Renormalisation in regularity structures

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Two mini-courses

- Felix Otto: *Regularity structures: Reconstruction and Integration*
- myself: *Renormalisation in regularity structures*

Both based on the same review paper:

- Martin Hairer, *Regularity structures and the dynamical Φ^4_3 model*, arXiv:1508.05261

The Φ^4_3 model seems to be a unifying theme for the four mini-courses.

Constructive Quantum Field Theory

Aim: finding a physically-relevant, relativistic quantum field theory that is also mathematically consistent.

Starting from the 60s, in particular for the Euclidean case:

- Symanzik
- Nelson
- Glimm and Jaffe
- Osterwalder and Schrader
- Guerra, Rosen, Simon

A major open problem is the Yang-Mills theory on the Minkowski 4-dimensional space.

The Φ^4_3 model

A large literature in the 70s and 80s on the 3-dimensional Φ^4_3 case:

- Glimm and Jaffe, Feldman and Osterwalder
- Magnen and Sénéor, Rivasseau
- Gallavotti, Benfatto, Cassandro, Nicolò, Olivieri, Presutti
- Bałaban
- Battle and Federbush
- Brydges, Frölich and Sokal

The Φ^4_3 model

Let us consider the torus $\mathbb{T}^3 := \mathbb{R}^3/\mathbb{Z}^3$. The aim is to construct the probability measure

$$\mu(\mathrm{d}\Phi) := \frac{1}{Z}\exp\left(-\frac{1}{2}\int_{\mathbb{T}^3}\left(|\nabla\Phi(x)|^2 + m\Phi(x)^2 + \frac{\lambda}{2}\Phi(x)^4\right)\mathrm{d}x\right)\mathrm{d}\Phi$$

where $\mathrm{d}\Phi$ is a (non-existent) Lebesgue measure on $\{\Phi : \mathbb{T}^3 \to \mathbb{R}\}$, $m \in \mathbb{R}$, $\lambda > 0$.

This measure is in fact ill-defined since...

The Φ^4_3 model

... if μ were easily well-posed, we would have

$$\mu(\mathrm{d}\Phi) = \frac{1}{Z} \exp\left(-\frac{\lambda}{4}\int_{\mathbb{T}^3} \Phi(x)^4\, \mathrm{d}x\right) \mathcal{N}(0, (m-\Delta)^{-1})(\mathrm{d}\Phi)$$

where Δ is the Laplace operator on $\mathbb{T}^3$. However it is well known that $\mathcal{N}(0, (m-\Delta)^{-1})$ is supported only by a space of distributions on $\mathbb{T}^3$ and therefore the quartic non-linearity is ill-defined.

This lack of local regularity is known as the phenomenon of ultra-violet divergence, opposed to that of infra-red divergence on large scales.

The higher the space dimension, the worse the lack of regularity of Φ.

Stochastic quantization

In 1981, Parisi and Wu proposed to construct μ as the invariant measure of a Markov process, solution to a SPDE

$$\partial_t \Phi = \frac{1}{2}\Delta\Phi - m\Phi - \lambda\Phi^3 + \xi$$

where ξ is a space-time white noise, namely a centered Gaussian random field on $\mathbb{R}_+ \times \mathbb{T}^3$ such that

$$\mathbb{E}[\xi(t,x)\,\xi(s,y)] = \delta(t-s)\,\delta(x-y).$$

The same problem on $\mathbb{T}^2$ has been succesfully studied by [Jona Lasinio-Mitter, 1985], [Albeverio-Röckner, 1991], [Da Prato-Debussche, 2003].

Stochastic quantization

However the 3-dimensional case had to wait until [Hairer, 2014] and the introduction of regularity structures.

Another succesful approach is based on paracontrolled distributions [Gubinelli-Imkeller-Perkowski, 2015], see [Catellier-Chouk, 2018].

Another approach, based on Wilson's renormalization group analysis, has been proposed by [Kupiainen, 2016].

In dimension 3, [Mourrat-Weber, 2017] prove strong estimates on the dynamical Φ^4_3 equation, implying in particular ergodicity.

Using the results of [Brydges-Frölich-Sokal, 1983] and a lattice discretisation, [Hairer-Matetski, 2018] prove that the invariant measure of the dynamical Φ^4_3 equation is the Φ^4_3 model constructed in the 70s.

[Mourrat-Weber, 2017] prove convergence of the two-dimensional dynamic Ising-Kac model to Φ^4_2.

Ultra-violet divergence

Of course the ultra-violet divergence appears also in the dynamical Φ^4_3 equation

$$\partial_t \Phi = \Delta \Phi - \Phi^3 + \xi.$$

We introduce the function space $\mathcal{C}^\alpha$ for $\alpha \in \mathbb{R}$, which is equal to the parabolic Besov space $B^\alpha_{\infty,\infty,\mathrm{loc}}$ and coincides with the usual parabolic $\mathcal{C}^\alpha$ space for positive non-integer α.

Then the solution to the linear equation $\partial_t u = \Delta u + \xi$ belongs to $\mathcal{C}^\alpha$ only for $\alpha < -\frac{1}{2}$. Therefore the cubic non-linearity Φ^3 is ill-defined.

Singular stochastic PDEs

Another famous equation with a similar phenomenon is

$$(\mathrm{KPZ}) \qquad \partial_t u = \Delta u + (\partial_x u)^2 + \xi, \quad x \in \mathbb{R},$$

where ξ now is a space-time white noise on $\mathbb{R}_+ \times \mathbb{R}$.

Even for polynomial non-linearities, we do not know how to properly define products of (random) distributions.

In this minicourse we are going to discuss Martin Hairer's approach to the renormalisation of singular equations of the form

$$\partial_t u = \Delta u + F(u, \nabla u, \xi)$$

but concentrating on the Φ^4_3 example.

Regularisation/renormalisation

Theorem (Hairer)

Let $\xi_\varepsilon := \varrho_\varepsilon * \xi$ *and* Φ_ε *given on* $\mathbb{T}^3$ *by*

$$\partial_t \Phi_\varepsilon = \Delta \Phi_\varepsilon + C_\varepsilon \Phi_\varepsilon - \Phi_\varepsilon^3 + \xi_\varepsilon.$$

There exist constants C_ε *diverging as* $\varepsilon \to 0$ *and a process* Φ *such that* $\Phi_\varepsilon \to \Phi$ *in probability and* Φ *does not depend on the mollifier* ϱ.

In fact, as in Antti's course this morning

$$C_\varepsilon \sim \frac{a_1}{\varepsilon} + a_2 \log \varepsilon, \qquad \varepsilon \to 0.$$

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$$C_\varepsilon \sim \frac{a_1}{\varepsilon} + a_2 \log \varepsilon, \qquad \varepsilon \to 0.$$

Questions to be discussed in this course:

- Is it legitimate to modify the equation in order to obtain a convergent solution?
- In other words: which one is the correct equation?

An analogous phenomenon: Wong-Zakai

Let us consider the ODE in $\mathbb{R}^d$ with smooth coefficients b, f

$$\dot{x}_\varepsilon = b(x_\varepsilon) + f(x_\varepsilon)\,\dot{B}_\varepsilon \tag{1}$$

where B_ε is a smooth approximation of a BM B. The Wong-Zakai result states that $x_\varepsilon \to x$ solution to the Stratonovich SDE

$$\mathrm{d}x = b(x)\,\mathrm{d}t + f(x) \circ \mathrm{d}B.$$

In order to obtain the Itô SDE in the limit, one has to define rather

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{x}_\varepsilon = b(\hat{x}_\varepsilon) - \frac{1}{2}Df(\hat{x}_\varepsilon)f(\hat{x}_\varepsilon) + f(\hat{x}_\varepsilon)\,\dot{B}_\varepsilon \tag{2}$$

and in this case $\hat{x}_\varepsilon \to \hat{x}$ solution to

$$\mathrm{d}\hat{x} = b(\hat{x})\,\mathrm{d}t + f(\hat{x})\,\mathrm{d}B.$$

Now, (2) is a renormalisation of (1).

Regularisation

Let $\xi_\varepsilon = \varrho_\varepsilon * \xi$ a regularisation of ξ and let u_ε solve

$$\partial_t u_\varepsilon = \Delta u_\varepsilon + F(u_\varepsilon, \nabla u_\varepsilon, \xi_\varepsilon).$$

What happens as $\varepsilon \to 0$?

We need a topology such that

- the map $\xi_\varepsilon \mapsto u_\varepsilon$ is continuous
- $\xi_\varepsilon \to \xi$ as $\varepsilon \to 0$.

For classical negative Sobolev spaces the first point fails.

For classical positive Sobolev spaces the second point fails.

The theory of regularity structures (RS) gives a framework to solve this problem.

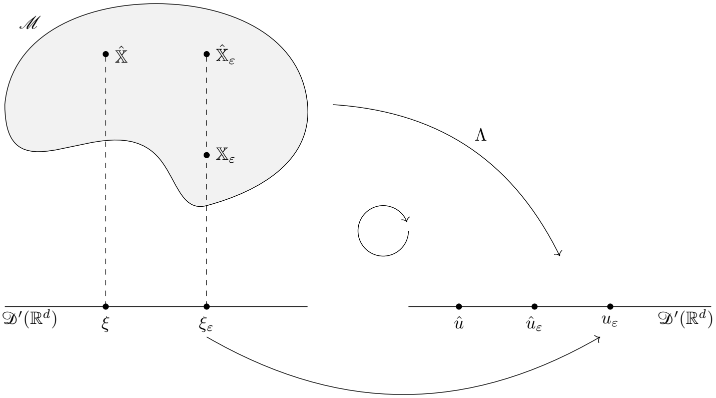
The Solution Map on models

Martin's theory gives

- a space of Models $(\mathscr{M}, \mathrm{d})$
- a canonical lift of every smooth ξ_ε to a model $\mathbb{X}^\varepsilon \in \mathscr{M}$
- a continuous function $\Lambda : \mathscr{M} \to \mathcal{D}'(\mathbb{R}^d)$ such that $u_\varepsilon = \Lambda(\mathbb{X}^\varepsilon)$ solves the regularised equation
$$\partial_t u_\varepsilon = \Delta u_\varepsilon + F(u_\varepsilon, \nabla u_\varepsilon, \xi_\varepsilon).$$
- a group action on $\mathscr{M}$ and a choice of $\hat{\mathbb{X}}^\varepsilon \in \mathscr{M}$ which lifts ξ_ε and converges to $\hat{\mathbb{X}} \in \mathscr{M}$ as $\varepsilon \to 0$.
- a renormalised solution $\hat{u} := \Lambda(\hat{\mathbb{X}})$, also the unique solution of a fixed point problem.

In general $\mathbb{X}^\varepsilon$ does not converge in $\mathscr{M}$.

Regularity stuctures in an image



$\mathcal{D}'(\mathbb{R}^d) \ni \xi_\varepsilon \mapsto u_\varepsilon \in \mathcal{D}'(\mathbb{R}^d)$ is not continuous

$\mathcal{D}'(\mathbb{R}^d) \ni \xi_\varepsilon \mapsto \mathbb{X}_\varepsilon \in \mathscr{M}$ is not continuous

$\Lambda : \mathscr{M} \to \mathcal{D}'(\mathbb{R}^d)$ is continuous, like the Itô-Lyons map in rough paths.

Renormalised equations

The renormalised function $\hat{u}_\varepsilon := \Lambda(\hat{\mathbb{X}}_\varepsilon)$ is also the solution to a modified equation.

For instance, in the case of Φ^4_3, we have

$$\partial_t u_\varepsilon = \Delta u_\varepsilon - u_\varepsilon^3 + \xi_\varepsilon$$

and for a suitable choice of $\hat{\mathbb{X}}_\varepsilon$

$$\partial_t \hat{u}_\varepsilon = \Delta \hat{u}_\varepsilon + C_\varepsilon \hat{u}_\varepsilon - \hat{u}_\varepsilon^3 + \xi_\varepsilon.$$

Generalised Taylor expansions

The theory of regularity structures is based on a deep generalisation of the classical Taylor expansion technique.

Quote: "while solutions to equations like KPZ or Φ^4_3 may appear to be very rough, they can actually considered to be smooth, provided that one looks at them in the right way".

Following Terry Lyons' and Massimiliano Gubinelli's ideas on rough paths, one considers a set of generalised local monomials which are explicit functions of the noise ξ and then writes a generalised Taylor expansion of u or Φ with respect to these generalised monomials.

(However the theory is not perturbative!)

The space of models $\mathscr{M}$ is the space of such generalised monomials, with a suitable topology.

Divergent products

The model $\mathbb{X}^\varepsilon \in \mathscr{M}$ contains a finite number of relevant explicit products. For instance

$$\xi_\varepsilon(G * \xi_\varepsilon)$$

(with G the heat kernel). These products can be ill-defined in the limit $\varepsilon \to 0$:

$$\mathbb{E}[\xi_\varepsilon(G * \xi_\varepsilon)] = \varrho_\varepsilon * G * \varrho_\varepsilon(0) \to G(0) = +\infty.$$

Therefore in general $\mathbb{X}^\varepsilon$ does not converge in $(\mathscr{M}, \mathrm{d})$ as $\varepsilon \to 0$.

Renormalised products

The theory identifies a class of equations, called subcritical, for which it is enough to modify a finite number of products in order to obtain a convergent lift $\hat{\mathbb{X}}^\varepsilon \in \mathscr{M}$ of ξ_ε. For instance

$$\xi_\varepsilon(G * \xi_\varepsilon) \to \xi_\varepsilon(G * \xi_\varepsilon) - \mathbb{E}[\xi_\varepsilon(G * \xi_\varepsilon)].$$

The model $\hat{\mathbb{X}}^\varepsilon \in \mathscr{M}$ contains all these modified (renormalised) products.

Convergence in $(\mathscr{M}, \mathrm{d})$ means (simplifying a lot) convergence of all these objects as distributions.

Then we define the renormalised solution by $\hat{u}_\varepsilon := \Lambda(\hat{\mathbb{X}}^\varepsilon)$.

Critical equations / superrenormalisable theories

This works for a class of equations called subcritical.

This corresponds to physicists' superrenormalisable theories.

It is still impossible to apply these techniques to critical equations.

For example, major problems which remain open are

- Φ^4_4
- Quantum Electrodynamics
- Yang-Mills in dimension 4
- Fluctuating Hydrodynamics

Four papers

- Martin Hairer (2014),
  A theory of regularity structures, Inventiones.
- Yvain Bruned, M.H., L.Z. (2016),
  Algebraic renormalisation of regularity structures, arXiv.
- Ajay Chandra, M.H. (2016),
  An analytic BPHZ theorem for regulariy structures, arXiv.
- Y.B., A.C., Ilya Chevyrev, M.H. (2017),
  Renormalising SPDEs in regularity structures, arXiv.

This quartet of papers "gives a completely automatic black box for local existence and uniqueness theorems for a wide class of SPDEs".

The general procedure

One can summarize the procedure into four steps:

- Analytic step Construction of the space of models $(\mathscr{M}, \mathrm{d})$ and continuity of the solution map $\Lambda : \mathscr{M} \to \mathcal{D}'(\mathbb{R}^d)$,
- Algebraic step Renormalisation of the canonical model $\mathbb{X}^\varepsilon \to \hat{\mathbb{X}}^\varepsilon \in \mathscr{M}$,
- Probabilistic step Convergence in probability of the renormalised model $\hat{\mathbb{X}}^\varepsilon$ to $\hat{\mathbb{X}}$ in $(\mathscr{M}, \mathrm{d})$,
- Combinatorial step Computation of the equation solved by $\hat{u}_\varepsilon$.

We obtain a renormalised solution $\hat{u} := \Lambda(\hat{\mathbb{X}})$, also the unique solution of a fixed point problem.

This works for very general noises, far beyond the Gaussian case.

A technical detail

For technical reasons it is convenient to split the Green function of the heat operator $G : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R}$ as follows

$$G = K + R, \qquad K, R : \mathbb{R}^4 \times \mathbb{R}^4 \to \mathbb{R},$$

where

- R is smooth
- K has compact support and is such that

$$\int_{\mathbb{R}^4} K(x,y)\,\mathrm{d}y = \int_{\mathbb{R}^4} K(x,y)\,y_i\,\mathrm{d}y = 0, \qquad x = (x_0,\dots,x_3), y = (y_0,\dots,y_3) \in \mathbb{R}^4.$$

The singular part of G is contained in K since R is smooth.

Non-classical monomials

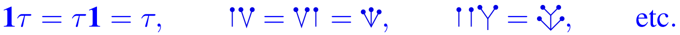
We fix a family of symbols which are relevant for Φ^4_3:

These represent combinations of products (branching points) and integrations (edges) which are relevant for the equation (to be explained later). Leaves represent the noise.

Similar to Feynman integrals, although we have only trees for the moment.

Using the notations given by Felix, they can be expressed also as follows:

The symbol $\mathcal{I}$ represents a convolution with the truncated green kernel K and there is a commutative product of $\tau_1, \tau_2 \in T$ given by the identification of the roots:



Homogeneities

We define homogeneities $|\cdot| : T \to \mathbb{R}$

$$|\mathcal{I}(\tau)| = |\tau| + 2\,, \qquad |\mathbf{1}| = 0, \qquad |X_i| = 1, \qquad |\Xi| = -\frac{5}{2} - \kappa$$

with $\kappa \in (0, 1/100)$. Then

$|$$| = -\frac{1}{2} - \kappa$, $|$$| = -1 - 2\kappa$, $|$$| = -\frac{3}{2} - 3\kappa$, $|$$| = -\frac{1}{2} - 5\kappa$,

$|$$| = |$$| = -4\kappa$, $|X_i$$| = -2\kappa$, $|$$| = |$$| = \frac{1}{2} - 3\kappa$, $|$$| = 1 - 2\kappa$.

For $x, y \in \mathbb{R}^4$ we define (and then forget) the parabolic distance

$$|x - y| := |x_0 - y_0|^{1/2} + \sum_{i=1}^{3} |x_i - y_i|.$$

Canonical model

Then we define for all $x \in \mathbb{R}^4$ the map $\Pi^\varepsilon_x : T \to C(\mathbb{R}^4)$

$$\Pi^\varepsilon_x \mathbf{1}(y) = 1, \quad \Pi^\varepsilon_x X_i(y) = (y - x)_i, \quad \Pi^\varepsilon_x \tau_1 \tau_2 = (\Pi^\varepsilon_x \tau_1)(\Pi^\varepsilon_x \tau_2),$$

$$\Pi^\varepsilon_x \mathcal{I}(\tau) = K * (\Pi^\varepsilon_x \tau)(y) - \mathbb{1}_{(|\mathcal{I}(\tau)|>0)} K * (\Pi^\varepsilon_x \tau)(x).$$

Note that these are already Taylor expansions centered at x. Then

$$\Pi^\varepsilon_x \Xi(y) = \xi_\varepsilon(y), \quad \Pi^\varepsilon_x \mathsf{I}(y) = K * \xi_\varepsilon(y), \quad \Pi^\varepsilon_x \mathsf{V}(y) = (K * \xi_\varepsilon(y))^2, \quad \Pi^\varepsilon_x \Psi(y) = (K * \xi_\varepsilon(y))^3,$$

$$\Pi^\varepsilon_x X_i \mathsf{V}(y) = (y - x)_i (K * \xi_\varepsilon(y))^2, \quad \Pi^\varepsilon_x \mathsf{Y}(y) = K * (\Pi^\varepsilon_x \Psi)(y) - K * (\Pi^\varepsilon_x \Psi)(x).$$

Canonical model

Finally

$$\Pi_x^{\varepsilon}\Upsilon(y) = K*((K*\xi_\varepsilon(\cdot))^2)(y) - K*((K*\xi_\varepsilon(\cdot))^2)(x),$$

$$\Pi_x^{\varepsilon}\Upsilon^{\bullet}(y) = K*\xi_\varepsilon(y)\left[K*((K*\xi_\varepsilon(\cdot))^2)(y) - K*((K*\xi_\varepsilon(\cdot))^2)(x)\right],$$

$$\Pi_x^{\varepsilon}{}^{\bullet}\Upsilon^{\bullet}(y) = (K*\xi_\varepsilon(y))^2\left[K*((K*\xi_\varepsilon(\cdot))^2)(y) - K*((K*\xi_\varepsilon(\cdot))^2)(x)\right],$$

$$\Pi_x^{\varepsilon}\Psi^{\bullet}(y) = K*\xi_\varepsilon(y)\left[K*((K*\xi_\varepsilon(\cdot))^3)(y) - K*((K*\xi_\varepsilon(\cdot))^3)(x)\right],$$

$$\Pi_x^{\varepsilon}{}^{\bullet}\Psi^{\bullet}(y) = (K*\xi_\varepsilon(y))^2\left[K*((K*\xi_\varepsilon(\cdot))^3)(y) - K*((K*\xi_\varepsilon(\cdot))^3)(x)\right].$$

Canonical model

In order to construct a model, we need another ingredient: the family of linear operators

$$\Gamma_{xy} : \mathcal{T} \to \mathcal{T}$$

where $\mathcal{T}$ is the linear span of T. We define

$$\Gamma^\varepsilon_{xy}\mathbf{1} = \mathbf{1}, \qquad \Gamma^\varepsilon_{xy}X_i(y) = X_i + (x-y)_i\mathbf{1}, \qquad \Gamma^\varepsilon_{xy}\Xi = \Xi,$$

$$\Gamma^\varepsilon_{xy}\tau_1\tau_2 = (\Gamma^\varepsilon_{xy}\tau_1)(\Gamma^\varepsilon_{xy}\tau_2).$$

Further (a recursive formula due to Yvain Bruned)

$$\Gamma^\varepsilon_{xy}\mathcal{I}(\tau) = \mathcal{I}(\Gamma^\varepsilon_{xy}\tau) - \mathbb{1}_{(|\mathcal{I}(\tau)|>0)}\,\Pi^\varepsilon_x\mathcal{I}\Gamma^\varepsilon_{xy}\tau(y)\mathbf{1}.$$

Then we can compute

$$\Gamma^{\varepsilon}_{xy}\mathcal{I}(\Xi) = \mathcal{I}(\Xi), \qquad \Gamma^{\varepsilon}_{xy}\mathcal{I}(\Xi)^2 = (\Gamma^{\varepsilon}_{xy}\mathcal{I}(\Xi))^3 = \mathcal{I}(\Xi)^2, \qquad \Gamma^{\varepsilon}_{xy}\mathcal{I}(\Xi)^3 = (\Gamma^{\varepsilon}_{xy}\mathcal{I}(\Xi))^2 = \mathcal{I}(\Xi)^3,$$

$$\Gamma^{\varepsilon}_{xy}\mathcal{I}(\mathcal{I}(\Xi)^2) = \mathcal{I}(\mathcal{I}(\Xi)^2) - \left[K * (K * \xi_\varepsilon)^2(y) - K * (K * \xi_\varepsilon)^2(x)\right]\mathbf{1}.$$

With the same formulae we can compute $\Gamma^{\varepsilon}_{xy}\tau$ for $\tau \in \{\mathcal{I}(\mathcal{I}(\Xi)^2)\mathcal{I}(\Xi), \mathcal{I}(\mathcal{I}(\Xi)^2)\mathcal{I}(\Xi)^2, \mathcal{I}(\mathcal{I}(\Xi)^3), \mathcal{I}(\mathcal{I}(\Xi)^3)\mathcal{I}(\Xi), \mathcal{I}(\mathcal{I}(\Xi)^3)\mathcal{I}(\Xi)^2\}$. For example

$$\Gamma^{\varepsilon}_{xy}\mathcal{I}(\mathcal{I}(\Xi)^2)\mathcal{I}(\Xi) = \mathcal{I}(\mathcal{I}(\Xi)^2)\mathcal{I}(\Xi) - \left[K * (K * \xi_\varepsilon)^2(y) - K * (K * \xi_\varepsilon)^2(x)\right]\mathcal{I}(\Xi),$$

$$\Gamma^{\varepsilon}_{xy}\mathcal{I}(\mathcal{I}(\Xi)^2)\mathcal{I}(\Xi)^2 = \mathcal{I}(\mathcal{I}(\Xi)^2)\mathcal{I}(\Xi)^2 - \left[K * (K * \xi_\varepsilon)^2(y) - K * (K * \xi_\varepsilon)^2(x)\right]\mathcal{I}(\Xi)^2.$$

Canonical model

Then we can compute

$$\Gamma^\varepsilon_{xy}\treeI = \treeI, \qquad \Gamma^\varepsilon_{xy}\treeIII = (\Gamma^\varepsilon_{xy}\treeI)^3 = \treeIII, \qquad \Gamma^\varepsilon_{xy}\treeII = (\Gamma^\varepsilon_{xy}\treeI)^2 = \treeII,$$

$$\Gamma^\varepsilon_{xy}\treeY = \treeY - \left[K * (K * \xi_\varepsilon)^2(y) - K * (K * \xi_\varepsilon)^2(x)\right]\mathbf{1}.$$

With the same formulae we can compute $\Gamma^\varepsilon_{xy}\tau$ for $\tau \in \{\treeYI, \treeYII, \treeIY, \treeIYI, \treeIYII\}$. For example

$$\Gamma^\varepsilon_{xy}\treeYI = \treeYI - \left[K * (K * \xi_\varepsilon)^2(y) - K * (K * \xi_\varepsilon)^2(x)\right]\treeI,$$

$$\Gamma^\varepsilon_{xy}\treeYII = \treeYII - \left[K * (K * \xi_\varepsilon)^2(y) - K * (K * \xi_\varepsilon)^2(x)\right]\treeII.$$

Theorem

For all x, y, z

$$\Pi^\varepsilon_x\Gamma^\varepsilon_{xy} = \Pi^\varepsilon_y, \qquad \Gamma^\varepsilon_{xy} \circ \Gamma^\varepsilon_{yz} = \Gamma^\varepsilon_{xz}, \qquad \Gamma^\varepsilon_{xx} = \mathrm{Id}.$$

These relations make the space of models non-linear.

Analytical estimates

For $\sigma \in T$ we set $\sigma^* : \mathcal{T} \to \{0,1\}$ as the linear functional such that $\sigma^*(\tau) = \mathbb{1}_{(\sigma=\tau)}$, $\tau \in T$.

Felix explained that, in a model (Π_x, Γ_{xy}), one requires analytical estimates like

$$|\Pi_x\tau(y)| \le C|x-y|^{|\tau|}, \qquad |\sigma^*(\Gamma_{xy}\tau)| \le C|x-y|^{|\tau|-|\sigma|}, \quad |\sigma| < |\tau|.$$

In the case of classical monomials these formulae are intuitive since

$$\Pi_x X^k(y) = (y-x)^k, \qquad \Gamma_{xy} X^k = (X + (x-y)\mathbf{1})^k = \sum_{i \le k} \binom{k}{i} (x-y)^{k-i} X^i$$

In our canonical non-classical setting the analytical estimates are trivially true, and are really significant only if they are uniform in ε (which they will not be if we do not renormalise!).

Classical Taylor expansion

If we have a function $f \in C^\gamma$, classical Hölder space, then we can set

$$U(x) = \sum_{k<\gamma} \frac{X^k}{k!} f^{(k)}(x),$$

and it is easy to check that

$$U(x) - \Gamma_{xy} U(y) = \sum_{k<\gamma} \frac{X^k}{k!} \left[f^{(k)}(x) - \sum_{i \le \gamma - k} \frac{(x-y)^i}{i!} f^{(k+i)}(y) \right] = \sum_{k<\gamma} \frac{X^k}{k!} O(|x-y|^{\gamma-k}),$$

so that $U \in \mathcal{D}^\gamma$ since $f^{(k)} \in C^{\gamma-k}$ for all $k \le \gamma$.

Then the reconstruction theorem simply states that

$$\mathcal{R}U(y) = \Pi_y U(y)(y) = f(y), \qquad \left| f(y) - \sum_{k<\gamma} \frac{(y-x)^k}{k!} f^{(k)}(x) \right| \le C|x-y|^\gamma.$$

Non-classical Hölder functions

Let $\gamma = 1 + 3\kappa$. We denote by $\mathcal{D}^\gamma$ the set of functions $U : \mathbb{R}^4 \to \mathcal{T}$ such that

$$|\sigma^*[U(x) - \Gamma^\varepsilon_{xy} U(y)]| \leq C|x - y|^{\gamma - |\sigma|}.$$

Note that the space $\mathcal{D}^\gamma$ depends on the model (Π, Γ) ! It is called the space of modelled distributions, in the sense that they are modelled by (Π, Γ).

Martin's reconstruction theorem in this setting says the following: setting

$$\mathcal{R}_\varepsilon U(x) = \Pi^\varepsilon_x U(x)(x),$$

then for all x, y

$$|\mathcal{R}_\varepsilon U(y) - \Pi^\varepsilon_x U(x)(y)| \leq C|x - y|^\gamma.$$

Non-classical Hölder functions

Let us make two examples of functions in $\mathcal{D}^\gamma$.

If $U(x) = \Gamma^\varepsilon_{xz}\tau$ for some fixed z and $\tau \in \mathcal{T}$, then

$$U(x) - \Gamma^\varepsilon_{xy}U(y) = \Gamma^\varepsilon_{xz}\tau - \Gamma^\varepsilon_{xy}\Gamma^\varepsilon_{yz}\tau = 0.$$

If $U(x) = a(x)\upharpoonleft + b(x)\Upsilon$ then

$$U(x) - \Gamma^\varepsilon_{xy}U(y) = \big(a(x) - a(y) - b(y)\big[K * (K * \xi_\varepsilon)^2(y) - K * (K * \xi_\varepsilon)^2(x)\big]\big)\upharpoonleft$$
$$+ (b(x) - b(y))\Upsilon.$$

Since $|\upharpoonleft| = -\frac{1}{2} - \kappa$ and $|\Upsilon| = \frac{1}{2} - 3\kappa$ then $U \in \mathcal{D}^\gamma$ is equivalent to

$$\big|a(x) - a(y) - b(y)\big[K * (K * \xi_\varepsilon)^2(x) - K * (K * \xi_\varepsilon)^2(y)\big]\big| \le C|x - y|^{\frac{3}{2}+2\kappa},$$
$$|b(x) - b(y)| \le C|x - y|^{\frac{1}{2}+4\kappa}.$$

This is exactly the analog of Massimiliano's definition of controlled paths.

Models

A model of a regularity structure $\mathscr{T} = (A, \mathcal{T}, G)$ is a pair $(\Pi_x, \Gamma_{xz})_{x,z\in\mathbb{R}^d}$ of operators s.t.

- $\Pi_x : \mathcal{T} \to \mathcal{D}'(\mathbb{R}^d)$ is linear for all $x \in \mathbb{R}^d$
- $\Gamma_{xz} : \mathcal{T} \to \mathcal{T}$ is an element of G for all $x, z \in \mathbb{R}^d$
- for all $x, y, z \in \mathbb{R}^d$

$$\Gamma_{yx} \circ \Gamma_{xz} = \Gamma_{yz}, \qquad \Pi_x \Gamma_{xz} \tau(y) = \Pi_z \tau(y), \qquad \tau \in \mathcal{T}.$$

- $(\Pi_x, \Gamma_{xz})_{x,z\in\mathbb{R}^d}$ satisfy several analytical requirements.

The distributions $\Pi_x(y)$ are our generalised monomials in the variable y and centered at x.

The operators Γ_{xz} are used to compare coefficients at x and z.

Until now we have only seen one model $(\Pi^\varepsilon, \Gamma^\varepsilon)$ which is in particular multiplicative, but the theory is built in order to work without this assumption.

Reconstruction

Theorem

We fix a model $(\Pi, \Gamma) \in \mathscr{M}$ *and* $\gamma > 0$. *Then there exists a unique linear map* $\mathcal{R} : \mathcal{D}^\gamma \to \mathcal{D}'(\mathbb{R}^d)$ *such that*

$$|(\mathcal{R}U - \Pi_x U(x))(\varphi_x^\lambda)| \lesssim \lambda^\gamma$$

for all $\varphi \in C_0^\infty(\mathbb{R}^d)$, $\lambda > 0$ *and*

$$\varphi_x^\lambda(y) := \lambda^{-d} \varphi(\lambda^{-1}(y - x)), \qquad x, y \in \mathbb{R}^d.$$

Informally one can interpret the above result as saying that

$$|\mathcal{R}U(y) - \Pi_x U(x)(y)| \lesssim |x - y|^\gamma.$$

Reconstruction

Recall that $U(x)$ has the form

$$U(x) = \sum_{\tau \in T} a_\tau(x)\, \tau$$

where $x \mapsto a_\tau(x)$ is a function and the τ's are just symbols.

The distributions are inside $\Pi_x \tau(\cdot)$, while even Γ_{xy} contains only proper functions of x, y.

When we compute

$$\Pi_x U(x)(\cdot) = \sum_{\tau \in T} a_\tau(x)\, \Pi_x \tau(\cdot) \in \mathcal{D}'(\mathbb{R}^d),$$

there is no problem since we have decoupled the x, y variables.

When all $\Pi_x \tau(\cdot)$ are continuous functions, then in fact the reconstruction operator is simply

$$\mathcal{R}U(x) = \Pi_x U(x)(x) = \sum_{\tau \in T} a_\tau(x)\, \Pi_x \tau(x).$$

However in general the existence of $\mathcal{R}U$ is a much deeper result.

Theorem

Let $\gamma > 0$. There exists a unique linear operator $\mathcal{K} : \mathcal{D}^\gamma \to \mathcal{D}^{\gamma+2}$ such that

$$\mathcal{R}\mathcal{K}U = K * \mathcal{R}U, \qquad \forall\, U \in \mathcal{D}^\gamma.$$

For us $\gamma = 1 + 3\kappa$ (see below why). Then $\mathcal{K}U \in \mathcal{D}^\gamma$ has an explicit expression

$$\mathcal{K}U(x) = \Pi_{<\gamma}\mathcal{I}U(x) + f(x)\mathbf{1} + \langle \nabla f(x), (X_1, X_2, X_3)\rangle$$

for some explicit functions $f : \mathbb{R}^4 \to \mathbb{R}$ and $\nabla f : \mathbb{R}^4 \to \mathbb{R}^4$, where by convention

$$\mathcal{I}\mathbf{1} = \mathcal{I}X_i = 0$$

(this explains why K has to integrate to 0 the functions 1 and $y \mapsto y_i$) and $\Pi_{<\gamma}$ is a projection onto the linear span of trees with homogeneity strictly less than γ.

Multiplication

Question: Let $f_1 \in C^{\gamma_1}$ and $f_2 \in C^{\gamma_2}$ (classical Hölder spaces with $\gamma_i > 0$).
What is γ such that $f_1 f_2 \in C^{\gamma}$?

Choose:

- $\gamma = \gamma_1 + \gamma_2$
- $\gamma = \gamma_1 \gamma_2$
- $\gamma = \gamma_1 \wedge \gamma_2$
- $\gamma = \cosh(\gamma_1 + \gamma_2)$.

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- $\gamma = \cosh(\gamma_1 + \gamma_2)$.

Answer: $\gamma = \gamma_1 \wedge \gamma_2$.

Multiplication

Let $\alpha \in]-\infty, 0]$. We say that $F \in \mathcal{D}^\gamma_\alpha$ if $F \in \mathcal{D}^\gamma$ and

$$F(x) = \sum_k a_k(x)\, \tau_k, \qquad |\tau_k| \geq \alpha.$$

Theorem

Suppose that for $i = 1, 2$ *we have* $F_i \in \mathcal{D}^{\gamma_i}_{\alpha_i}$. *Define* $\gamma := (\gamma_1 + \alpha_2) \wedge (\gamma_2 + \alpha_1)$ *and*

$$F_1 F_2(x) := \sum_{k_1, k_2} \mathbb{1}_{(|\tau^1_{k_1}| + |\tau^2_{k_2}| < \gamma)}\, a^1_{k_1}(x)\, a^2_{k_2}(x)\, \tau^1_{k_1} \tau^2_{k_2}.$$

Then $F_1 F_2 \in \mathcal{D}^\gamma_{\alpha_1 + \alpha_2}$.

Again in good (but not necessarily canonical) cases we have, setting $f_i := \mathcal{R} F_i$,

$$f_1 \cdot f_2(x) := \mathcal{R}(F_1 F_2)(x) = \sum_{k_1, k_2} \mathbb{1}_{(|\tau^1_{k_1}| + |\tau^2_{k_2}| < \gamma)}\, a^1_{k_1}(x)\, a^2_{k_2}(x)\, \Pi_x(\tau^1_{k_1} \tau^2_{k_2})(x).$$

Non-classical Taylor expansion

Now if u_ε is the solution to the regularised (non-renormalised) Φ^4_3 equation

$$\partial_t u_\varepsilon = \Delta u_\varepsilon - u_\varepsilon^3 + \xi_\varepsilon$$

then we want to find a $U \in \mathcal{D}^\gamma$ modelled by the canonical model $(\Pi^\varepsilon, \Gamma^\varepsilon)$ such that

$$\mathcal{R}_\varepsilon U = u_\varepsilon .$$

We'll see that $U \in \mathcal{D}^\gamma$ is of the form

$$U = \mathcal{I}(\Xi) + \varphi\,\mathbf{1} - \mathcal{I}(\mathcal{I}(\Xi)^3) - 3\varphi\,\mathcal{I}(\mathcal{I}(\Xi)^2) + \langle \nabla\varphi, X\rangle,$$

where φ is real-valued and $\nabla\varphi$ is $\mathbb{R}^3$-valued.

Lift of the regularised PDE

Let us suppose that $U \in \mathcal{D}^{1+3\kappa}$ satisfies

$$U = \mathcal{K}(\Xi - U^3).$$

By the definitions we must have

$$U = \sum_{\tau \in T} a_\tau \, \tau.$$

Then

$$\sum_{\tau \in T} a_\tau \, \tau = \mathcal{I}(\Xi) - \Pi_{<\gamma} \sum_{\tau_1, \tau_2, \tau_3} a_{\tau_1} a_{\tau_2} a_{\tau_3} \, \mathcal{I}(\tau_1 \, \tau_2 \, \tau_3) + \varphi \, \mathbf{1} + \langle \nabla \varphi, X \rangle.$$

This implies

$$U = \mathcal{I}(\Xi) + \varphi \, \mathbf{1} + a_1 \mathcal{I}(\mathcal{I}(\Xi)^3) + a_2 \mathcal{I}(\mathcal{I}(\Xi)^2) + \langle \nabla \varphi, X \rangle.$$

Lift of the regularised PDE

Recall that

$$U = \mathcal{I}(\Xi) + \varphi\,\mathbf{1} + a_1\,\mathcal{I}(\mathcal{I}(\Xi)^3) + a_2\,\mathcal{I}(\mathcal{I}(\Xi)^2) + \langle \nabla\varphi, X\rangle.$$

Let us compute $\Xi - U^3$. Let us start from U^2. Note that $U \in \mathcal{D}^{1+3\kappa}_{-\frac{1}{2}-\kappa}$.

Then $U^2 \in \mathcal{D}^{\frac{1}{2}+2\kappa}_{-1-2\kappa}$.

Now we multiply $U^2 \in \mathcal{D}^{\frac{1}{2}+2\kappa}_{-1-2\kappa}$ and $U \in \mathcal{D}^{1+3\kappa}_{-\frac{1}{2}-\kappa}$: we have then that $U^3 \in \mathcal{D}^{\kappa}$ and

$$\Xi - U^3 = \Xi - \mathcal{I}(\Xi)^3 - 3\varphi\,\mathcal{I}(\Xi)^2 - 3a_1\,\mathcal{I}(\mathcal{I}(\Xi)^3)\mathcal{I}(\Xi)^2 - 3\varphi^2\,\mathcal{I}(\Xi) - 6a_1\,\mathcal{I}(\mathcal{I}(\Xi)^3)\mathcal{I}(\Xi) - 3a_2\,\mathcal{I}(\mathcal{I}(\Xi)^2)\mathcal{I}(\Xi)^2 - 3\langle \nabla\varphi, X\rangle - \varphi^3\,\mathbf{1}$$

belongs to $\mathcal{D}^{\kappa}$ with $\kappa > 0$, so that we can apply the integration operator to it.

Here we see why $\gamma = 1 + 3\kappa$ and why $T = \langle \Xi, \mathcal{I}(\Xi)^3, \mathcal{I}(\Xi)^2, \mathcal{I}(\mathcal{I}(\Xi)^3)\mathcal{I}(\Xi)^2, \mathcal{I}(\Xi), \mathcal{I}(\mathcal{I}(\Xi)^3)\mathcal{I}(\Xi), \mathcal{I}(\mathcal{I}(\Xi)^2)\mathcal{I}(\Xi)^2, X_i\mathcal{I}(\Xi)^2, \mathbf{1}, \mathcal{I}(\mathcal{I}(\Xi)^3), \mathcal{I}(\mathcal{I}(\Xi)^2)\mathcal{I}(\Xi), \mathcal{I}(\mathcal{I}(\Xi)^2), X_i\rangle$.

Lift of the regularised PDE

Recall that

$$\Xi - U^3 = \Xi - \Psi - 3\varphi \vee - 3a_1 \dot{\Psi} - 3\varphi^2 \upharpoonleft - 6a_1 \dot{\Upsilon} - 3a_2 \ddot{\vee} - 3\langle \nabla\varphi, X\rangle - \varphi^3\, \mathbf{1}.$$

Now we apply $\mathcal{K}$ to $\Xi - U^3 \in \mathcal{D}^\kappa$. We obtain $\mathcal{K}(\Xi - U^3) \in \mathcal{D}^{1+3\kappa}$

$$\mathcal{K}(\Xi - U^3) = \upharpoonleft - \Psi - 3\varphi\, \Upsilon + \psi\, \mathbf{1} + \langle \nabla\psi, X\rangle.$$

Suppose now that $U = \mathcal{K}(\Xi - U^3)$. Then

$$\upharpoonleft + \varphi\, \mathbf{1} + a_1 \Psi + a_2\, \Upsilon + \langle \nabla\varphi, X\rangle = \upharpoonleft + \psi\, \mathbf{1} - \Psi - 3\varphi\, \Upsilon + \langle \nabla\psi, X\rangle$$

namely $a_1 = -1$, $a_2 = -3\varphi$, $\psi = \varphi$ and $\nabla\psi = \nabla\varphi$. We have shown that

$$U = \upharpoonleft + \varphi\, \mathbf{1} - \Psi - 3\varphi\, \Upsilon + \langle \nabla\varphi, X\rangle,$$

The solution map

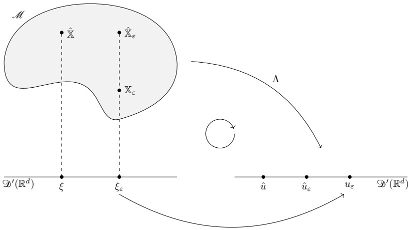
We note that until here these computations hold for any model (Π, Γ).

For any such model, we construct the associated $\mathcal{D}^\gamma$ space, and we set the fixed point problem

$$U = \mathcal{K}(\Xi - U^3).$$

We define a map on $\mathcal{D}^\gamma$, which will be a contraction on a small time inverval: this is the reason why in general solutions are only local in time. Then we define $u := \mathcal{R}U$.

In this way we have constructed the solution map $\Lambda : \mathscr{M} \to \mathcal{D}'(\mathbb{R}^d)$.



Lift of the regularised PDE

By reconstruction we obtain

$$u_\varepsilon := \mathcal{R}_\varepsilon U = \mathcal{R}_\varepsilon \mathcal{K}(\Xi - U^3) = K * (\mathcal{R}_\varepsilon \Xi - \mathcal{R}_\varepsilon U^3) = K * (\xi_\varepsilon - u_\varepsilon^3).$$

This is the regularised PDE. Well, almost in fact, since K replaces the heat kernel G, but we neglect this detail. Therefore

$$u_\varepsilon := \mathcal{R}_\varepsilon U = \mathcal{R}_\varepsilon \mathcal{K}(\Xi - U^3) = G * (\xi_\varepsilon - u_\varepsilon^3)$$

which is equivalent to

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In the formula $\mathcal{R}_\varepsilon U^3 = u_\varepsilon^3$ we use the property of the canonical model of being multiplicative.

The next step is to renormalise the canonical model and compute the associated PDE.