BV functions in Hilbert spaces.

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Introduction

Let H be a separabile Hilbert space and ν a Borel probability measure on H. Our main assumption is the following

Hypothesis 1 (DP–Debussche).

There exists $R \in L(H)$ symmetric and positive such that for all $p > 1$ there is $C_p > 0$ such that

$$\left| \int_H \langle RD\varphi(x), z \rangle \, \nu(dx) \right| \le C_p \, |z| \, \|\varphi\|_{L^p(H,\nu)}, \; \forall \, \varphi \in C_b^1(H), \; \forall \, z \in H.$$

Hypothesis 1 is clearly fulfilled when $\nu = N_Q$ is Gaussian with $R = Q^{1/2}$.

Moreover, has been proved by DP-Debussche when:

• ν is the invariant measure of the Burgers equation, Ann. Poincaré 2016

• ν is the invariant measure for a family of Reaction–Diffusion equations. J. Ev. Equ, 2017

• ν is the invariant measure of the Φ_2^4 model. Paper in preparation

Closability of M

Under Hypothesis 1 it is not difficult to show that $M := RD$ is closable in $L^p(H,\nu)$ for all $p \in (1,\infty)$.

We denote by M_p the closure of M, by $W^{1,p}(H,\nu)$ its domain, by M_p^* its adjoint. By definition

$$\int_H \langle M_p u, F\rangle \, d\nu = \int_H u \, M_p^*(F)\, d\nu, \quad \forall\, u \in W^{1,p}(H,\nu),\; F \in D(M_p^*).$$

When no confusion can arise, we shall omit the sub–index p.

M is the natural generalisation of the Malliavin derivative and M^* of the Gaussian divergence, or Skorokhod integral, for the probability measure ν.

BV functions with respect to ν

Let ν be a Borel probability measure on H, $p > 1$ and $u \in L^p(H,\nu)$. We say that u belongs to $BV(H,\nu)$ if there exists a vector measure $\boldsymbol{m} : \mathscr{B}(H) \to H$ with finite total variation such that,

$$\int_H u\, M^*(F)\, d\nu = \int_H \langle F, d\boldsymbol{m}\rangle, \quad \forall\, F \in \mathscr{E},$$

where $\mathscr{E}$ is the following space of test functions

$$\mathscr{E} := \{F = \textstyle\sum_{i=1}^n u_i\, z_i,\ n \in \mathbb{N},\ u_i \in \mathscr{F}C_b^1(H),\ z_i \in H\}$$

and $\mathscr{F}C_b^1(H)$ is the subset of $C_b^1(H)$ of functions depending only in a finite number of variables.

Perimeters

We say that a Borel subset A of H has a finite perimeter with respect to ν if $\mathbb{1}_A \in BV(H,\nu)$.

In this case there is a vector measure $\boldsymbol{m}_A$ with finite total variation such that

$$\int_A M^*(F)\,d\nu = \int_H \langle F, d\boldsymbol{m}_A\rangle, \quad \forall\, F \in \mathscr{E}.$$

$\boldsymbol{m}_A$ is called the perimeter measure of A and its total variation

$$|\boldsymbol{m}_A|(H)$$

the perimeter of A.

Outline

(i) In the first part of the talk we shall show, following

DP–Lunardi, ArXiv 2018,

that a function $u \in L^p(H, \nu)$ (for some $p > 1$) belongs to $BV(H, \nu)$ if and only if there is $K > 0$ such that

$$\left| \int_H u\, M^*(F)\, d\nu \right| \le K \|F\|_\infty, \quad \forall\, F \in \mathscr{E}. \qquad (BV)$$

where $\mathscr{E}$ is the space of test functions introduced before.

$$\mathscr{E} := \left\{ F = \textstyle\sum_{i=1}^n u_i\, z_i,\ n \in \mathbb{N},\ u_i \in \mathscr{F}C_b^1(H),\ z_i \in H \right\}.$$

(ii) The second part will be devoted to examples of sets of finite perimeters. We shall concentrate on the sub–level sets of functions $g : H \to \mathbb{R}$.

A particular attention will be payed to the special function

$$g : H \to (-\infty, 0], \quad g = \inf_{t\in[0,1]} \xi(t),$$

where

$$d\xi = b(\xi)dt + dB_t, \quad \xi(0) = 0,$$

which arises in reflexion problems.

This fact is well know when $b = 0$ (in this case ξ reduces to the Brownian motion) thanks to the reflexion principle. The general case seems to be open.

Some comments about literature

Functions of bounded variation in abstract Wiener spaces, have been first studied by M. Fukushima and M. Hino in JFA-2000 by using Dirichlet forms.

Then L. Ambrosio, S. Maniglia, M. Miranda and D. Pallara see JFA-2010, have improved these results (always in abstract Wiener spaces) using purely analytical tools, in particular, disintegration of Gaussian measures.

The tool of disintegration seems difficult to handle for non Gaussian measures. We were able instead to generalise the Riesz theorem in H, thanks to a suitable integration by parts formula which is a consequence of Hypothesis 1.

For abstract results for a function ν possessing the Fomin derivative in enough many directions see V. Bogachev, AMS 2010 and his school.

Summary on vector measures

A vector measure $\boldsymbol{m}$ in H is a σ–additive mapping

$$\boldsymbol{m} : \mathscr{B}(H) \to H, \quad B \mapsto \mu(B).$$

We define the total variation of $\boldsymbol{m}$ setting

$$|\boldsymbol{m}|(I) := \sup_{(I_k)\subset\Pi_I} \sum_{k=1}^{\infty} |m(I_k)|, \quad \forall\, I \in \mathscr{B}(H),$$

where Π_I denotes the set of all decompositions of I.

One can show that $|\boldsymbol{m}|$ is a positive measure (not necessarily finite). If $|\boldsymbol{m}|(H) < \infty$ we say that $\boldsymbol{m}$ has a finite total variation and we call $|\boldsymbol{m}|(H)$ the total variation of $\boldsymbol{m}$.

We shall denote by $\mathscr{M}(H, H)$ the set of all vector measures with finite total variation. Here is a simple example.

Example

Let ν be a scalar Borel measure and $F \in L^1(H, \nu; H)$. Then

$$\boldsymbol{m}(dx) := F(x)\,\nu(dx)$$

is a vector measure whose total variation is

$$|\boldsymbol{m}|(H) = \int_H \|F(x)\|\,\nu(dx).$$

Let $\boldsymbol{m} \in \mathscr{M}(H, H)$ and let (e_h) be an orthonormal basis in H. Then we define the projection m_h of $\boldsymbol{m}$ to be the scalar measure

$$m_h(B) = \langle \boldsymbol{m}(B), e_h \rangle, \quad \forall\, B \in \mathscr{B}(H),\ h \in \mathbb{N}.$$

It is easy to see that

$$\boldsymbol{m}(B) = \sum_{h=1}^\infty m_h(B)\, e_h, \quad \forall\, B \in \mathscr{B}(H).$$

Conversely, given a sequence (m_h) of signed measures, we can construct a vector measure

$$\boldsymbol{m}(B) := \sum_{h=1}^\infty \mu_h(B)\, e_h, \quad \forall\, B \in \mathscr{B}(H),$$

provided $\sup\limits_{n\to\infty} |m_n|(H) < \infty$.

A basic integration by parts formula

Proposition

(i) Constant vectors fields belong to $D(M^*)$. *As a consequence, for any* $z \in H$ *there exists* $v_z \in L^p(H,\nu), \forall\, p \in [1,\infty)$, *such that* $M^*(z) = v_z$.

(ii) Moreover, the followig integration by parts formula holds

$$\int_H \langle M\varphi, z\rangle \, d\nu = \int_H v_z \varphi \, d\nu, \quad \forall\, \varphi \in C_b^1(H), \qquad (IBP)$$

v_z coincides with the Fomin derivative of ν in the direction Rz.

Proof

Fix $z \in H$ and consider the constant vector field

$$F(x) = z, \quad \forall\, x \in H.$$

Then by Hypothesis 1, for all $p > 1$ there exists $C_p > 0$ such that

$$\left| \int_H \langle M\varphi(x), F(x)\rangle \, \nu(dx) \right| \le C_p \, |z| \, \|\varphi\|_{L^p(H,\nu)}, \; \forall\, \varphi \in C_b^1(H).$$

This implies $F \in D(M_q^*)$, $q = \frac{p}{p-1}$, and setting $M_q^*(F) = v_z$ we have $v_z \in L^q(H,\nu)$ and (IBP) follows by the arbitrariness of p.

Remark

Replacing φ by $u\,\varphi$ we find the following useful generalisation of (IBP), for all $z \in H$,

$$\int_H u\langle M\varphi, z\rangle\, d\nu = -\int_H \varphi\langle Mu, z\rangle\, d\mu + \int_H v_z\, u\,\varphi\, d\nu, \quad \forall\, u, \varphi \in C_b^1(H). \tag{IBP1}$$

The basic characterisation result

Theorem (DP-Lunardi-ArXiv-2018)

Assume Hypothesis 1 and let $u \in L^p(H,\nu)$ for some $p > 1$. Then $u \in BV(H,\nu)$ if and only if there is $K > 0$ such that

$$\left| \int_H u\, M^*(F)\, d\nu \right| \le K \|F\|_\infty, \quad \forall\, F \in \mathscr{E}. \qquad (BV)$$

where $\mathscr{E}$ is the space of test functions defined before

$$\mathscr{E} := \left\{ F = \textstyle\sum_{i=1}^n u_i\, z_i,\ n \in \mathbb{N},\ u_i \in \mathscr{F}C_b^1(H),\ z_i \in H \right\}.$$

Sketch of the proof. We claim that there exists $\boldsymbol{m} \in \mathscr{M}(H,H)$ such that,

$$\int_H u\, M^*(F)\, d\nu = \int_H \langle F, d\boldsymbol{m}\rangle, \quad \forall\, F \in \mathscr{E}.$$

First we fix an orthonormal basis (e_h) on H and for any $h \in \mathbb{N}$ we show, by a suitable generalisation of Riesz's theorem, the existence of a scalar measure m_h, such that (thanks to (IBP1)) for all $\varphi \in C_b^1(H)$ we have

$$\int_H u\, M^*(\varphi\, e_h)\, d\nu = \int_H u\,(-\langle M\varphi, e_h\rangle + \varphi\, v_{e_h})\, d\nu = \int_H \varphi\, dm_h.$$

Then we shall define $\boldsymbol{m} \in BV(H,\nu)$ as

$$\boldsymbol{m}(B) := \textstyle\sum_{h=1}^n \mu_h(B)\, e_h, \quad \forall\, n \in \mathbb{N},\ B \in \mathscr{B}(H),$$

after proving that $\sup_{n\to\infty} |m_n|(H) < \infty$. ∎

Sets of finite perimeter

A Borel set $A \subset H$ has a finite perimeter if $\mathbb{1}_A \in BV(H,\nu)$. In this case there exists a vector measure $\boldsymbol{m}_A \in \mathscr{M}(H,H)$ such that

$$\int_A M^*(F)\,d\nu = \int_H \langle F, d\boldsymbol{m}_A\rangle, \quad \forall\, F \in \mathscr{E}.$$

By Theorem 1, $\mathbb{1}_A$ belongs to $BV(H,\nu)$ if and only if there exists $K > 0$ such that

$$\int_A M^*(F)\,d\nu \le K\|F\|_\infty, \quad \forall\, F \in \mathscr{E}.$$

When $A = \{g \ge r\}$, where $g : H \to \mathbb{R}$, the following lemma provides an useful expression for

$$\int_{\{g\ge r\}} M^*(F)\,d\nu.$$

Lemma

Assume that $g \in W^{1,2}(H,\nu)$ *and let* $z \in H$. *Then for all* $r \in \mathbb{R}$ *we have*

$$\int_{\{g\geq r\}} M^*(F)\,d\nu = \lim_{\epsilon\to 0} \frac{1}{2\epsilon} \int_{\{r-\epsilon\leq g\leq r+\epsilon\}} \langle Mg, F\rangle\,d\nu, \quad \forall\, F \in \mathscr{E}.$$

Proof

Fix $\epsilon > 0$ and apply the identity

$$\int_H \langle M\varphi, F\rangle \, d\nu = \int_H \varphi \, M^*(F) \, d\nu$$

where $F \in \mathscr{E}$ and $\varphi = \theta_\epsilon(g)$, where

$$\theta_\epsilon(\xi) = \begin{cases} 0 & \text{if } \xi \le r - \epsilon \\ \frac{\xi - r}{\epsilon} & \text{if } \xi \in [r-\epsilon, r+\epsilon] \\ 1 & \text{if } \xi \ge r + \epsilon \end{cases}$$

Then it follows that $\theta_\epsilon(g) \in W^{1,2}(H,\nu)$ and

$$M(\theta_\epsilon(g)) = \theta'_\epsilon(g) Mg.$$

see DP-Lunardi-Tubaro, Trans. AMS 2018,

Since

$$\theta'_\epsilon(\xi) = \begin{cases} 0 & \text{if } \xi \le r-\epsilon \\ \frac{1}{\epsilon} & \text{if } \xi \in (r-\epsilon, r+\epsilon] \\ 0 & \text{if } \xi > r+\epsilon, \end{cases}$$

we have

$$\frac{1}{2\epsilon}\int_{\{r\le g\le r-\epsilon\}} \langle Mg, F\rangle\, \varphi\, d\nu = \int_{\{g\ge r-\epsilon\}} M^*(F)\, d\nu,$$

and as $\epsilon \to 0$

$$\lim_{\epsilon\to 0}\frac{1}{2\epsilon}\int_{\{r-\epsilon\le g\le r-\epsilon\}} \langle Mg, F\rangle\, \varphi\, d\nu = \int_{\{g\ge r\}} M^*(F)\, d\nu.$$

■

Proposition

Assume that $g \in W^{1,2}(H,\nu)$ and $Mg \in L^\infty(H,\nu)$. Then $\{g \ge r\}$ has a finite perimeter for almost all $r \in I$.

Proof. From the lemma we have

$$\int_{\{g\ge r\}} M^*(F)\,d\nu \le \|F\|_\infty\,\|Mg\|_\infty\,\liminf_{\epsilon\to 0}\frac{1}{2\epsilon}(\nu\circ g^{-1})([r-\epsilon,r+\epsilon]).$$

Set

$$H(r)=\nu(\{g\ge r\})=(\nu\circ g^{-1})((-\infty,r]).$$

Note that H is increasing and so, right differentiable for almost all $r\in[-\infty,0)$. For such an r we have

$$\int_{\{g\ge r\}} M^*(F)\,d\nu \le \|F\|_\infty\,D^+H(r)$$

so that $\{g \ge r\}$ has a finite perimeter for almost all $r \in I$. ∎

But it is important to prove that the perimeter is finite for all r.

A typical case is when a surface measure σ_r can be defined with respect to ν. This happens if the following holds

Hypothesis 2

Let $g \in W^{1,2}(H,\nu)$ *and*

$$\frac{Mg}{\|Mg\|^2} \in D(M^*).$$

A similar hypothesis was introduced in Airault–Malliavin, Bull. Sci. Math. 88, when ν is Gaussian.

Let us recall the following result, proved in DP-Lunardi-Tubaro, Trans. AMS 2018.

Theorem

Assume that ν *fulfills Hypotheses 1 and 2. Then for all* $\varphi \in C_b(H) \cup W^{1,2}(H,\mu)$ *and for all* $r \in \mathbb{R}$ *there exists the limit*

$$\int_{\{g=r\}} \varphi \, d\sigma_r^\nu := \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int_{\{r-\epsilon \le g \le r+\epsilon\}} \varphi \, d\nu.$$

As a consequence $\{g \ge r\}$ *has a finite perimeter for all* $r < 0$.

Example

Let ν be the invariant measure of the Burgers, a reaction diffusion equation or of the Φ_2^4 model.

Then balls and half–spaces (and several other sets) have finite perimeters.

See DP-Lunardi-Tubaro, Trans. AMS, 2018.

Hypothesis 2 is too strong for several applications. The next proposition provides a sufficient condition implying that the perimeter of $\{g \geq r\}$ is finite for all $r \in (-\infty, 0)$.

Proposition

Assume that $g \in W^{1,2}(H, \nu)$, $Mg \in L^\infty(H, \nu)$, $\nu \circ g^{-1}$ *is absolutely continuous with respect to the Lebesgue measure in* $(-\infty, 0)$ *and that the density*

$$\frac{d(\nu \circ g^{-1})}{d\lambda}(r) = \rho(r), \quad r \in H,$$

is continuous. Then $\{g \geq r\}$ *has a finite perimeter for all* $r \in (-\infty, 0)$.

Proof. Recall that

$$\int_{\{g\geq r\}} M^*(F)\,d\nu = \lim_{\epsilon\to 0}\frac{1}{2\epsilon}\int_{\{r-\epsilon\leq g\leq r+\epsilon\}}\langle Mg, F\rangle\,d\nu, \quad \forall\, F\in\mathscr{E}.$$

Let $r\in(-\infty,0)$ and $\rho\leq C$ near r. Then

$$\int_{\{g\geq r\}} M^*(F)\,d\nu \leq \frac{1}{2\epsilon}\,\|Mg\|_\infty\,\|F\|_\infty\,\liminf_{\epsilon\to 0}\frac{1}{2\epsilon}\int_{\{r-\epsilon<g\leq r+\epsilon\}} d\nu$$

$$\leq \|Mg\|_\infty\,\|F\|_\infty\,\liminf_{\epsilon\to 0}\frac{1}{\epsilon}\int_{r-\epsilon}^{r+\epsilon}\rho(s)ds \leq \rho(0)\|Mg\|_\infty\,\|F\|_\infty.$$

So, we can apply the last Theorem and the conclusion follows. ∎

Now we consider a one dimensional SDE

$$d\xi = b(\xi)dt + dB(t), \quad \xi(0) = 0,$$

where $b \in C_b^2(\mathbb{R})$, moreover

$$v(\eta) = \int_0^\eta b(r)dr, \quad \eta \in \mathbb{R},$$

is bounded and $B(\cdot)$ is a Brownian motion in $(\Omega, \mathscr{F}, \mathbb{P})$. We denote by ν the law of $\xi(\cdot)$ in $C([0,1])$.

Theorem (BoDaLuTu)

The law of

$$g := \min_{t\in[0,1]} \xi(t)$$

in $(-\infty, 0]$ *is absolutely continuous with respect to the Lebesgue measure with a continuous density.*

Therefore $\{g \geq r\}$ *has a finite perimeter for all* $r \in (-\infty, 0)$.

Proof

Step 1. We show that $\mathbb{P} \circ (\xi(\cdot)^{-1}) << N_Q$ and

$$\frac{d\mathbb{P} \circ (\xi(\cdot)^{-1})}{dN_Q} = \Psi,$$

with Ψ smooth.

In fact, let us consider the Girsanov transform

$$d\mathbb{Q} = \rho(1)d\mathbb{P},$$

where

$$\rho(1) = e^{-\frac{1}{2}\int_0^1 |b(\xi(s))|^2 ds - \int_0^1 b(\xi(s))\, dB(s)}.$$

Then $\xi(\cdot)$ is a Brownian motion in $(\Omega, \mathscr{F}, \mathbb{Q})$.

Therefore $\mathbb{Q} \circ (\xi(\cdot)^{-1}) = N_Q$, and so, for any $F : C([0,1]) \to \mathbb{R}$ bounded and Borel one has

$$\int_\Omega F(\xi(\cdot))\, d\mathbb{Q} = \int_{C([0,1])} F(h)\, N_Q(dh).$$

We note that by Ito's formula applied to

$$v(\eta) = \int_0^\eta b(r)dr, \quad \eta \in \mathbb{R},$$

we have

$$\rho(1) = e^{-v(\xi(1)) + \frac{1}{2} \int_0^1 b^2(\xi(s))ds + \frac{1}{2} \int_0^1 b'(\xi(s))ds}.$$

Since

$$\mathbb{E}_{\mathbb{P}}(F(\xi(\cdot))) = \mathbb{E}_{\mathbb{Q}}(F(\xi(\cdot))\rho(1)^{-1}),$$

we have

$$\int_\Omega F(\xi(\cdot))\, d\mathbb{P} = \int_{C([0,1]} F(h(\cdot))\Psi(h)\, N_Q(dh),$$

where

$$\Psi(h) = e^{v(h(1)) - \frac{1}{2} \int_0^1 b^2(h(s))ds - \frac{1}{2} \int_0^1 b'(h(s))ds}.$$

So, $\mathbb{P} \circ (\xi(\cdot))^{-1} = \Psi\, N_Q$ and Step 1 is proved.

In particular, we have

$$\int_\Omega \alpha(\min_{t\in[0,1]} \xi(\cdot))\, d\mathbb{P} = \int_{C([0,1]} \alpha(\min_{t\in[0,1]} h(\cdot))\Psi(h)\, N_Q(dh),$$

for any $\alpha : \mathbb{R} \to \mathbb{R}$ bounded and Borel.

Claim. $\Psi\, N_Q \circ g^{-1} << \lambda$ and

$$\frac{d(\Psi\, N_Q \circ g^{-1})}{d\lambda}\,(dr) =: \rho_\Psi(r)dr$$

with $\rho_\Psi(r)$ continuous on r. So,

$$\int_\Omega \alpha(\min_{t\in[0,1]} \xi(\cdot))\, d\mathbb{P} = \int_{-\infty}^{0} \alpha(\min_{t\in[0,1]})\rho_\Psi(r)dr,$$

This shows that the law of $\min_{t\in[0,1]} \xi(\cdot)$ is absolutely continuous with respect to the Lebesgue measure λ on $(-\infty, 0)$ with a continuos density proving the theorem.

Finally, the claim follows thanks to some recent results from Florit and Nualart, Stat. Probab Lett. 15.

because, these results make possible to construct a surface integral for $\min_{t\in[0,1]} h(t)$ with respect to N_Q.

See Bonaccorsi, DP and Tubaro, J. Evol. Equ. 18.

Recalling some results on surface integrals

Let X be a separable Banach space, μ a Gaussian measure on X and $g : X \to \mathbb{R}$ Borel. For any $r \in I$ set

$$F_r(\varphi) = \int_{\{g \geq r\}} \varphi \, d\nu, \quad \varphi \in C_b(H).$$

If for all $\varphi \in C_b(H)$ there exists the limit

$$F_r'(\varphi) = \lim_{\epsilon \to 0} \frac{1}{2\epsilon} \int_{r-\epsilon \leq g \leq r+\epsilon} \varphi \, d\nu$$

and a probability measure σ_r^g in H such that

$$F_r'(\varphi) = \int_H \varphi \, \sigma_r^g,$$

we say that σ_r^g is a surface measure with respect to g and ν.

The theory of surface measures in an infinite dimensional space was initiated by Airault and Malliavin, see [AiMa88] for Gaussian measures and developed by several authors.

However, in all these papers some regularity of g are assumed.

In particular that

$$\int_X |Mg|^{-p} d\nu < \infty.$$

This assumption is not fulfilled in the case under consideration

$$g = \min_{t\in[0,1]} \xi(t)$$

The following weaker assumption was given in Nualart [Nu06]

Hypothesis 2

Given $g : X \to I$ belonging to the domain of the Malliavin derivative, there exist two random variables $u : X \to X$ and $\gamma : X \to \mathbb{R}$ such that

$$\langle Mg(x), u(x)\rangle = \gamma(x), \quad \forall\, x \in g^{-1}(I)$$

and

$$\frac{u}{\gamma} \in D(M_p^*) \quad \forall\, p \geq 1,$$

where M_p is the Malliavin derivative and M_p^* its dual.

Remark

This assumption is fulfilled when

$$g = \inf_{t \in [0,1]} B(t),$$

where $B(\cdot)$ is a standard Brownian motion, see

Florit and Nualart, Stat. Probab Lett. 15

and

Bonaccorsi, DP and Tubaro, J. Evol. Equ. 18.

Theorem

Under Hypothesis 2 for every $r \in I$ there exists a unique Borel measure σ_r^g on H Moreover,

$$F'_\varphi(r) = \rho_\varphi(r) = \int_X \varphi(x)\,\sigma_r^g(dx), \quad \forall\,\varphi \in UC_b(X).$$

is continuous and

$$\frac{d(\varphi\,\mu)\circ g^{-1}}{d\lambda}(r) = F'_\varphi(r).$$

See Bonaccorsi–DP–Tubaro, J. Evol. Equ. 2018.