Regularity structures
Reconstruction and Integration

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following M. Hairer's
"Regularity structures and the dynamical Φ^4_3 model"

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Toy example, input

Parabolic $\partial_t - \triangle$ in $d = 3 \longleftrightarrow$ elliptic $\triangle$ in $d = 2 + 3 = 5$.

Analyze $-\triangle\phi = \eta\xi$

with $\xi =$ white noise , η smooth and bounded support.

Relate to standard *stationary* centered Gaussian fields:

$-\triangle v = \xi$	v stationary	$C^{-\frac{1}{2}}$,
$-\triangle v_i = 2\partial_i v$	v_i-$v_i(x)$ stationary	$C^{\frac{1}{2}}$,
$-\triangle v_{ij} = 2(v\delta_{ij} + \partial_i v_j + \partial_j v_i)$	v_{ij}-$v_{ij}(x)$-$(\cdot$-$x) \cdot \nabla v_{ij}(x)$ stationary	$C^{\frac{3}{2}}$.

2 goals: display locality effect in presence of roughness, introduce regularity structure in simple context.

Toy example, outcome

Recall: standard stationary centered Gaussian fields:

$-\triangle v = \xi$	v stationary	$C^{-\frac{1}{2}}$,
$-\triangle v_i = 2\partial_i v$	v_i-$v_i(x)$ stationary	$C^{\frac{1}{2}}$,
$-\triangle v_{ij} = 2(v\delta_{ij} + \partial_i v_j + \partial_j v_i)$	v_{ij}-$v_{ij}(x)$-$(\cdot\text{-}x)\cdot\nabla v_{ij}(x)$ stationary	$C^{\frac{3}{2}}$.

These provide (non-stationary) building blocks:

$$-\triangle((\cdot\text{-}x)_i v + v_i) = (\cdot\text{-}x)_i\xi,$$

$$-\triangle((\cdot\text{-}x)_i(\cdot\text{-}x)_j v + (\cdot\text{-}x)_i v_j + (\cdot\text{-}x)_j v_i + v_{ij}) = (\cdot\text{-}x)_i(\cdot\text{-}x)_j\xi.$$

Singular part is local, regular (=polynomial) part nonlocal:

$$\phi = \eta(x)v + \partial_i\eta(x)((\cdot\text{-}x)_i v + v_i)$$
$$+ \tfrac{1}{2}\partial_{ij}\eta(x)((\cdot\text{-}x)_i(\cdot\text{-}x)_j v + (\cdot\text{-}x)_i v_j + (\cdot\text{-}x)_j v_i + v_{ij})$$
$$+ p_x + O_{\text{weak}}(|\cdot - x|^{\frac{5}{2}})$$

p_x of degree ≤ 2.

Toy example: standard Schauder theory

Recall $(-\triangle)v=\xi,\ (-\triangle)v_i=2\partial_i v,$, one step further
$\frac{1}{2}(-\triangle)v_{ij}=\delta_{ij}v+\partial_i v_j+\partial_j v_i\in C^{-\frac{1}{2}}\implies v_{ij}\in C^{\frac{3}{2}}$,
more precisely $v_{ij}-v_{ij(x)}-(\cdot\text{-}x)\cdot\nabla v_{ij(x)}=O(|\cdot\text{-}x|^{\frac{3}{2}})$.

Leibniz rule:

$$(-\triangle)\Big(\phi-\eta v-\partial_i\eta(v_i\text{-}v_{i(x)})-\frac{1}{2}\partial_{ij}\eta(v_{ij}\text{-}v_{ij(x)}\text{-}(\cdot\text{-}x)\cdot\nabla v_{ij(x)})\Big)$$
$$=(\triangle\partial_i\eta)(v_i\text{-}v_{i(x)})+\nabla\partial_{ij}\eta\cdot(\nabla v_{ij}\text{-}\nabla v_{ij(x)})$$
$$+\tfrac{1}{2}(\triangle\partial_{ij}\eta)(v_{ij}\text{-}v_{ij(x)}\text{-}(\cdot\text{-}x)\cdot\nabla v_{ij(x)})\in C^{\frac{1}{2}}.$$

(standard) Schauder theory:

$$\phi-\eta v-\partial_i\eta(v_i\text{-}v_{i(x)})-\frac{1}{2}\partial_{ij}\eta(v_{ij}\text{-}v_{ij(x)}\text{-}(\cdot\text{-}x)\cdot\nabla v_{ij(x)})\in C^{\frac{5}{2}}.$$

Toy example: post-process by Taylor

Recall: $\phi - \eta v - \partial_i\eta(v_i\text{-}v_i(x)) - \frac{1}{2}\partial_{ij}\eta(v_{ij}\text{-}v_{ij}(x)\text{-}(\cdot\text{-}x)\cdot\nabla v_{ij}(x)) \in C^{\frac{5}{2}}$

Taylor expansion 1: $\phi = \eta v + \partial_i\eta(v_i\text{-}v_{i(x)})$
$+ \frac{1}{2}\partial_{ij}\eta(v_{ij}\text{-}v_{ij(x)}\text{-}(\cdot\text{-}x)\cdot\nabla v_{ij(x)})$
$+ p_x + O(|\cdot\text{-}x|^{\frac{5}{2}})$.

Taylor expansion 2: $\phi = \eta_{(x)}v + \partial_i\eta_{(x)}((\cdot\text{-}x)_i v + v_i)$
$+ \frac{1}{2}\partial_{ij}\eta_{(x)}((\cdot\text{-}x)_i(\cdot\text{-}x)_j v + (\cdot\text{-}x)_i v_j + (\cdot\text{-}x)_j v_i + v_{ij})$
$+ p_x + O_{\text{weak}}(|\cdot\text{-}x|^{\frac{5}{2}})$.

Recall:

$(-\triangle)((\cdot\text{-}x)_i(\cdot\text{-}x)_j v + (\cdot\text{-}x)_i v_j + (\cdot\text{-}x)_j v_i + v_{ij}) = (\cdot\text{-}x)_i(\cdot\text{-}x)_j\xi$.

Definition of regularity structure

Set of homogeneities: $A \subset \mathbb{R}$ finite.

Model space: T finite dimensional vector space;
graded i. e. $T = \oplus_{\alpha \in A} T_\alpha$.

Structure group: G continuum subgroup of Aut(T);
nilpotent in sense of $(\Gamma\text{-}id)\tau \in \oplus_{\alpha<\beta} T_\alpha$ for $\tau \in T_\alpha$

Model space in toy example

Model space T spanned by the homogeneous

$$\mathbf{1},\ X_i,\ X_iX_j,\quad \Xi,\ X_i\Xi,\ X_iX_j\Xi,\quad \mathcal{I}\Xi,\ \mathcal{I}X_i\Xi,\ \mathcal{I}X_iX_j\Xi.$$

Homogeneities:

$|\mathbf{1}| = 0,\qquad |X_i| = 1,\qquad |X_iX_j| = 1+1 = 2,$

$|\Xi| = -\frac{5}{2},\qquad |X_i\Xi| = -\frac{5}{2}+1 = -\frac{3}{2},\qquad |X_iX_j\Xi| = -\frac{5}{2}+2 = -\frac{1}{2},$

$|\mathcal{I}\Xi| = -\frac{5}{2}+2 = -\frac{1}{2},\qquad |\mathcal{I}X_i\Xi| = -\frac{3}{2}+2 = \frac{1}{2},\qquad |\mathcal{I}X_iX_j\Xi| = -\frac{1}{2}+2 = \frac{3}{2}.$

$$A = \{-\tfrac{5}{2} < -\tfrac{3}{2} < -\tfrac{1}{2} < 0 < \tfrac{1}{2} < 1 < \tfrac{3}{2} < 2\}.$$

Structure group in toy example

Structure group G in the matrix representation w. r. t. ordered basis Ξ, $X_i\Xi$, $X_iX_j\Xi$, $\mathbf{1}$, X_i, X_iX_j, $\mathcal{I}\Xi$, $\mathcal{I}X_i\Xi$, $\mathcal{I}X_iX_j\Xi$:

$$G \ni \Gamma = \begin{pmatrix} 1 & a & a^2 & & & & & & \\ & 1 & 2a & & & & & & \\ & & 1 & & & & & & \\ & & & 1 & a & a^2 & & b & 2ab+d \\ & & & & 1 & 2a & & & 2b+c \\ & & & & & 1 & & & \\ & & & & & & 1 & a & a^2 \\ & & & & & & & 1 & 2a \\ & & & & & & & & 1 \end{pmatrix}.$$

Non-Abelian group structure: $\bar{\Gamma} = \Gamma\Gamma'$ with
$\bar{a} = a + a'$, $\bar{b} = b + b'$, $\bar{c} = c + c'$, $\bar{d} = d + d' + ac'$.

Definition of model, Π-part

$\Pi : \mathbb{R}^d \to \mathcal{L}(T, \mathcal{S}'(\mathbb{R}^d)) : \mathbb{R}^d \ni x \mapsto \big(T \ni \tau \mapsto \Pi_x \tau \in \mathcal{S}'(\mathbb{R}^d)\big).$

"To every base point x, attach a distribution with values in T^*."

"To every base point x and placeholder τ, attach a distribution."

Postulate:

(local) order of distribution $\Pi_x \tau$ = homogeneity of τ:

$$|\int \phi_x^\lambda(z) \Pi_x \tau(dz)| \le C \lambda^\alpha \quad \text{for } \tau \in T_\alpha,\ \lambda \in (0,1].$$

Recall $\phi_x^\lambda(z) := \frac{1}{\lambda^d} \phi(\frac{z-x}{\lambda})$,

ϕ smooth and supported in $B_1(0)$ of C^{77}-norm ≤ 1.

Π-part of model in toy example

Recall $(-\triangle)v = \xi$, $(-\triangle)v_i = 2\partial_i v$, $\frac{1}{2}(-\triangle)v_{ij} = \delta_{ij}v + \partial_i v_j + \partial_j v_i$,

so that $(-\triangle)((\cdot\text{-}x)_i v + v_i) = (\cdot\text{-}x)_i \xi$

and $(-\triangle)((\cdot\text{-}x)_i(\cdot\text{-}x)_j v + (\cdot\text{-}x)_i v_j + (\cdot\text{-}x)_j v_i + v_{ij}) = (\cdot\text{-}x)_i(\cdot\text{-}x)_j \xi$.

$\Pi \colon \mathbb{R}^d \to \mathcal{L}(T, \mathcal{S}'(\mathbb{R}^d)) \colon \mathbb{R}^d \times T \ni (x, \tau) \mapsto \Pi_x \tau \in \mathcal{S}'(\mathbb{R}^d)$.

$\Pi_x \mathbf{1} = 1$, $\Pi_x X_i = (\cdot\text{-}x)_i$, $\Pi_x X_i X_j = (\cdot\text{-}x)_i(\cdot\text{-}x)_j$,
$\Pi_x \Xi = \xi$, $\Pi_x X_i \Xi = (\cdot\text{-}x)_i \xi$, $\Pi_x X_i X_j \Xi = (\cdot\text{-}x)_i(\cdot\text{-}x)_j \xi$,
$\Pi_x \mathcal{I}\Xi = v$, $\Pi_x \mathcal{I} X_i \Xi = (\cdot\text{-}x)_i v + v_i\text{-}v_{i(x)}$,
$\Pi_x \mathcal{I} X_i X_j \Xi = (\cdot\text{-}x)_i(\cdot\text{-}x)_j v + (\cdot\text{-}x)_i(v_j\text{-}v_{j(x)}) + (\cdot\text{-}x)_j(v_i\text{-}v_{i(x)})$
$+ v_{ij}\text{-}v_{ij(x)} - (\cdot\text{-}x) \cdot \nabla v_{ij(x)}$,

Definition of model, Γ-part

$\Gamma\colon \mathbb{R}^d \times \mathbb{R}^d \to G \subset \mathrm{Aut}(T)$:

$\mathbb{R}^d \times \mathbb{R}^d \ni (x,y) \;\mapsto\; \big(T \ni \tau \mapsto \Gamma_{xy}\tau \in T\big)$.

"Γ_{xy} expresses change of base point $y \rightsquigarrow x$ in the model on the level of the placeholders."

Defining properties: $\Pi_y = \Pi_x \Gamma_{xy}$, $\Gamma_{xz} = \Gamma_{xy}\Gamma_{yz}$.

Postulate:

For $\tau \in T_\alpha$ decompose $T \ni \Gamma_{xy}\tau = \sum_\beta (\Gamma_{xy}\tau)_\beta \in \oplus_\beta T_\beta$

then $|(\Gamma_{xy}\tau)_\beta| \le C|x-y|^{\alpha-\beta}|\tau|$ for $\beta < \alpha$, $|x-y| \le 1$.

Γ-part of model in toy example

$\Gamma\colon \mathbb{R}^d \times \mathbb{R}^d \to G\colon \mathbb{R}^d \times \mathbb{R}^d \times T \ni (x, y, \tau) \mapsto \Gamma_{xy}\tau \in T.$

$$(\Gamma_{xy}\text{-id})X_i = (x\text{-}y)_i\mathbf{1},$$
$$(\Gamma_{xy}\text{-id})X_iX_j = (x\text{-}y)_iX_j + (x\text{-}y)_jX_i + (x\text{-}y)_i(x\text{-}y)_j\mathbf{1},$$

$$(\Gamma_{xy}\text{-id})X_i\Xi = (x\text{-}y)_i\Xi,$$
$$(\Gamma_{xy}\text{-id})X_iX_j\Xi = (x\text{-}y)_iX_j\Xi + (x\text{-}y)_jX_i\Xi + (x\text{-}y)_i(x\text{-}y)_j\Xi,$$

$$(\Gamma_{xy}\text{-id})\mathcal{I}X_i\Xi = (x\text{-}y)_i\mathcal{I}\Xi + (v_{i(x)}\text{-}v_{i(y)})\mathbf{1},$$

$$\begin{aligned}
&(\Gamma_{xy}\text{-id})\mathcal{I}X_iX_j\Xi \\
&= (x\text{-}y)_i\mathcal{I}X_j\Xi + (x\text{-}y)_j\mathcal{I}X_i\Xi + (x\text{-}y)_i(x\text{-}y)_j\mathcal{I}\Xi \\
&+(v_{i(x)}\text{-}v_{i(y)})X_j + (v_{j(x)}\text{-}v_{j(y)})X_i + (\nabla v_{ij(x)}\text{-}\nabla v_{ij(y)}) \cdot X \\
&+\Big((x\text{-}y)_i(v_{j(x)}\text{-}v_{j(y)}) + (x\text{-}y)_j(v_{i(x)}\text{-}v_{i(y)}) \\
&+(v_{ij(x)}\text{-}v_{ij(y)} - \nabla v_{ij}(y) \cdot (x\text{-}y)\Big)\mathbf{1}.
\end{aligned}$$

Back to structure group

Structure group G in the matrix representation w. r. t.
ordered basis Ξ, $X_i\Xi$, $X_iX_j\Xi$, $\mathbf{1}$, X_i, X_iX_j, $\mathcal{I}\Xi$, $\mathcal{I}X_i\Xi$, $\mathcal{I}X_iX_j\Xi$:

$$\Gamma = \begin{pmatrix} 1 & a & a^2 & & & & & & \\ & 1 & 2a & & & & & & \\ & & 1 & & & & & & \\ & & & 1 & a & a^2 & & b & ab + ba + d \\ & & & & 1 & 2a & & & 2b + c \\ & & & & & 1 & & & \\ & & & & & & 1 & a & a^2 \\ & & & & & & & 1 & 2a \\ & & & & & & & & 1 \end{pmatrix}.$$

$a = x\text{-}y$, $b = v(x)\text{-}v(y)$, $c_{ij} = \nabla v_{ij}(x)\text{-}\nabla v_{ij}(y)$,
$d_{ij} = v_{ij}(x)\text{-}v_{ij}(y) - (x\text{-}y) \cdot \nabla v_{ij}(y)$.

Recall group multiplication $\bar{d} = d + d' + a \cdot c'$.

Just what is needed for $\Gamma_{xy}\Gamma_{yz} = \Gamma_{xz}$.

Modelled distribution

$f:\mathbb{R}^d\to T \quad \in \mathcal{D}^\gamma$ provided

$f(x)\in\otimes_{\alpha<\gamma}T_\beta$ and

$|(f(x)-\Gamma_{xy}f(y))_\alpha|\le \|f\|_\gamma|x-y|^{\gamma-\alpha}$ for $\beta<\gamma$, $|x-y|\le 1$.

Example 1: For $\eta\in C_0^3(\mathbb{R}^d)$

$f(x)=\eta(x)\mathbf{1}+\partial_i\eta(x)X_i+\frac{1}{2}\partial_{ij}\eta(x)X_iX_j$ defines $f\in\mathcal{D}^3$.

This follows from $f(x)-\Gamma_{xy}f(y)$

$$=\Big(\eta(x)-\eta(y)-(x\text{-}y)_i\partial_i\eta(y)-\frac{1}{2}(x\text{-}y)_i(x\text{-}y)_j\frac{1}{2}\partial_{ij}\eta(y)\Big)\mathbf{1}$$
$$+\Big(\partial_i\eta(x)-\partial_i\eta(y)-(x\text{-}y)_j\partial_{ij}\eta(y)\Big)X_i+\Big(\partial_{ij}\eta(x)-\partial_{ij}\eta(y)\Big)X_iX_j.$$

Example 2: For $\eta\in C_0^3(\mathbb{R}^d)$

$f(x)=\eta(x)\Xi+\partial_i\eta(x)X_i\Xi+\frac{1}{2}\partial_{ij}\eta(x)X_iX_j\Xi$

defines $f\in\mathcal{D}^{3-\frac{5}{2}=\frac{1}{2}}$.

Modelled distribution, toy example

Recall: $f\colon \mathbb{R}^d \to T \quad \in \mathcal{D}^\gamma$ provided

$f(x) \in \otimes_{\beta<\gamma} T_\beta$ and

$|(f(x) - \Gamma_{xy} f(y))_\beta| \le \|f\|_\gamma |x-y|^{\gamma-\beta}$ for $\beta < \gamma$, $|x-y| \le 1$.

Example 3: For $\eta \in C_0^3(\mathbb{R}^d)$ and $\phi = (_{m^2}-\triangle)^{-1}\eta\xi$

$f(x) = \eta(x)\mathcal{I}\Xi + \partial_i\eta(x)\mathcal{I}X_i\Xi$ "algebra", local

$+(\phi - \eta v)(x)\mathbf{1} + \partial_i\big(\phi - \eta v - \partial_j\eta(v_j\text{-}v_j(x))\big)(x)X_i$ "analysis"

defines $f \in \mathcal{D}^{-\frac{1}{2}+2=\frac{3}{2}}$.

This follows from $f(x) - \Gamma_{xy} f(y)$

$= \big(\eta(x) - \eta(y) - (x\text{-}y)_i \partial_i \eta(y)\big)\mathcal{I}\Xi + \big(\partial_i\eta(x) - \partial_i\eta(y)\big)\mathcal{I}X_i\Xi$

$+\big(w_x(x) - w_x(y) - (x\text{-}y)_j\partial_j w_x(y) - (x\text{-}y)_i(v_j(x)\text{-}v_j(y))\partial_{ij}\eta(y)\big)\mathbf{1}$

$+ \big(\partial_i w_x(x) - \partial_i w_x(y) - (v_j(x)\text{-}v_j(y))\partial_{ij}\eta(y)\big)X_i$,

where $w_x := \phi - \eta v - \partial_i(v_i\text{-}v_i(x)) \in C^{\frac{3}{2}}$.

Reconstruction of distribution given modelled distribution

Theorem 1 (Reconstruction) Let $\gamma > 0$.
For $f \in \mathcal{D}^\gamma$ $\exists ! \, \mathcal{R}f \in \mathcal{S}'(\mathbb{R}^d)$ s. t.
$\int \varphi_x^\lambda(z)\big(\mathcal{R}f - \Pi_x f(x)\big)(dz) = O(\lambda^\gamma)$ for $x \in \mathbb{R}^d$ and $\lambda \in (0,1]$.

Crucial for existence:
Graded continuity of $\mathbb{R}^d \ni x \mapsto F_x := \Pi_x f(x) \in \mathcal{S}'(\mathbb{R}^d)$:
$|\int \varphi_x^\lambda(z)(F_x - F_y)(dz)| \lesssim \sum_{\alpha<\gamma} \lambda^\alpha |x-y|^{\gamma-\alpha}$.

Based on
$F_x - F_y = \Pi_x(f(x)\text{-}\Gamma_{xy}f(y)) = \sum_{\alpha<\gamma} \Pi_x(f(x)\text{-}\Gamma_{xy}f(y))_\alpha$,
$\int \varphi_x^\lambda(z)(F_x - F_y)(dz) = \sum_{\alpha<\gamma} \int \varphi_x^\lambda(z)\Pi_x(f(x)\text{-}\Gamma_{xy}f(y))_\alpha(dz)$,
$|\int \varphi_x^\lambda(z)\Pi_x(f(x)\text{-}\Gamma_{xy}f(y))_\alpha(dz)|$
$\le C\lambda^\alpha|(f(x)\text{-}\Gamma_{xy}f(y))_\alpha| \le C\lambda^\alpha \|f\|_\gamma |x\text{-}y|^{\gamma-\alpha}$.

Short proof of reconstruction

Theorem 1' Given $\mathbb{R}^d \ni x \mapsto F_x \in \mathcal{S}'(\mathbb{R}^d)$ with
$|\int \varphi_x^\lambda(z)(F_x - F_y)(dz)| \leq \sum_{\alpha<\gamma} \lambda^\alpha |x-y|^{\gamma-\alpha}$
for $\gamma > 0$ there exists $\mathcal{R}F \in \mathcal{S}'(\mathbb{R}^d)$ s. t.
$\int \varphi_x^\lambda(z)(\mathcal{R}F - F_x)(dz) = O(\lambda^\gamma)$.

Enough to show latter for φ *Gaussian*; advantage:
semi-group property $\int \varphi_x^{\sqrt{T-t}}(y)\varphi_y^{\sqrt{t}}(z)dy = \varphi_x^{\sqrt{T}}$.

Consider $I_{x,T}^t := \int \varphi_x^{\sqrt{T-t}}(y)\int \varphi_y^{\sqrt{t}}(z)F_y(dz)dy$.

Have $I_{x,T}^{t=T} = \int \varphi_x^{\sqrt{T}}(z)F_x(dz)$.

Goal $\lim_{t\downarrow 0} I_{x,T}^t$ exists $=: \int \varphi_x^{\sqrt{T}}(y)(\mathcal{R}F)(dy)$.

Task $|I_{x,T}^{\frac{t}{2}} - I_{x,T}^t| \lesssim \sqrt{t}^\gamma \overset{\gamma>0}{\Longrightarrow} |\lim_{t\downarrow 0} I_{x,T}^t - I_{x,T}^T| \lesssim \sqrt{T}^\gamma$

Short proof of reconstruction

Have cont. in base-point: $|\int \varphi_x^\lambda(z)(F_x\text{-}F_y)(dz)| \le \sum_{\alpha<\gamma} \lambda^\alpha |x-y|^{\gamma-\alpha}$.

Have semi-group property: $\int \varphi_x^{\sqrt{T-t}}(y)\varphi_y^{\sqrt{t}}(z)dy = \varphi_x^{\sqrt{T}}$.

Consider $I_{x,T}^t := \int \varphi_x^{\sqrt{T-t}} \int \varphi_y^{\sqrt{t}}(z)F_y(dz)dy$.

Task $|I_{x,T}^{t/2} - I_{x,T}^t| \lesssim \sqrt{t}^\gamma$.

Indeed $|I_{x,T}^{t/2} - I_{x,T}^t|$

$$= \left| \int \varphi_x^{\sqrt{T-t}} \int \varphi_y^{\sqrt{t/2}}(y') \int \varphi_{y'}^{\sqrt{t/2}}(z)(F_{y'} - F_y)(dz)dy'dy \right|$$

$$\le \sum_{\alpha<\gamma} \int \varphi_x^{\sqrt{T-t}} \int \varphi_y^{\sqrt{t/2}}(y')(\sqrt{t/2})^\alpha |y'-y|^{\gamma-\alpha} dy'dy$$

$$\lesssim \int \varphi_x^{\sqrt{T-t}}(\sqrt{t/2})^\gamma dy = (\sqrt{t/2})^\gamma.$$

from joint work with Hendrik Weber (arXiv:1605.09744)
and Jonas Sauer, Scott Smith (arXiv:1803.07884)

Integration

$$\text{Affects the three levels} \begin{cases} \text{regularity structure} & (A,T,G) \\ \text{model} & (\Pi,\Gamma) \\ \text{modelled distribution} & \mathcal{D}^\gamma \end{cases}$$

Standing assumption: (A,T,G) and (Π,Γ) contain the polynomials in d variables (up to some order)

On level of regularity structures: $\exists\, \mathcal{I} \in \mathcal{L}(T,T)$ s. t.

$$\mathcal{I}T_\alpha \subset T_{\alpha+2},$$
$$\langle\{1, X_i, \cdots\}\rangle \subset \ker\mathcal{I},$$
$$\mathrm{im}(\Gamma\mathcal{I} - \mathcal{I}\Gamma) \subset \langle\{1, X_i, \cdots\}\rangle \quad \forall\Gamma \in G.$$

Check for toy example.

First two properties are build-in (ignoring $\mathcal{I}^2$); check last.

Abstract integration for toy example

Recall commutator prop.: $\text{im}(\Gamma\mathcal{I} - \mathcal{I}\Gamma) \subset \langle\{1, X_i, \cdots\}\rangle \ \forall\Gamma \in G$.

Recall form of $\Gamma \in G$ on Ξ, $X_i\Xi$, $X_iX_j\Xi$; $\mathcal{I}\Xi$, $\mathcal{I}X_i\Xi$, $\mathcal{I}X_iX_j\Xi$ (parameterized by a_i, b_i, c_{ijk}, $d_{ij} \in \mathbb{R}$):

$(\Gamma\text{-id})\Xi = 0$, $(\Gamma\text{-id})X_i\Xi = a_i\Xi$,

$(\Gamma\text{-id})X_iX_j\Xi = a_iX_j\Xi + a_jX_i\Xi + a_ia_j\Xi$;

$(\Gamma\text{-id})\mathcal{I}\Xi = 0$, $(\Gamma\text{-id})\mathcal{I}X_i\Xi = a_i\Xi + b_i\mathbf{1}$,

$(\Gamma\text{-id})\mathcal{I}X_iX_j\Xi = a_iX_j\Xi + a_jX_i\Xi + a_ia_j\Xi + c_{ijk}X_k + d_{ij}\mathbf{1}$.

Get as desired for commutator

$$\begin{aligned} (\Gamma\text{-id})\mathcal{I}\Xi &- \mathcal{I}(\Gamma\text{-id})\Xi &&= 0, \\ (\Gamma\text{-id})\mathcal{I}X_i\Xi &- \mathcal{I}(\Gamma\text{-id})X_i\Xi &&= b_i\mathbf{1}, \\ (\Gamma\text{-id})\mathcal{I}X_iX_j\Xi &- \mathcal{I}(\Gamma\text{-id})X_iX_j\Xi &&= c_{ijk}X_k + d_{ij}\mathbf{1}. \end{aligned}$$

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Integration on level of model

More convenient to assume that solution operator is given by well-decaying convolution kernel K, e. g. inverse of $(m^2 - \triangle)$

$\exists\ \mathcal{J} : \mathbb{R}^d \to \mathcal{L}(T,T)$ s. t. for $x \in \mathbb{R}^d$, $\tau \in T_\alpha$

$\mathcal{J}(x)\tau \in \langle\{1, \cdots, X^{[\alpha+2]}\}\rangle$ i. e. polynomials of degree $< \alpha + 2$,

$(m^2 - \triangle)\Pi_x(\mathcal{I} + \mathcal{J}(x))\tau = \Pi_x\tau$.

For toy example:

$\mathcal{J}(x)\Xi = 0$, $\mathcal{J}(x)X_i\Xi = v_i(x)\mathbf{1}$,

$\mathcal{J}(x)X_iX_j\Xi = v_{ij}(x)\mathbf{1} + (v_i(x)X_j + v_j(x)X_i + \nabla v_{ij}(x)\cdot X)$.

Integration on level of model for toy example

Recall: $\mathcal{J}(x)\tau \in \langle\{1,\cdots,X^{[\alpha+2]}\}\rangle$, $(m^2-\triangle)\Pi_x(\mathcal{I}+\mathcal{J}(x))\tau = \Pi_x\tau$.

Claim: $\mathcal{J}(x)\Xi = 0$, $\mathcal{J}(x)X_i\Xi = v_i(x)\mathbf{1}$,

$\mathcal{J}(x)X_iX_j\Xi = v_{ij}(x)\mathbf{1} + (v_i(x)X_j + v_j(x)X_i + \nabla v_{ij}(x)\cdot X)$.

Follows from definition of $|\tau|$ and Π_x in form of:

$[|\Xi|+2] = -1 < 0$, $[|X_i\Xi|+2] = 0$, $[|X_iX_j\Xi|+2] = 1$;

$\Pi_x\mathcal{I}\Xi = v \quad vs. \quad \Pi_x\Xi = \xi$;

$\Pi_x\mathcal{I}X_i\Xi = (\cdot\text{-}x)_i v + (v_i\text{-}v_i(x)) \quad vs. \quad \Pi_x X_i\Xi = (\cdot\text{-}x)_i\xi$;

$\Pi_x\mathcal{I}X_iX_j\Xi = (\cdot\text{-}x)_i(\cdot\text{-}x)_j v + (\cdot\text{-}x)_i(v_j\text{-}v_j(x)) + (\cdot\text{-}x)_j(v_i\text{-}v_i(x))$
$+v_{ij} - v_{ij}(x) - (\cdot\text{-}x)\cdot\nabla v_{ij}(x) \quad vs. \quad \Pi_x X_iX_j\Xi = (\cdot\text{-}x)_i(\cdot\text{-}x)_j\xi$.

Recall $(m^2-\triangle)v = \xi$, $(m^2-\triangle)((\cdot\text{-}x)_i v + v) = (\cdot\text{-}x)_i\xi$,

$(m^2-\triangle)((\cdot\text{-}x)_i(\cdot\text{-}x)_j v + (\cdot\text{-}x)_i v_j + (\cdot\text{-}x)_j v_i + v_{ij}) = (\cdot\text{-}x)_i(\cdot\text{-}x)_j\xi$.

Integration on level of model, intertwining

Corollary of postulates: $(\mathcal{I}+\mathcal{J}(x))\Gamma_{xy} = \Gamma_{xy}(\mathcal{I}+\mathcal{J}(y))$.

Since $\Gamma_{xy} \in G$ and thus $\mathrm{im}[\Gamma_{xy}, \mathcal{I}]$, $\mathrm{im}J(x)$, $\mathrm{im}\Gamma_{xy}\langle\{1, X_i, \cdots\}\rangle \subset \langle\{1, X_i, \cdots\}\rangle$

$$(\mathcal{I}+\mathcal{J}(x))\Gamma_{xy}\tau - \Gamma_{xy}(\mathcal{I}+\mathcal{J}(y))\tau \in \langle\{1, X_i, \cdots\}\rangle,$$

so that enough to show $(\Pi_x\Gamma_{xy} = \Pi_y)$

$$0 \stackrel{!}{=} \Pi_x\big((\mathcal{I}+\mathcal{J}(x))\Gamma_{xy}\tau - \Gamma_{xy}(\mathcal{I}+\mathcal{J}(y))\tau\big)$$
$$= \Pi_x\mathcal{I}+\mathcal{J}(x))\Gamma_{xy}\tau - \Pi_y(\mathcal{I}+\mathcal{J}(y))\tau,$$

so that enough to show $((m^2-\triangle)\Pi_x(\mathcal{I}+\mathcal{J}(x))\tau = \Pi_x\tau)$

$$0 \stackrel{!}{=} (m^2-\triangle)\big(\Pi_x(\mathcal{I}+\mathcal{J}(x))\Gamma_{xy}\tau - \Pi_y(\mathcal{I}+\mathcal{J}(y))\tau\big)$$
$$= \Pi_x\Gamma_{xy}\tau - \Pi_y\tau.$$

Integration on level of modelled distribution

For $f \in \mathcal{D}^\gamma$ with $\gamma > 0$ consider $\mathcal{R}f \in \mathcal{S}'(\mathbb{R}^d)$ and

$(m^2 - \triangle)\phi = \mathcal{R}f$ Goal: $\phi = \mathcal{R}\mathcal{K}f$ with $\mathcal{K}f \in \mathcal{D}^{\gamma+2}$.

By definition of $\mathcal{J}$ and $\mathcal{R}$ for every $x \in \mathbb{R}^d$

$(m^2 - \triangle)\big(\phi \text{-} \Pi_x(\mathcal{I} + \mathcal{J}(x))f(x)\big) = \mathcal{R}f \text{-} \Pi_x f(x) = O_{\text{weak}}(|\cdot \text{-} x|^\gamma)$.

PDE argument: $\exists!$ $(\mathcal{N}f)(x)$ s. t.

$(\mathcal{N}f)(x) \in \langle \{1, \cdots, X^{[\gamma+2]}\} \rangle$,

$\phi - \Pi_x\big((\mathcal{I} + \mathcal{J}(x))f(x) + (\mathcal{N}f)(x)\big) = O_{\text{weak}}(|\cdot \text{-} x|^{\gamma+2})$.

In particular $\phi = \mathcal{R}\mathcal{K}f$ where

$(\mathcal{K}f)(x) := (\mathcal{I} + \mathcal{J}(x))f(x) + (\mathcal{N}f)(x)$

Theorem 2 (Integration): For $f \in \mathcal{D}^\gamma$ have $\mathcal{K}f \in \mathcal{D}^{\gamma+2}$.

Integr. on level of modelled distr. , toy example

Recall: $(m^2 - \triangle)\phi = \mathcal{R} f$

$(\mathcal{N} f)(x) \in \langle\{1, \cdots, X^{[\gamma+2]}\}\rangle$,

$\phi - \Pi_x\Big(\mathcal{I}f(x) + \mathcal{J}(x)f(x) + (\mathcal{N}f)(x)\Big) = O_{\mathrm{weak}}(|\cdot\text{-}x|^{\gamma+2})$;

$(\mathcal{K}f)(x) = \mathcal{I}f(x) + \mathcal{J}(x)f(x) + (\mathcal{N}f)(x)$.

In case of $f(x) = \eta(x)\Xi + \partial_i\eta(x)X_i\Xi \in \mathcal{D}^{2-\frac{5}{2}=-\frac{1}{2}}$ get

$$\mathcal{J}(x)f(x) + (\mathcal{N}f)(x)$$

$$= (\phi - \eta v)(x)\mathbf{1} + \partial_i\Big(\phi - \eta v - \partial_k\eta(v_k\text{-}v_{k(x)})\Big)(x)X_i$$

and showed above $\mathcal{K}f \in \mathcal{D}^{-\frac{1}{2}+2=\frac{3}{2}}$.

Proof of integration

Recall $(\mathcal{K}f)(x) = (\mathcal{I}+\mathcal{J}(x))f(x)+(\mathcal{N}f)(x)$.

Intertwining $\Gamma_{xy}(\mathcal{I}+\mathcal{J}(y)) = (\mathcal{I}+\mathcal{J}(x))\Gamma_{xy}$ yields

$$(\mathcal{K}f)(x) - \Gamma_{xy}(\mathcal{K}f)(y)$$
$$= \mathcal{I}(f(x) - \Gamma_{xy}f(y)) + (r_{x,y} \in \langle\{1, X, \cdots\}\rangle).$$

First part ok (tautological): $|(\mathcal{I}(f(x) - \Gamma_{xy}f(y)))_{\alpha+2}|$
$\leq |(f(x) - \Gamma_{xy}f(y))_\alpha| \leq \|f\|_\gamma |x\text{-}y|^{\gamma-\alpha}$.

Remains to show $|(r_{x,y})_k| = O(|x\text{-}y|^{\gamma+2-k})$ for $k = 0, \cdots, [\gamma+2]$.

$\iff \quad \left|\int \varphi_x^\lambda(z)\Pi_x r_{x,y}(dz)\right| = O(\lambda^{\gamma+2})$ for $\lambda = |x\text{-}y|$.

Since $\mathcal{I}(f(x) - \Gamma_{xy}f(y))$ ok

$\iff \quad \left|\int \varphi_x^\lambda(z)\Pi_x\big((\mathcal{K}f)(x) - \Gamma_{xy}(\mathcal{K}f)(y)\big)(dz)\right| = O(\lambda^{\gamma+2})$.

Proof of integration, cont.

Recall that $(\mathcal{N}f)(x)$ was defined

s. t. $(\mathcal{K}f)(x) = (\mathcal{I}+\mathcal{J}(x))f(x) + (\mathcal{N}f)(x)$ satisfies

$$\int \varphi_x^\lambda(z)(\phi - \Pi_x(\mathcal{K}f)(x))(dz) = O(\lambda^{\gamma+2}).$$

Recall task: for $\lambda = |x\text{-}y|$ want

$$\Big|\int \varphi_x^\lambda(z)\Pi_x\big((\mathcal{K}f)(x) - \Gamma_{xy}(\mathcal{K}f)(y)\big)(dz)\Big| = O(\lambda^{\gamma+2}).$$

Ok since $\Pi_x\big((\mathcal{K}f)(x) - \Gamma_{xy}(\mathcal{K}f)(y)\big)$

$= -(\phi\text{-}\Pi_x(\mathcal{K}f)(x)) + (\phi\text{-}\Pi_y(\mathcal{K}f)(y))$ by $\Pi_x\Gamma_{xy} = \Pi_y$

and since $\lambda = |x\text{-}y|$ we have $\varphi_x^\lambda = \tilde{\varphi}_y^\lambda$.