The unreasonable effectiveness of Nonstandard Analysis

Sam Sanders

HIFW03, Newton Institute, Dec. 2015

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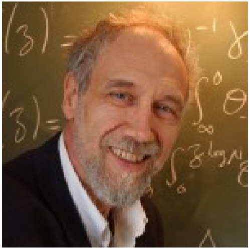
Alexander von Humboldt Stiftung/Foundation

Connes and Bishop on NSA



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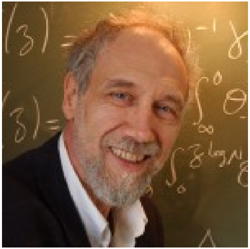

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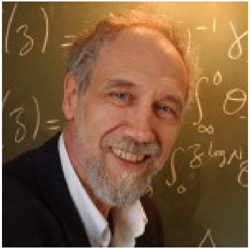

The Connes-Bishop view: NSA has NO numerical / constructive / computable content.

This talk: Nonstandard Analysis has TONS of numerical / constructive / computable content.

Aim and motivation

Technical aim: Show that theorems of PURE Nonstandard Analysis produce effective theorems not involving NSA, and vice versa.

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Motivation: Many authors have observed the 'constructive nature' of the practice of NSA. (Horst Osswald's local constructivity)

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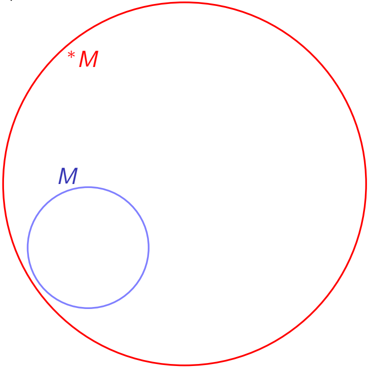

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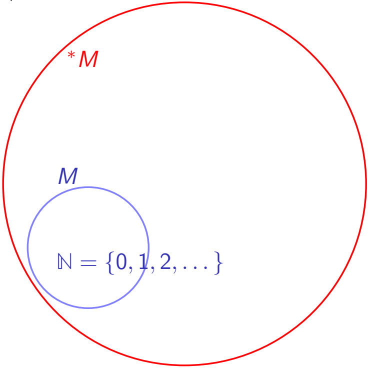

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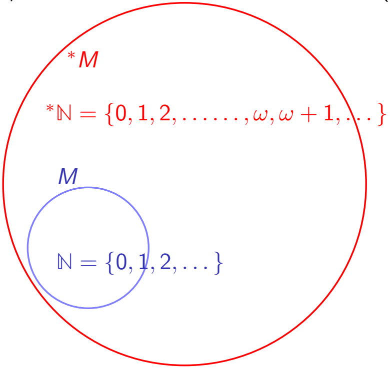

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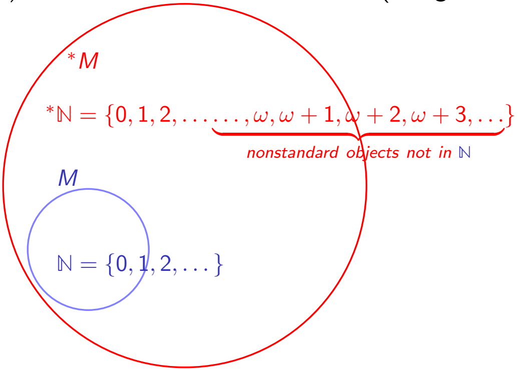

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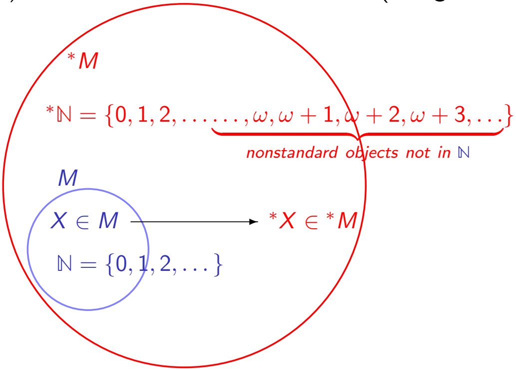

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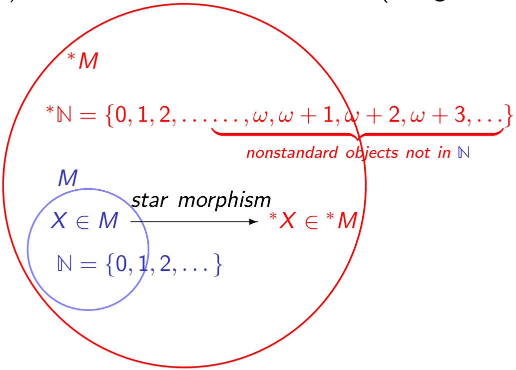

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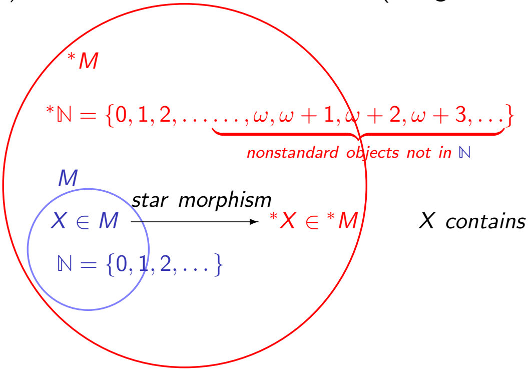

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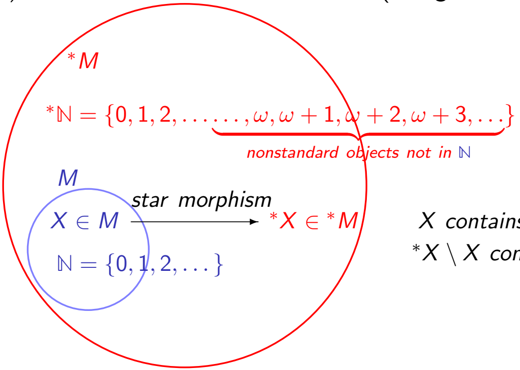

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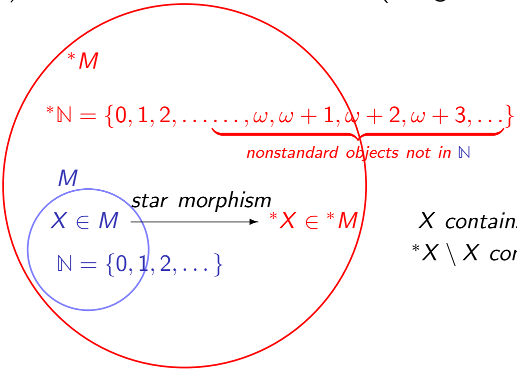

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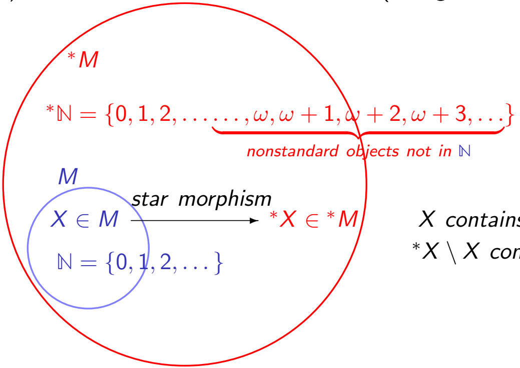

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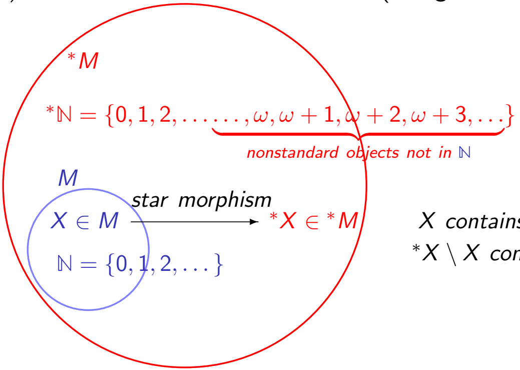

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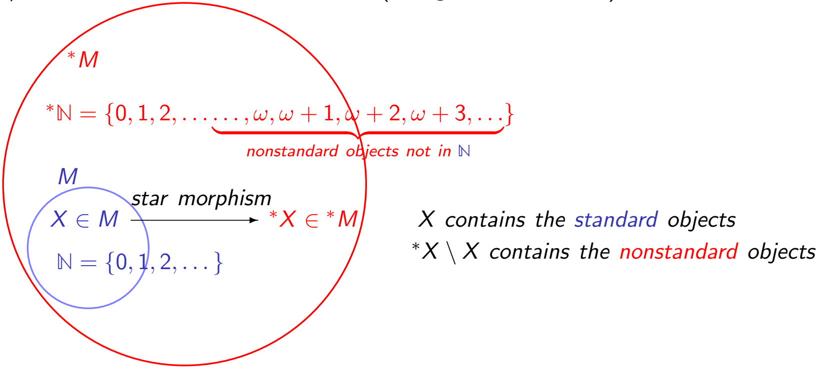

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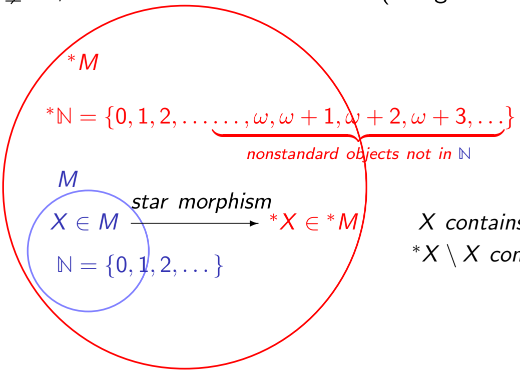

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3) Idealization/Saturation ...

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And analogous results for fragments of IST.

A fragment based on Gödel's T

van den Berg, Briseid, Safarik, A functional interpretation of nonstandard arithmetic, APAL2012

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A new computational aspect of NSA

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All theorems of PURE Nonstandard Analysis can thus be 'proof mined' using the term extraction result (of P and H).

The unreasonable effectiveness of NSA

Example I: Continuity.

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we can extract a term t^1 (from Gödel's T) such that E-PA$^\omega$ proves

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Et pour les constructivists: la même chose!

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Example II: Continuity implies Riemann integration

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$$(\forall f:\mathbb{R}\to\mathbb{R})\big[(\forall x,y\in[0,1])[x\approx y\to f(x)\approx f(y)]$$

$$\downarrow$$

$$(\forall \pi,\pi'\in P([0,1]))(\|\pi\|,\|\pi'\|\approx 0\to S_{\pi}(f)\approx S_{\pi'}(f))\big],$$

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From a proof that nonstandard uniformly continuity implies nonstandard Riemann integration in P, i.e.

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But (3) is the theorem expressing continuity implies Riemann integration from constructive analysis and computable math.

Explicit Reverse Mathematics

Example III: The monotone convergence theorem

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From a proof in P of the following equivalence:

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The unreasonable effectiveness of NSA

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$=$ the EXPLICIT version of FAN $\leftrightarrow$ (cont $\to$ Rieman int. on $[0,1]$).

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Example VI: Compactness bis

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totally bounded $=$ the preferred notion of compactness in constructive/comp. math.

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It would seem Connes and Bishop were wrong...

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Non-terminating recursion! Non-well-founded self-reference!

Γ is not primitive recursive (in the sense of Gödel's T).

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$$G(Y,s,M) := \begin{cases} Y(s*00\ldots) & |s| \geq M \\ Y\big(s*0*(\lambda n)G(Y,s*(n+1),M)\big) & \text{otherwise} \end{cases}.$$

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($X_{\text{ns}} = \mathsf{P}$ plus nonstandard continuity and nonstandard WKL)

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From the previous proof, a (primitive recursive) term t can be extracted such that E-PA$^\omega$ proves

$$(\forall Y^2, s^0)\big[G(Y, s, t(\Omega, Y, s, X)) = \Gamma(Y, s)\big],$$

where Ω is the fan functional and X witnesses the continuity of Y.

Beyond Analysis III: Partiality in Computability theory

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The previous results on the Gandy-Hyland functional (defined on associates) can be sharpened using the above.

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Nonstandard Analysis will however unify these fields in hitherto unseen ways.

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Any questions?