Beyond the van Hove Time Scale

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- This is joint work with **Jacob S. Møller (U Aarhus)** and **Matthias Westrich (Allianz Stuttgart)**.
- We consider an L-level atom, $L \geq 2$, coupled to a quantized scalar field.
- $L = 2$ is the Spin-Boson model, the simplest nontrivial model of matter interacting with quantized fields, work horse of quantum optics (lasers, qubits,...)
- The Hilbert space of the model is

$$\mathcal{H} = \mathbb{C}^L \otimes \mathcal{F}, \quad \mathcal{F} = \mathcal{F}_b[L^2(\mathbb{R}^3)].$$

- The Hamiltonian $H_g = H_g^*$ is

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$$\begin{aligned} H_0 &= H_{\mathrm{at}} \otimes \mathbf{1}_{\mathrm{ph}} + \mathbf{1}_{\mathrm{at}} \otimes H_{\mathrm{ph}}, \\ H_{\mathrm{at}} &= \mathrm{diag}\left[E_0^{(1)} < E_0^{(2)} < \ldots < E_0^{(L)}\right], \\ H_{\mathrm{ph}} &= \int |k|\, a^*(k)\, a(k)\, d^3k, \\ W &= M \otimes \left\{a^*(G) + a(G)\right\}. \end{aligned}$$

- The atomic interaction matrix $M = M^* \in \mathbb{C}^{L \times L}$ has zero diagonal, $M_{\ell,\ell} = 0$ (no self-interactions).
- The coupling function G is UV- and IR-regular and dilation analytic, e.g.,

$$G(k) = \frac{e^{-k^2/\Lambda^2}}{|k|^{\frac{1}{2}-\mu}}, \quad \text{for some } \mu > 0.$$

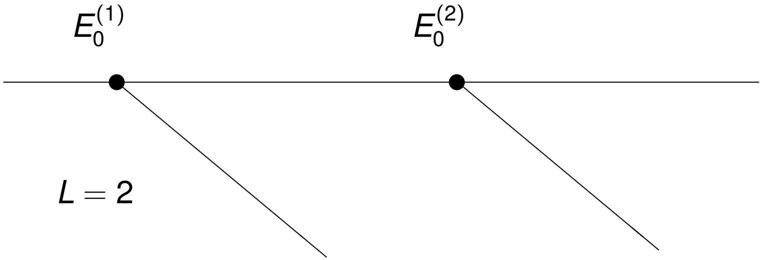
- Solution $\psi_t = e^{-itH_g}\psi_0$ of the Schrödinger Eq. is given by propagator $\exp[-itH_g]$.
- Here study semigroup generated by dilated (non-s.a.) Hamiltonian $\exp[-itH_g(\vartheta)]$, with dilation parameter $0 < \vartheta \ll 1$ and

$$H_g(\vartheta) = H_{\mathrm{at}} \otimes \mathbf{1}_{\mathrm{ph}} + \mathbf{1}_{\mathrm{at}} \otimes e^{-i\vartheta} H_{\mathrm{ph}} + gM \otimes \big\{ a^*(G_\vartheta) + a(G_{-\vartheta}) \big\}.$$

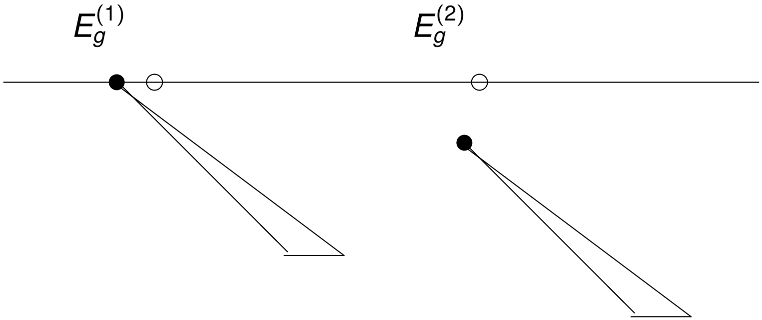
- Spectral analysis of the uncoupled system, $g = 0$:

$$\sigma[H_{\mathrm{at}}] = \{E_0^{(1)} < E_0^{(2)} < \ldots < E_0^{(L)}\}, \quad \sigma[H_{\mathrm{ph}}] = \mathbb{R}_0^+$$

$$\Rightarrow \sigma[H_0(\vartheta)] = \sigma[H_{\mathrm{at}}] + e^{-i\vartheta}\sigma[H_{\mathrm{ph}}] = \bigcup_{\ell=1}^{L} \Big(E_0^{(\ell)} + e^{-i\vartheta}\mathbb{R}_0^+ \Big).$$


$E_0^{(1)}$
$E_0^{(2)}$
$L = 2$

Spectrum of $H_g(\vartheta)$, $g > 0$



Spectral analysis of the interacting system, $g > 0$, $\mu > 0$: [B+Fröhlich+Sigal 98] ($\mu = 0$: [Reker 18]):

$$\sigma[H_g(\vartheta)] \;\subseteq\; \bigcup_{\ell=1}^{L} \Big(E_g^{(\ell)} + e^{-i\vartheta}\mathcal{C} \Big) ,$$

$$E_g^{(\ell)} \in \sigma_{\mathrm{pp}}[H_g(\vartheta)]$$

$$\forall\, \ell \geq 1 : \;\; E_g^{(\ell)} = E_0^{(\ell)}(0) + \mathcal{O}(g^2) ,$$

$$\forall\, \ell \geq 2 : \;\; \mathrm{Im}\, \big\{ E_g^{(\ell)} \big\} = -\Gamma^{(\ell)} g^2 + \mathcal{O}(g^3) , \quad E_g^{(1)} \in \mathbb{R} ,$$

$$\mathcal{C} \;:=\; \big\{ r(1+ib) \in \mathbb{C} \big| r \geq 0,\; |b| < g^{\mu} \big\}$$

The full propagator $e^{-itH_g(\vartheta)}$ is approximated by $\sum_{\ell=1}^{L} U^{(\ell)}(t)$, where

$$e^{-itH_g(\vartheta)} = \frac{-1}{2\pi i} \int_{\mathbb{R}+i\varepsilon} \frac{e^{-itz}\,dz}{H_g(\vartheta) - z}$$

$$\forall\,\ell: \quad U^{(\ell)}(t) = \frac{-1}{2\pi i} \int_{\gamma^{(\ell)}} \frac{e^{-itz}\,dz}{H_g(\vartheta) - z}, \quad \gamma^{(\ell)} \subset D(E_0^{(\ell)}, \delta),$$

Lemma [B+Møller+Westrich 18]: $\exists\, 0 < c < C < \infty$:

$$\Big\| e^{-itH_g(\vartheta)} - \sum_{\ell=1}^{L} U^{(\ell)}(t) \Big\| \leq \frac{C\,L}{\vartheta} \exp\big[-c\,\vartheta^3\, t \big].$$

Theorem [B+Møller+Westrich 18]:

$\exists\, c, p > 0,\ C < \infty \quad \forall\, \ell \in \{1, 2, \ldots, L\},\ \vartheta \ll 1$

$\exists$ bdd. ops. $A^{(\ell)}$, $B^{(\ell)} = P_{\mathrm{at}}^{(\ell)} + \mathcal{O}(g)$ and

fns. $T^{(\ell)} \in C^1(\mathbb{R}_0^+)$, $T^{(\ell)}[r] \sim r$, as $r \to 0$, $\quad \forall\, t > 1$:

$$\Big\| U^{(\ell)}(t) \ - \ A^{(\ell)} \exp\big(-it(E_g^{(\ell)} + e^{-i\vartheta} T^{(\ell)}[H_{\mathrm{ph}}]\big) B^{(\ell)} \Big\| \leq \frac{C}{\vartheta}\, e^{t\,\mathrm{Im}\, E_g^{(\ell)}}\ t^{-p},$$

and

$$\Big\| A^{(\ell)} \exp\big[-it(E_g^{(\ell)} + e^{-i\vartheta} T^{(\ell)}[H_{\mathrm{ph}}]\, B^{(\ell)} \Big\| \ \geq \ c\ e^{t\,\mathrm{Im}\, E_g^{(\ell)}}\ .$$

- Statement of Thm. is well-known for $t \sim g^{-2} \ln[g^{-1}]$, e.g. [B+Fröhlich+Sigal 99, Mück 00, Hasler+Herbst+Huber 08].
- Statement of Thm. has sharper version for isolated resonances of ops. [Hunziker 90, Klein+Rama+Wüst 07]
- Statement of Thm. for non-isolated resonances of ops. [Klein+Rama 10] with Gevrey-type decay,

$$\Big\| U^{(\ell)}(t) \; - \; A^{(\ell)} \exp\big(- it(E_g^{(\ell)} + e^{-i\vartheta} T^{(\ell)}[H_{\mathrm{ph}}]\big) B^{(\ell)} \Big\| \le \; C \, g^2 \, e^{-ct^{1-\varepsilon}}$$

- Proof uses RG based on Smooth Feshbach-Schur map [B+Chen+Fröhlich+Sigal 03].

Note that the relative error tends to 0,

$$\frac{\left\| U^{(\ell)}(t) \ - \ A^{(\ell)} \, e^{-it(E_g^{(\ell)} + e^{-i\vartheta} T^{(\ell)}[H_{\mathrm{ph}}])} \, B^{(\ell)} \right\|}{\left\| A^{(\ell)} \, e^{-it(E_g^{(\ell)} + e^{-i\vartheta} T^{(\ell)}[H_{\mathrm{ph}}])} \, B^{(\ell)} \right\|} \ = \ \mathcal{O}\big(t^{-p}\big),$$

as $t \to \infty$.

The estimates leading to this error bound are delicate and require to make a time-dependent choice $\rho \equiv \rho(t)$ of the scale step parameter $0 < \rho \leq \frac{1}{32}$ in the Feshbach-Schur RG such that $\rho \cdot t \in \mathbb{N}$ is a integer.