Renormalized quantum BV operator and observables in gauge theories and gravity

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INI, 24.10.2018

Outline of the talk

Algebraic quantum field theory

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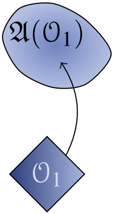

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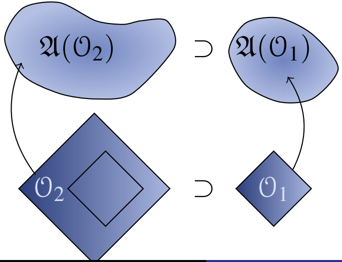
- The physical notion of subsystems is realized by the condition of isotony, i.e.: $\mathcal{O}_1 \subset \mathcal{O}_2 \Rightarrow \mathfrak{A}(\mathcal{O}_1) \subset \mathfrak{A}(\mathcal{O}_2)$. We obtain a net of algebras.



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- **Time-slice axiom:** If $\mathcal{N}$ is a neighborhood of a Cauchy-surface in $\mathcal{O}$, then $\mathfrak{A}(\mathcal{N})$ is isomorphic to $\mathfrak{A}(\mathcal{O})$.
- This is a QFT version of the initial value problem (or local constancy in the time direction).

Generalizations

- Replace $\mathbb{M}$ with an arbitrary Lorentzian gloabally hyperbolic (has a Cauchy surface) manifold (M, g): locally covariant QFT on curved spacetimes ([Brunetti-Fredenhagen-Verch 03, Hollands-Wald 01, Fewster-Verch 12].

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- We can also follow the spirit of AQFT in perturbation theory,
- pAQFT is a mathematically rigorous framework that can be used to make precise calculations done in perturbative QFT.

Mathematical methods and results

- Free theory obtained by the formal deformation quantization of the Poisson (Peierls) bracket: $\star$-product ([Dütsch-Fredenhagen 00, Brunetti-Fredenhagen 00, Brunetti-Dütsch-Fredenhagen 09, …]).

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- Thermal states are treated using the results of [Fredenhagen-Lindner 2014, Drago-Hack-Pinamonti 2016].

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- Dynamics: we use a modification of the Lagrangian formalism (fully covariant).

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- A functional is regular, $F \in \mathcal{F}_{\mathrm{reg}}$ if $F^{(n)}(\varphi)$ is as smooth section (in general it would be distributional).

Dynamics

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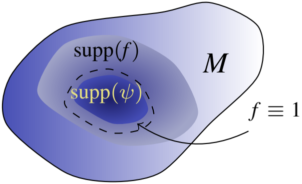

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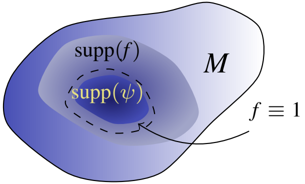
- The field equation is: $dS(\varphi) = 0$, so geometrically, the solution space is the zero locus of the 1-form dS.



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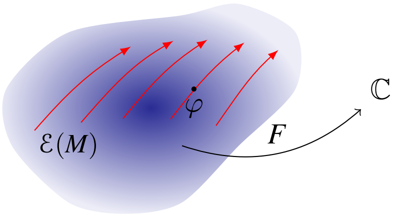

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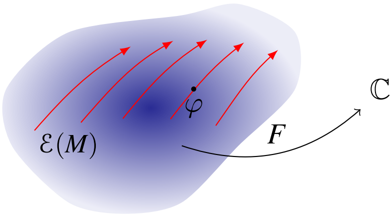

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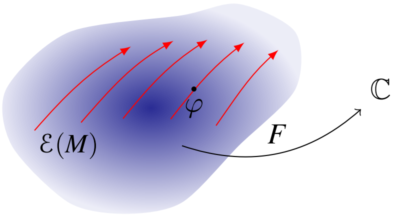

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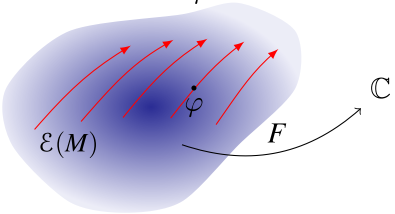
- A symmetry of S is a direction in $\mathcal{E}$ in which the action is constant, i.e. it is a vector field $X \in \mathcal{V}$ such that $\forall \varphi \in \mathcal{E}$: $0 = \langle dS(\varphi), X(\varphi)\rangle$.



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- BV allows us to do this in the classical theory and in quantization we can use standard methods that worked for the scalar field.
- We keep track of where things belong using grading.

Quotients and homology

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- Denote functionals that vanish on $\mathcal{E}_S$ by $\mathcal{F}_0$ and assume that they are of the form: $\delta_S(X)$ for some $X \in \mathcal{V}$.

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- We obtain a sequence:

$$\begin{array}{ccccccccc} 0 & \rightarrow & Sym & \hookrightarrow & \mathcal{V} & \xrightarrow{\delta_S} & \mathcal{F} & \rightarrow & 0 \\ & & 2 & & 1 & & 0 & & \end{array}$$

where the numbers below mean grading and they help to keep track of where things belong.

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- If $F \in \mathcal{F}^{\mathrm{inv}}$ then $\gamma F \equiv 0$, so the $H^0(\gamma)$ characterizes the gauge invariant functionals.

BV complex

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- This way we obtain the BV complex: $\mathcal{B}\mathcal{V}(M)$. Its underlying algebra is the algebra of multivector fileds on $\overline{\mathcal{E}}$.
- $\mathcal{B}\mathcal{V}$ is equipped with the BV differential, which in simple cases is just $s = \delta + \gamma$ (in general, more work needed).
- We have $H^0(s) = H^0(H_0(\delta), \gamma) = \mathcal{F}_S^{\mathrm{inv}}$, which is the reason to work with $\mathcal{B}\mathcal{V}$ as it contains the same information as $\mathcal{F}_S^{\mathrm{inv}}$, but has a simpler algebraic structure (quotients and spaces of orbits are resolved).

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- The BV differential s has to be nilpotent, i.e.: $s^2 = 0$, which leads to the classical master equation (CME):

$$\{S^{\text{ext}}(f), S^{\text{ext}}(f)\} = 0\,,$$

modulo terms that vanish in the limit of constant f.

Poisson structure and the $\star$-product

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$$\{F, G\} \doteq \left\langle F^{(1)}, \Delta G^{(1)} \right\rangle ,$$

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where $\Delta = \Delta^{\mathrm{R}} - \Delta^{\mathrm{A}}$ is the Pauli-Jordan function for the $\#\mathrm{af} = 0$ part of S_0.
- Define the $\star$-product (deformation of the pointwise product):

$$(F \star G)(\varphi) \doteq \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} \left\langle F^{(n)}(\varphi), W^{\otimes n} G^{(n)}(\varphi) \right\rangle ,$$

where W is the 2-point function of a Hadamard state and it differs from $\frac{i}{2}\Delta$ by a symmetric bidistribution: $W = \frac{i}{2}\Delta + H$.

Interlude: free scalar field example

- **Smeared fields**: Let $\mathcal{D}(M) = \mathcal{C}^\infty_c(M,\mathbb{R})$ and $f, f' \in \mathcal{D}(M)$.

$$\Phi(f)[\varphi] \doteq \int f(x)\varphi(x)d\mu_g(x), \quad \Phi(f')[\varphi] \doteq \int f'(x)\varphi(x)d\mu_g(x)$$

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- Formally, we can consider $\Phi_x \doteq \Phi(\delta_x)$, where δ_x is the Dirac delta supported at some $x \in M$.
- for $M = \mathbb{M}$ (Minkowski spacetime):
  $[\Phi_{(0,\mathbf{x})}, \Phi_{(0,\mathbf{y})}]_\star = \Delta(0, \mathbf{x}; 0, \mathbf{y}) = 0.$
  $[\Phi_{(0,\mathbf{x})}, \dot{\Phi}_{(0,\mathbf{y})}]_\star = \partial_{y^0}\Delta(0, \mathbf{x}; 0, \mathbf{y}) = i\hbar\delta(\mathbf{x} - \mathbf{y})$, where dot denotes the time derivative.

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$$\mathcal{T}F(\varphi) \doteq \sum_{n=0}^{\infty} \frac{1}{n!} \left\langle F^{(2n)}(\varphi), (\tfrac{\hbar}{2}\Delta^{\mathrm{F}})^{\otimes n} \right\rangle ,$$

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- Define the time-ordered product $\cdot_{\mathcal{T}}$ on $\mathcal{F}_{\text{reg}}[[\hbar]]$ by:

$$F \cdot_{\mathcal{T}} G \doteq \mathcal{T}(\mathcal{T}^{-1}F \cdot \mathcal{T}^{-1}G)$$

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- We define the interacting star product as:
$$F \star_{int} G \doteq R_V^{-1}\left(R_V(F) \star R_V(G)\right)\,,$$

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- $\cdot_{\mathcal{T}}$ is the time-ordered version of $\star$, in the sense that

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if the support of F is later than the support of G.

- Interaction is a functional V, for the moment $V \in \mathcal{F}_{\mathrm{reg}}$.
- We define the formal S-matrix, $\mathcal{S}(\lambda V) \in \mathcal{F}_{\mathrm{reg}}((\hbar))[[\lambda]]$ by

$$\mathcal{S}(\lambda V) \doteq e_{\mathcal{T}}^{i\lambda V/\hbar} = \mathcal{T}(e^{\mathcal{T}^{-1}(i\lambda V/\hbar)})\,.$$

- Interacting fields are elements of $\mathcal{F}_{\mathrm{reg}}[[\hbar, \lambda]]$ given by

$$R_{\lambda V}(F) \doteq (e_{\mathcal{T}}^{i\lambda V/\hbar})^{\star -1} \star (e_{\mathcal{T}}^{i\lambda V/\hbar} \cdot_{\mathcal{T}} F) = -i\hbar \frac{d}{d\mu} \mathcal{S}(\lambda V)^{-1} \mathcal{S}(\lambda V + \mu F)\Big|_{\mu=0}$$

- We define the interacting star product as:

$$F \star_{int} G \doteq R_V^{-1}\left(R_V(F) \star R_V(G)\right)\,,$$

- Renormalization problem: extend $\cdot_{\mathcal{T}}$ to V local and non-linear.

QME on regular functionals

- The quantum master equation is the condition that the S-matrix is invariant under the quantum Koszul operator:

$$\{e_{\mathcal{T}}^{iV/\hbar}, S_0\}_\star = 0\,,$$

where $\{.,.\}_\star$ is the antibracket where the pointwise product is replaced by $\star$.

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- This should be understood as a condition on V, which guarantees that the S-matrix on-shell doesn't depend on the gauge fixing.

Quantum BV operator I

- The linearized BV operator is defined by

$$\hat{s}_0 X = \{X, S_0\}_\star \,.$$

Under appropriate conditions on the 2-point function W, $\hat{s}_0 = s_0$.

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- The 0th cohomology of $\hat{s}$ characterizes quantum gauge invariant observables.

Quantum BV operator II

- Assuming QME, $\hat{s}X = e_{\mathcal{T}}^{-iV/\hbar} \cdot_{\mathcal{T}} \left(\{ e_{\mathcal{T}}^{iV/\hbar} \cdot_{\mathcal{T}} X, S_0 \}_\star \right)$.

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- $\hat{s}$ on regular functionals can also be written as:

$$\hat{s} = \{., S + V\}_{\mathcal{T}} - i\hbar\triangle ,$$

where $\triangle$ is the BV Laplacian, which on regular functionals is

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- In our framework this is a mathematically rigorous result, no path integral needed (in contrast to other approaches).

Towards renormalization

To extend QME and $\hat{s}$ to local observables, we need to replace $\cdot_{\mathcal{T}}$ with the renormalized time-ordered product.

Theorem (K. Fredenhagen, K.R. 2011)

The renormalized time-ordered product $\cdot_{\mathcal{T}_\mathrm{r}}$ is an associative product on $\mathcal{T}_\mathrm{r}(\mathcal{F})$ given by

$$F \cdot_{\mathcal{T}_\mathrm{r}} G \doteq \mathcal{T}_\mathrm{r}(\mathcal{T}_\mathrm{r}^{-1}F \cdot \mathcal{T}_\mathrm{r}^{-1}G)\,,$$

where $\mathcal{T}_\mathrm{r} : \mathcal{F}[[\hbar]] \to \mathcal{T}_\mathrm{r}(\mathcal{F})[[\hbar]]$ is defined as

$$\mathcal{T}_\mathrm{r} = (\oplus_n \mathcal{T}_\mathrm{r}^n) \circ \beta\,,$$

where $\beta : \mathcal{T}_\mathrm{r} : \mathcal{F} \to S^\bullet \mathcal{F}_{\mathrm{loc}}^{(0)}$ is the inverse of multiplication m.

Renormalized QME and the quantum BV operator

- Since $\cdot_{\mathcal{T}_r}$ is an associative, commutative product, we can use it in place of $\cdot_{\mathcal{T}}$ and define the renormalized QME and the quantum BV operator as:

$$\{e_{\mathcal{T}_r}^{iV/\hbar}, S_0\}_\star = 0$$

$$\hat{s}(X) \doteq e_{\mathcal{T}_r}^{-iV/\hbar} \cdot_{\mathcal{T}_r} \left(\{e_{\mathcal{T}_r}^{iV/\hbar} \cdot_{\mathcal{T}_r} X, S_0\}_\star\right) \, ,$$

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- These formulas get even simpler if we use the anomalous Master Ward Identity ([Brenecke-Dütsch 08, Hollands 07]).

Renormalized QME and the quantum BV operator

- Using the MWI we obtain following formulas:

$$0 = \frac{1}{2}\{V + S_0, V + S_0\}_{\mathcal{T}_r} - \triangle_V \,,$$
$$\hat{s}X = \{X, V + S_0\} - \triangle_V(X)\,,$$

where $\triangle_V$ is identified with the anomaly term and $\triangle_V(X) \doteq \frac{d}{d\lambda} \triangle_{V+\lambda X}\big|_{\lambda=0}$.

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- Hence, by using the renormalized time ordered product $\cdot_{\mathcal{T}_r}$, we obtained in place of $\triangle(X)$, the interaction-dependent operator $\triangle_V(X)$ (the anomaly). It is of order $\mathcal{O}(\hbar)$ and local.
- In the renormalized theory, $\triangle_V$ is well-defined on local vector fields, in contrast to $\triangle$.

Conclusions

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- The renormalized QME and the quantum BV operator are defined in a natural way and don't suffer from divergent terms,
- Example applications: Yang-Mills theories, bosonic string, perturbative quantum gravity.

Thank you for your attention!