *Let* $\mathrm{Epi}(S, T)$ *be collection of all those surjections.*

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Theorem (Dual Ramsey Theorem; Graham and Rothschild)

For every finite linearly ordered sets S and T, and $r \in \mathbb{N}$ there exists n such that every r-coloring of $\mathrm{Epi}(n, S)$ has a monochromatic set of the form $\mathrm{Epi}(T, S) \circ \sigma$ for some $\sigma \in \mathrm{Epi}(n, T)$.

Sketch of Proof of Ramsey Degree of Full Rank matrices

Definition

We order $\mathbb{F}$ in a way that 0 and 1 are the first two elements, and then each power $\mathbb{F}^k$ antilexicographically. Let $\Phi : \mathrm{Epi}(n, \mathbb{F}^k) \to \mathcal{I}_{n\times k}$ be defined for $\sigma \in \mathrm{Epi}(n, \mathbb{F}^k)$ as the matrix $A = \Phi(\sigma)$ whose ξ-row vector is $\sigma(\xi)$, for each $\xi < n$. $\Phi(\sigma)$ is the σ-Matrix.

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For fixed integers k, m and r, let n be the Graham-Rothschild number for $\mathbb{F}^k$, $\mathbb{F}^m$, and number of colors $r^{\mathrm{GL}_k(\mathbb{F})}$.

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Fix $f : \mathcal{I}_{n\times m} \to r$. Let $c : \mathrm{Epi}(n, \mathbb{F}^k) \to r^{\mathrm{GL}_k(\mathbb{F})}$ be the mapping

$$\begin{aligned}\sigma \mapsto c(\sigma) : \mathrm{GL}_k(\mathbb{F}) &\to r\\ B &\mapsto f(\Phi(\sigma)\cdot B)\end{aligned}$$

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$$\sigma \mapsto c(\sigma) : \mathrm{GL}_k(\mathbb{F}) \to r$$
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Let $\varrho \in \mathrm{Epi}(n, \mathbb{F}^m)$ be such that c is constant on $\mathrm{Epi}(\mathbb{F}^m, \mathbb{F}^k) \circ \varrho$ with value $g \in r^{\mathrm{GL}_k(\mathbb{F})}$.

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Let R be the ϱ-matrix

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For $A \in \mathcal{I}_{m\times k}$ one has that

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It is also easy that

$$\Phi(\varrho)\cdot\mathrm{Red}(A) = \Phi(\mathrm{Red}(A)^{\mathrm{t}}\circ\varrho).$$

Sketch of Proof of GLR

Hence, setting $\sigma := \mathrm{Red}(A)^{\mathrm{t}}$,

$$f(R \cdot A) = f(\Phi(\sigma \circ \varrho) \cdot \tau(A)) = c(\sigma \circ \varrho)(\tau(A)) = g(\tau(A)).$$

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It is not difficult to see the minimality of $\tau : \mathcal{I}_{n\times k} \to \mathrm{GL}_k(\mathbb{F})$;

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It is not difficult to see the minimality of $\tau : \mathcal{I}_{n\times k} \to \mathrm{GL}_k(\mathbb{F})$; that is,

given any $R \in \mathcal{I}_{n\times m}$ and any $B \in \mathrm{GL}_k(\mathbb{F})$ there is $A \in \mathcal{I}_{m\times k}$ such that $\tau(R \cdot A) = B$.

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Given k, n, we consider $\mathcal{M}_{n\times k}(\mathbb{F})$ as a metric space by considering the matrix norm

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Example

Given $v = (v_i)_{i<n} \in \mathbb{F}^n$,

$$N_\infty(v) = \|v\|_\infty := \max_{i<n} |v_i|$$

$$N_p(v) = \|v\|_p := \left(\sum_{i<n} |v_i|^p\right)^{\frac{1}{p}}, \ p \geq 1.$$

Definition

We say that $M \leq N$ *if* $M(v) \leq N(v)$ *for every* $v \in \mathbb{F}^k$*. Let*

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Observe that $\bigcup_{0<a\leq b<\infty}[aM, bM]$ for every norm $M \in \mathcal{N}_k$.

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$$d_0(M,N) := d_{\mathcal{H}}^{\ell_1}(B_{(\mathbb{F}^k, M^*)}, B_{(\mathbb{F}^k, N^*)})$$
$$d_1(M,N) := \log \max\{\|\mathrm{Id}\|_{M,N}, \|\mathrm{Id}\|_{N,M}\},$$

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where

(a) *$d_{\mathcal{H}}^{\ell_1}(K,L)$ is the Hausdorff distance between two compacta $K, L \subseteq \mathbb{F}^k$ when considering $\mathbb{F}^k$ endowed with the ℓ_1-metric.*

(b) *M^* is the dual norm of M, defined by*

$$M^*(v) := \max_{M(w)=1} w^* v$$

and $B_{(\mathbb{F}^k, P)} := \{v \in \mathbb{F}^k \,:\, P(v) \le 1\}$ is the unit ball of $(\mathbb{F}^k, P)$.

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Proposition

$\tau_\infty : (\mathcal{I}_{n\times k}, N_\infty) \to (\mathcal{N}_k, d_0)$ *is 1-Lipschitz, and*

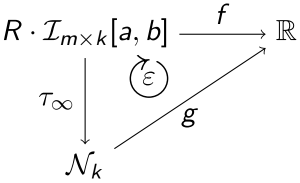
$$\bigcup_{n\geq k} \tau_\infty(\mathcal{I}_{n\times k}) \text{ is dense in } \mathcal{N}_k.$$

Theorem

For every k, m, $0 < a \leq b$ *and every* $\varepsilon > 0, L > 0$ *there exists* n *such that for every Lipschitz coloring*

$$f : \mathcal{I}_{n \times k} \to \mathbb{R}$$

with $\mathrm{Lip}(f) \leq L$ *there exists* $R \in \mathcal{E}_{n \times m}$ *and* $g : \mathcal{N}_k \to \mathbb{R}$ *such that* $\mathrm{Lip}(g) \leq (1 + \varepsilon)L$ *and*



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$\tau_\infty : \mathcal{I}_{n\times k} \to \mathcal{N}_k$ *is unique in some precise sense.*

Definition

Given $V, W \in \mathrm{Gr}(k, \mathbb{F}^n)$, let $d(V, W)$ be the Hausdorff distance between $B_{(V,N_\infty)}$ and $B_{(W,N_\infty)}$ with respect to the ∞-distance on $\mathbb{F}^n$.

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Let

$$d_{\mathrm{BM}}(M, N) := \log \inf_{A \in \mathrm{GL}_k} \|A\|_{M,N} \cdot \|A^{-1}\|_{N,M}$$

be the Banach-Mazur pseudo metric on $\mathcal{N}_K$. Let

$$\mathcal{B}_k := \mathcal{N}_k / \{d = 0\}.$$

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$$\nu_\infty(V) = [\tau_\infty(A)]_{\mathrm{BM}}$$

where $A \in \mathcal{I}_{n\times k}$ *is such that* $\mathrm{Im} A = V$*.*

Proposition

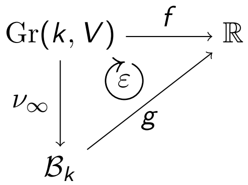
ν_∞ *is* $\sim k \log k$*-Lipschitz.*

Theorem

For every k, m, and every $\varepsilon > 0, L > 0$ there exists n such that for every Lipschitz coloring

$$f : \mathrm{Gr}(k, \mathbb{F}^n) \to \mathbb{R}$$

with $\mathrm{Lip}(f) \leq L$ there exists $V \in \mathrm{Gr}(m, \mathbb{F}^n)$ with (V, N_∞) linearly isometric to ℓ_∞^m and $g : \mathrm{Gr}(k, V) \to \mathbb{R}$ such that $\mathrm{Lip}(g) \leq (1 + \varepsilon)L$ and



Since the ℓ_∞^n's are almost-universal, for large N, ℓ_∞^N has an almost-isometrical V copy of ℓ_2^m, that is, (V, N_∞) is almost isometric to ℓ_2^m. Hence, $\nu_\infty(\mathrm{Gr}(k, V))$ as a small diameter.

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Corollary

For every k, m, and every $\varepsilon > 0, L > 0$ there exists n such that for every Lipschitz coloring

$$f : \mathrm{Gr}(k, \mathbb{F}^n) \to \mathbb{R}$$

with $\mathrm{Lip}(f) \leq L$ there exists $V \in \mathrm{Gr}(m, \mathbb{F}^n)$ such that

$$\mathrm{osc}(f \upharpoonright \mathrm{Gr}(k, V)) < \varepsilon.$$

Definition

We say that a collection of Banach spaces $\mathcal{F}$ has the Approximate Ramsey Property (ARP) when for every $F, G \in \mathcal{F}$, $K > 0$ and $\varepsilon > 0$ there exists $H \in \mathcal{F}$ such that $\mathrm{Emb}(G, H) \neq \emptyset$ and such that for every Lipschitz map $f : \mathrm{Emb}(F, H) \to \mathbb{R}$ with $\mathrm{Lip}(f) \leq L$ there exists

$$\varrho \in \mathrm{Emb}(G, H).$$

such that

$$\mathrm{osc}(f \restriction \varrho \circ \mathrm{Emb}(F, G)) < \varepsilon.$$

Theorem

The class of f.d. normed spaces have the ARP.

Corollary

For every f.d. spaces X and Y and every $\varepsilon, L > 0$ there is a f.d. Z such that for every Lipschitz map $f : \binom{Z}{X} \to \mathbb{R}$, $\mathrm{Lip}(f) \leq L$ there is $\bar{Y} \in \binom{Z}{Y}$ such that

$$\mathrm{Osc}(f \upharpoonright \binom{\bar{Y}}{X}) < \varepsilon.$$

Here the metric on $\binom{Z}{X}$ is the Hausdorff metric (wrt the norm on Z) between the unit balls B_{X_0}, B_{X_1} of X_0 and X_1, respectively.

$$d(X_0, X_1) := d_{\mathcal{H}}^{Z}(B_{X_0}, B_{X_1}).$$

$(\mathbb{F} = \mathbb{R})$

Definition

Recall that a finite dimensional space F is called polyhedral when its unit ball is a polyhedron; that is, it has finitely many extreme points.

It is well-known that a f.d. space is polyhedral if and only if can be isometrically embedded into some ℓ_∞^n.

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It is well-known that a f.d. space is polyhedral if and only if can be isometrically embedded into some ℓ_∞^n. Polyhedral spaces are dense in the class of finite dimensional spaces. Polyhedral spaces are isometric to $(\mathbb{F}^k, \tau_\infty(A))$ for some k and $A \in \mathcal{I}_{n\times k}$.

Theorem

The polyhedral spaces have the ARP.

Proof.

This is a particular case of our result over injective matrices. □

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In fact, the ARP of polyhedral spaces is equivalent to our result on full rank matrices.
ARP property of pol spaces is a consequence of this particular case:

Theorem

The class of isometric spaces to some ℓ_∞^n have the ARP.

Polyhedral spaces are dense in the class of f.d. spaces. So, an isometric embedding T between two f.d. spaces X and Y will induce a θ-embedding T' ($\theta^{-1}\|x\| \leq \|T'x\| \leq \theta\|x\|$) between polyhedral spaces X' and Y' appropriately closed to X and Y.

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Proposition

Let $(X_i)_{i\leq n}$ be f.d. spaces, and $1<\theta<\tau$. Then there is a f.d. space Y having isometric copies of each X_i and an isometric embedding $J: X_n \to Y$ such that for every θ-embedding $T: X_i \to X_n$ there is an isometric embedding $I: X_i \to Y$ such that $\|I - J\circ T\| < \tau - 1$.

Corollary

The group of (linear) isometries $\mathrm{Iso}(\mathbb{G})$ of the Gurarij space $\mathbb{G}$ (the metric Fraïssé limit of f.d. normed spaces), with the pointwise topology is extremely amenable.

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The universal minimal flow of the group of affine homeomorphism of the Poulsen simplex (the metric inverse Fraïssé limit of f.d. simplexes) with the uniform topology is the Poulsen simplex with its natural action.

Corollary

The finite metric spaces have the ARP.

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1. The unitary group $\mathrm{Iso}(\ell_2)$ (Gromov-Milman).
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Definition

Let (X,d) be a metric space, $\varepsilon > 0$. We say that an open covering $\mathcal{U}$ of X is ε-fat when for every $U \in \mathcal{U}$ there is V_U open such that $(V_U)_\varepsilon \subseteq U$ and $\{V_U\}_{U\in\mathcal{U}}$ is still a covering of X.

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It is not difficult to see that if X is compact, then every open covering is ε-fat for some $\varepsilon > 0$.

The ARP of ℓ_p^n's is restated as follows:

Theorem

For every $1 \leq p \leq \infty$, every integers d, m and r and every ε there is some $\mathbf{n}_p(d, m, r, \varepsilon)$ such that for every ε-fat open covering $\mathcal{U}$ of $\mathcal{I}_{d,n}^p$ with cardinality at most r there exists some $A \in \mathcal{I}_{m,n}^p$ such that

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For example, Borsuk-Ulam Theorem is the statement

$$\mathbf{n}_2(1, 1, r, \varepsilon) = r \text{ for all } \varepsilon > 0, \quad (2)$$

because $\mathcal{I}_{1,n}^2$ consists on 1-column-matrices (v) of vectors v of the sphere of ℓ_2^n, and $\mathcal{I}_{1,1}^2 = \{(1), (-1)\}$, so $(v) \cdot \mathcal{I}_{1,1}^2 = \{(-v), (-v)\}$.

Problem

Is there a multi-dimensional Borsuk-Ulam Theorem?

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Definition

Let $(S,<_S)$ and $(T,<_T)$ be two linearly ordered sets. A surjection $\theta : S \to T$ is called a min-surjection when $\min \theta^{-1}(t_0) < \min \theta^{-1}(t_1)$ for every $t_0 < t_1$ in T. Let $\mathrm{Epi}(S,T)$ be collection of all those surjections.

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Theorem (Dual Ramsey Theorem; Graham and Rothschild)

For every finite linearly ordered sets S and T, and $r \in \mathbb{N}$ there exists $n \geq \#T$ such that every r-coloring of $\mathrm{Epi}(n,S)$ has a monochromatic set of the form $\mathrm{Epi}(T,S) \circ \sigma$ for some $\sigma \in \mathrm{Epi}(n,T)$.

Definition

Let $\mathcal{E}_{n\times k}$ be the collection of all $n \times k$ matrices representing (in the unit bases of $\mathbb{F}^k$ ad $\mathbb{F}^n$) a linear isometry between ℓ_∞^k and ℓ_∞^n.

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$A \in \mathcal{E}_{n\times k}$ if and only if each column vector has ∞-norm one and each row vector has ℓ_1-norm at most 1.

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Given $\varepsilon > 0$, let $\mathcal{N}$ be a finite ε-dense subset of the unit ball $B_{\ell_1^k}$

1. containing 0 and the unit vectors u_i, and
2. such that for every non-zero $v \in B_{\ell_1^k}$ there is $w \in \mathcal{N}$ such that $\|v - w\|_1 < \varepsilon$ and $\|w\|_1 < \|v\|_1$. e.g., for large $l\varepsilon \geq 1$,

$$\mathcal{N} = (\{\pm \frac{i}{kl}\}_{i \leq kl})^k \cap B_{\ell_1^k}$$

Let $<$ be any total ordering on $\mathcal{N}$ such that $v < w$ when $\|v\|_1 < \|w\|_1$.
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Definition

Let $\Phi : \mathrm{Epi}(n, \mathcal{N}) \to \mathcal{E}_{n\times k}$ *be defined for a* $\sigma : n \to \mathcal{N}$ *as the* $n \times k$*-matrix* A_σ *whose* ξ*-row vector,* $\xi < n$*, is* $\sigma(\xi)$*.*

It is easy to see that $\Phi(\sigma) \in \mathcal{E}_{n\times k}$.

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It is easy to see that $\Phi(\sigma) \in \mathcal{E}_{n\times k}$. To simplify, suppose that $\mathbb{F} = \mathbb{R}$.

Proposition

There is a finite set $\Gamma \subseteq \mathcal{E}_{n\times k}$ such that for every other $A \in \mathcal{E}_{n\times k}$ there exists $B \in \Gamma$ such that

$$A^{\mathrm{t}} B = \mathrm{Id}_k.$$

We order now $\Delta := \mathcal{N} \times \Gamma$ lexicographically, where Δ is arbitrarily ordered. Given now k, m, a number of colors r, we use apply the DR theorem to $\mathcal{N}$ and Δ to find the corresponding n.

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$$c \circ \Phi : \mathrm{Epi}(n, \mathcal{N}) \to r$$

Let $\varrho \in \mathrm{Epi}(n, \Delta)$ such that c is constant on $\mathrm{Epi}(\Delta, \mathcal{N}) \circ \varrho$. Let now $R \in \mathcal{E}_{n\times m}$ be the matrix whose ξ-column is Av where $\varrho(\xi) = (v, A)$.

Proposition

For every $B \in \mathcal{E}_{m\times d}$ there exists $\sigma \in \mathrm{Epi}(\Delta, \mathcal{N})$ such that $\|RB - \Phi(\varrho \circ \sigma)\|_\infty < \varepsilon$.

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