A scaling limit from Euler to Navier-Stokes equations with random perturbation

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Subject of the talk

The talk deals with a special scaling limit connecting 2D Euler and Navier-Stokes equations, both with noise, but of different type:

$$\partial_t \omega + u \cdot \nabla \omega = \xi \circ \nabla \omega \qquad \text{Euler, multiplicative noise}$$

$$\partial_t \omega + u \cdot \nabla \omega = \Delta \omega + \eta \qquad \text{Navier-Stokes, additive noise}$$

where

$$\begin{aligned} u &= \text{velocity, } \operatorname{div} u = 0 \\ \omega &= \nabla^{\perp} u = \text{vorticity} \\ \xi, \eta &= \text{space-dependent noise.} \end{aligned}$$

Overview of some directions in stochastic fluid dynamics

Due to the interdisciplinary character of the meeting, let me spend a few introductory words on stochastic fluid dynamics.

- The most studied equation is the stochastic Navier-Stokes equation with additive noise:

$$\partial_t u + u \cdot \nabla u + \nabla p = \nu \Delta u + \dot{W}$$

- From the time of Kolmogorov it is indicated as a potential model to explain some feature of turbulence.
- For instance, the formal average energy balance

$$\frac{1}{2}\mathbb{E}\int |u_t|^2\, dx + \nu \int_0^t \mathbb{E}\int |\nabla u_s|^2\, dxds = \frac{1}{2}\mathbb{E}\int |u_0|^2\, dx + t \cdot Trace\,(Q)$$

  put the basis for investigating dissipation for small ν: energy input is under control, and dissipation is constant.
- Much work has been devoted to the attempt to prove well posedness in 3D, outstanding open problem.

Overview of some directions in stochastic fluid dynamics

- For Euler equation, a more natural noise is of transport type

$$\partial_t u + u \cdot \nabla u + \nabla p = \xi \circ \nabla u.$$

  It preserves certain physical quantities (energy, in the form above, enstrophy in the 2D model $\partial_t \omega + u \cdot \nabla \omega = \xi \circ \nabla \omega$).
- Intuitively, it may correspond to the Lagrangian motion of small scales, acting on larger scales.
- Similarly to additive noise for 3D Navier-Stokes equation, an open problem is whether this noise "regularizes", namely it produces better results of well posedness.
- For instance, it prevents point vortex collision (F.-Gubinelli-Priola '11).

Back to the subject of the talk

Thus the two models

$$\partial_t \omega + u \cdot \nabla \omega = \xi \circ \nabla \omega \qquad \text{Euler, multiplicative noise}$$

$$\partial_t \omega + u \cdot \nabla \omega = \Delta \omega + \eta \qquad \text{Navier-Stokes, additive noise}$$

have some physical and mathematical motivation.
(Here they are written in vorticity form)
The purpose of this talk is to discuss a special scaling limit which connects the two, based on a computation similar in spirit to a renormalization.

Disclaimer

There is an obvious misunderstanding that we have to declare.
In deterministic fluid mechanics, one of the most important open problems is the *vanishing viscosity limit*.
Is it true that the solution ω_ϵ of

$$\partial_t \omega_\epsilon + u_\epsilon \cdot \nabla \omega_\epsilon = \epsilon \Delta \omega_\epsilon \qquad \text{Navier-Stokes}$$

converges to a solution ω of

$$\partial_t \omega + u \cdot \nabla \omega = 0 \qquad \text{Euler}$$

as $\epsilon \to 0$? *This is not the limit investigated in this talk.*

From Euler to Navier-Stokes

Somewhat opposite, we aim to prove that solutions ω_ϵ of

$$\partial_t \omega_N + u_N \cdot \nabla \omega_N = \xi_N \circ \nabla \omega_N \qquad \text{Euler}$$

converge to a solution ω of

$$\partial_t \omega + u \cdot \nabla \omega = \Delta \omega + \eta \qquad \text{Navier-Stokes}$$

as $N \to \infty$. In other words, in a suitable scaling limit, a transport type noise

$$\xi_N \circ \nabla \omega_N$$

gives rise to

$$\Delta \omega + \eta.$$

The stochastic Euler equations

The equations will be considered on the torus $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$. The noise ξ_N in

$$\partial_t \omega_N + u_N \cdot \nabla \omega_N = \xi_N \circ \nabla \omega_N \qquad \text{Euler}$$

has the form

$$\xi_N (x, t) = \epsilon_N \sum_{|k| \leq N} \sigma_k (x) \frac{d\beta_k (t)}{dt}$$

where

$$k \in \mathbb{Z}^2_* = \mathbb{Z}^2 \setminus \{0\}$$

$$\sigma_k (x) : \mathbb{T}^2 \to \mathbb{R}^2 \text{ are divergence free fields}$$

$$\beta_k \text{ are independent scalar BM's.}$$

Coefficients on the noise

$$\begin{aligned} \partial_t \omega_N + u_N \cdot \nabla \omega_N &= \xi_N \circ \nabla \omega_N \\ \xi_N &= \epsilon_N \sum_{|k| \leq N} \sigma_k \frac{d\beta_k}{dt} \end{aligned}$$

We assume $\sigma_k(x) : \mathbb{T}^2 \rightarrow \mathbb{R}^2$, divergence free fields, of the form

$$\sigma_k(x) = \frac{k^\perp}{|k|^2} e_k(x)$$

where $(k_1, k_2)^\perp = (k_2, -k_1)$,

$$e_k(x) = \begin{cases} \sin(2\pi k \cdot x) & \text{for} \quad k \in \mathbb{Z}^2_+ \\ \cos(2\pi k \cdot x) & \text{for} \quad k \in \mathbb{Z}^2_- \end{cases}.$$

The case of infinite dimensional noise

Consider the divergence free noise (infinite series)

$$\xi^{(\gamma)}(x,t) = \sum_{k\in\mathbb{Z}_*^2} \frac{k^\perp}{|k|^\gamma} e_k(x) \frac{d\beta_k(t)}{dt}$$

parametrized by γ.

- The case $\gamma = 1$ is the divergence free analog of *space-time white noise*. Completely untractable as a transport noise.
- The case $\gamma = 2$ is the divergence free analog of *Gaussian free field noise*. It is our case but with truncated series. The infinite series (as a transport noise) is an open problem.
- Only the case $\gamma > 2$ (infinite series) can been treated by standard methods.

Itô-Stratonovich

The threshold $\gamma = 2$ appears when rewriting Stratonovich into Itô:

$$\xi^{(\gamma)} = \sum_{k \in \mathbb{Z}^2_*} \frac{k^\perp}{|k|^\gamma} e_k \frac{d\beta_k}{dt}$$

$$\xi^{(\gamma)} \circ \nabla\omega - \xi^{(\gamma)} \cdot \nabla\omega = \frac{1}{2} \sum_{k \in \mathbb{Z}^2_*} \left(e_k \frac{k^\perp}{|k|^\gamma} \cdot \nabla \right) \left(e_k \frac{k^\perp}{|k|^\gamma} \cdot \nabla\omega \right).$$

and it turns out that the second order differential operator is well defined only for $\gamma > 2$.
Let us see the details.

The Itô-Stratonovich corrector

$$\frac{1}{2}\sum_{k\in\mathbb{Z}^2_*}\left(e_k\frac{k^\perp}{|k|^\gamma}\cdot\nabla\right)\left(e_k\frac{k^\perp}{|k|^\gamma}\cdot\nabla\omega\right)$$

$$\overset{k^\perp\cdot\nabla e_k=0}{=}\frac{1}{2}\sum_{k\in\mathbb{Z}^2_*}e_k^2\left(\frac{k^\perp}{|k|^\gamma}\cdot\nabla\right)\left(\frac{k^\perp}{|k|^\gamma}\cdot\nabla\omega\right)$$

$$\overset{\sin^2+\cos^2=1}{=}\frac{1}{2}\sum_{k\in\mathbb{Z}^2_+}\left(\frac{k^\perp}{|k|^\gamma}\cdot\nabla\right)\left(\frac{k^\perp}{|k|^\gamma}\cdot\nabla\omega\right)$$

$$=\frac{1}{2}\sum_{k\in\mathbb{Z}^2_+}\frac{1}{|k|^{2\gamma}}\sum_{i,j=1}^{2}k_ik_j\partial_i\partial_j\omega$$

$$=\frac{1}{2}\left(\sum_{k\in\mathbb{Z}^2_+}\frac{1}{|k|^{2\gamma-2}}\right)\Delta\omega \qquad \text{finite only for } \gamma>2.$$

The Itô-Stratonovich corrector

- The previous computation shows that, for $\gamma > 2$, the Itô-Stratonovich corrector in $\frac{1}{2}\left(\sum_{k\in\mathbb{Z}^2_+}\frac{1}{|k|^{2\gamma-2}}\right)\Delta\omega$.
- For $\gamma = 2$ the coefficient diverges.
- But, for $\gamma = 2$, the Itô-Stratonovich corrector of the noise

$$\xi_N(x,t) = \epsilon_N \sum_{|k|\leq N} \sigma_k(x)\frac{d\beta_k(t)}{dt}$$

is $\frac{1}{2}\left(\epsilon_N^2\sum_{|k|\leq N}\frac{1}{|k|^2}\right)\Delta\omega$ which converges if we take

$$\epsilon_N \sim 1/\sqrt{\log N}.$$

Summary

For the Euler equation

$$\partial_t \omega_N + u_N \cdot \nabla \omega_N = \xi_N \circ \nabla \omega_N, \qquad \xi_N = \epsilon_N \sum_{|k| \leq N} \sigma_k \frac{d\beta_k}{dt}$$

with $\sigma_k(x) = \frac{k^\perp}{|k|^2} e_k(x)$ and $\epsilon_N \sim 1/\sqrt{\log N}$ we can write

$$\partial_t \omega_N + u_N \cdot \nabla \omega_N = \underbrace{\xi_N \cdot \nabla \omega_N}_{\text{Itô}} + \nu_N \Delta \omega_N$$

with $\nu_N \to \nu \in \mathbb{R}^+$.
In the case $\sigma_k(x) = \frac{k^\perp}{|k|^\gamma} e_k(x)$ with $\gamma > 2$ we get the same result without rescaling with ϵ_N.
What about the martingale term?

The martingale term

The martingale

$$M_N(t,x) := \epsilon_N \sum_{|k|\leq N} \int_0^t \sigma_k(x) \cdot \nabla \omega_N(s,x)\, d\beta_k(s)$$

in general *depends strongly on the solution* $\omega_N(s,x)$ and preserves this dependence in the limit $N \to \infty$.
This dependence spoils the dissipativity of $\Delta\omega_N$: the sum of the martingale and corrector is the Stratonovich term, which is not dissipative.
For $\gamma > 2$, not rescaled by ϵ_N, this remains true in the limit of the infinite dimensional noise: we get the stochastic Euler equation

$$\partial_t \omega + u \cdot \nabla \omega = \xi \circ \nabla \omega, \qquad \xi = \sum_{k \in \mathbb{Z}_*^2} \frac{k^\perp}{|k|^\gamma} e_k \frac{d\beta_k}{dt}.$$

The martingale term

In the case $\gamma = 2$, the martingale

$$M_N(t,x) := \frac{1}{\sqrt{\log N}} \sum_{|k|\leq N} \int_0^t \frac{k^\perp}{|k|^2} e_k(x) \cdot \nabla \omega_N(s,x)\, d\beta_k(s)$$

behaves in a special way for special solutions ω_N (described below) and converges to a Gaussian process, precisely of the form

$$\nabla^\perp \cdot W(t,x)$$

where $W'(t,x)$ is a divergence free space-time white noise. When true, the limit of the 2D Euler equation will be

$$\partial_t \omega + u \cdot \nabla \omega = \nu \Delta \omega + \nabla^\perp \cdot W'(t,x).$$

It remains to describe the special solutions when this happens.

Again

To avoid misunderstanding, let us insist on the fact that usually the Laplacian in the Itô-Stratonovich reformulation

$$\partial_t \omega_N + u_N \cdot \nabla \omega_N = \underbrace{\xi_N \cdot \nabla \omega_N}_{\text{Itô}} + \nu_N \Delta \omega_N$$

is a *fake* Laplacian, it does not dissipate. The sum of Itô + corrector is just Stratonovich. Thus, in general, both the approximating and the limit equation are of hyperbolic type.
But we have identified a special regime where the Laplacian (the corrector) converges to a Laplacian, and the Itô term converges to a Gaussian process. In this case the limit equation changes nature, becomes parabolic.

White noise solutions to 2D Euler equations

Consider, on $\mathbb{T}^2 = \mathbb{R}^2/\mathbb{Z}^2$, the equation

$$\begin{aligned} \partial_t \omega + u \cdot \nabla \omega &= \xi^{(\gamma)} \circ \nabla \omega \\ \omega &= \nabla^{\perp} u, \qquad \operatorname{div} u = 0 \end{aligned}$$

where $\xi^{(\gamma)}(x,t) = \sum_{k \in \mathbb{Z}^2_*} \frac{k^{\perp}}{|k|^{\gamma}} e_k(x) \frac{d\beta_k(t)}{dt}$.

Theorem (F.-Luo '18)

If $\gamma > 2$, there exists a solution, weak in the probabilistic and analytic sense, stationary process with trajectories of class $C([0,T]; H^{-1-})$, such $\omega(t)$ is a white noise in space, for every $t \geq 0$.

In spite of many attempts, we do not know how to extend this theorem to $\gamma = 2$.

Remarks

- For $\gamma > 3$, solutions in a more classical sense, precisely of class $\omega \in L^{\infty}$, have been obtained by Brzezniak-F.-Maurelli, ARMA 2016. In the class $\omega \in L^{\infty}$, also uniqueness is know, as a generalization of the deterministic result of Yudovich.
- Below $\omega \in L^{\infty}$, determinisc 2D Euler equations lack uniqueness and our hope is that this kind of transport noise may restore it.
- Therefore we investigate all possible ranges of noise, since perhaps those which may regularize are very rough. The range $\gamma > 2$ is covered by the previous result. $\gamma = 2$ is the threshold we do not know how to approach (with infinite dimensional noise).

The approximating system

Consider the equation

$$\partial_t \omega_N + u_N \cdot \nabla \omega_N = \xi_N \circ \nabla \omega_N$$

with

$$\xi_N(x,t) = \epsilon_N \sum_{|k| \leq N} \frac{k^\perp}{|k|^2} e_k(x) \frac{d\beta_k(t)}{dt}.$$

Similarly to above,

Lemma (F.-Luo '18)

There exists a solution (called below **White Noise solution***), weak in the probabilistic and analytic sense, stationary process with trajectories of class* $C([0,T]; H^{-1-})$*, such* $\omega_N(t)$ *is a white noise in space, for every* $t \geq 0$.

Main result

Theorem (F.-Luo '18)

For every $N \in \mathbb{N}$, let ω_N be a White Noise solution of

$$\partial_t \omega_N + u_N \cdot \nabla \omega_N = \xi_N \circ \nabla \omega_N, \qquad \xi_N = \epsilon_N \sum_{|k| \leq N} \frac{k^\perp}{|k|^2} e_k \frac{d\beta_k}{dt}$$

given by the previous lemma. Assume

$$\epsilon_N = \frac{1}{\sqrt{\log N}}.$$

Then ω_N converges weakly to ω, unique solution of the stochastic Navier-Stokes equation with space-time noise

$$\partial_t \omega + u \cdot \nabla \omega = \Delta \omega + \nabla^\perp \cdot \dot{W}.$$

Remark on the Navier-Stokes equations

2D Navier-Stoke equations with space-time white noise

$$\partial_t u + u \cdot \nabla u + \nabla p = \Delta u + \dot{W}$$

have been studied by Da Prato-Debussche, Albeverio-Ferrario and others. It is well posed also in the strong sense (for general initial conditions). The vorticity formulation, $\omega = \nabla^\perp u$, in the equation above:

$$\partial_t \omega + u \cdot \nabla \omega = \Delta \omega + \nabla^\perp \cdot \dot{W}.$$

It has a White Noise solution as a stationary solution. This is the regime considered by the theorem above.

Scheme of the proof

Tightness of the laws of solutions of the approximating Euler equations can be proved in $C\left([0,T];H^{-1-}\right)$, taking advantage of their stationarity in time and knowledge of the time-marginals.
The identification of the limit requires standard work for the nonlinear term (based on recent works on White Noise solutions of 2D Euler) and for the Laplacian arising as a corrector.
The difficult part is the convergence of the martingale term

$$M_N(t,x) := \frac{1}{\sqrt{\log N}} \sum_{|k|\leq N} \int_0^t \frac{k^\perp}{|k|^2} e_k(x) \cdot \nabla \omega_N(s,x)\, d\beta_k(s)$$

to the Gaussian process

$$\nabla^\perp \cdot W(t,x).$$

Quadratic variation

We have to prove, for

$$M_N(t,x) := \frac{1}{\sqrt{\log N}} \sum_{|k| \leq N} \int_0^t \frac{k^\perp}{|k|^2} e_k(x) \cdot \nabla \omega_N(s,x)\, d\beta_k(s)$$

that the joint variation

$$\left[\langle M_N, e_l \rangle, \langle M_N, e_m \rangle \right]_t$$

converges to

$$\left[\left\langle \nabla^\perp \cdot W, e_l \right\rangle, \left\langle \nabla^\perp \cdot W, e_m \right\rangle \right]_t = \int_0^t \langle \nabla e_l, \nabla e_m \rangle\, ds = t\, |l|^2\, \delta_{l,m}.$$

Quadratic variation

From

$$M_N(t,x) = \frac{1}{\sqrt{\log N}} \widetilde{M}_N(t,x)$$

$$\widetilde{M}_N(t,x) := \sum_{|k| \leq N} \int_0^t \frac{k^\perp}{|k|^2} e_k(x) \cdot \nabla \omega_N(s,x)\, d\beta_k(s)$$

one can show, similarly to the corrector above, that

$$\left[\left\langle \widetilde{M}_N, e_l \right\rangle, \left\langle \widetilde{M}_N, e_m \right\rangle \right]_t \text{ diverges.}$$

Precisely,

$$\left[\left\langle \widetilde{M}_N, e_l \right\rangle, \left\langle \widetilde{M}_N, e_m \right\rangle \right]_t$$

$$= \sum_{|k| \leq N} \frac{1}{|k|^4} \left(k^\perp \cdot l \right) \left(k^\perp \cdot m \right) \int_0^t \langle \omega, e_k e_{-l} \rangle \langle \omega, e_k e_{-m} \rangle\, ds$$

Quadratic variation

$$\left[\left\langle \widetilde{M}_N, e_l \right\rangle, \left\langle \widetilde{M}_N, e_m \right\rangle\right]_t$$

$$= \sum_{|k|\leq N} \frac{1}{|k|^4} \left(k^\perp \cdot l\right) \left(k^\perp \cdot m\right) \int_0^t \langle \omega, e_k e_{-l} \rangle \langle \omega, e_k e_{-m} \rangle \, ds$$

$$\mathbb{E}\left[\left\langle \widetilde{M}_N, e_l \right\rangle, \left\langle \widetilde{M}_N, e_m \right\rangle\right]_t$$

$$\overset{\omega(t)\ \text{WN}}{=} t \cdot \sum_{|k|\leq N} \frac{1}{|k|^4} \left|k^\perp \cdot l\right|^2 \delta_{l,m}$$

$$\sim \left(\sum_{|k|\leq N} \frac{1}{|k|^2}\right) t \, |l|^2 \, \delta_{l,m}$$

and the coefficient $\sum_{|k|\leq N} \frac{1}{|k|^2}$ diverges.

Analogy: renormalized energy

In the theory of 2D Euler with the enstrophy measure there is a well-known renormalization: the interaction kinetic energy.
The average kinetic energy of the fluid is (u_k= Fourier components of u)

$$\mathbb{E}\left[\frac{1}{2}\int |u(x)|^2 \, dx\right] = \frac{1}{2}\sum_{k\in\mathbb{Z}^2_*}\mathbb{E}\left[|u_k|^2\right] \overset{\omega=\nabla^\perp u=\mathrm{WN}}{=} \frac{1}{2}\sum_{k\in\mathbb{Z}^2_*}\frac{1}{|k|^2} = +\infty$$

but the renormalized partial sums (corresponding to interaction energy in the case of point vortices)

$$\sum_{|k|\leq N}|u_k|^2 - \mathbb{E}\left[\sum_{|k|\leq N}|u_k|^2\right]$$

converge in L^2_μ (μ=law of White Noise on H^{-1-}) to a well defined random variable

$$:\mathcal{E}:$$

called *renormalized kinetic energy*.

Renormalizing quadratic variation

- For $[\langle M_N, e_l\rangle, \langle M_N, e_m\rangle]_t$ it is similar.
- *Without* the term $\frac{1}{\log N}$, we have seen above that $\mathbb{E}\left[\left\langle \widetilde{M}_N, e_l\right\rangle, \left\langle \widetilde{M}_N, e_m\right\rangle\right]_t$ diverges.
- Moreover, it happens that

$$\left[\left\langle \widetilde{M}_N, e_l\right\rangle, \left\langle \widetilde{M}_N, e_m\right\rangle\right]_t - \mathbb{E}\left[\left\langle \widetilde{M}_N, e_l\right\rangle, \left\langle \widetilde{M}_N, e_m\right\rangle\right]_t$$

converges in L^2_μ (μ=law of White Noise on H^{-1-}) to a well defined random variable.

As a consequence, *including* the term $\frac{1}{\log N}$,

$$L^2_\mu - \lim_{N\to\infty} \left([\langle M_N, e_l\rangle, \langle M_N, e_m\rangle]_t - \mathbb{E}\left[\langle M_N, e_l\rangle, \langle M_N, e_m\rangle\right]_t \right) = 0.$$

And

$$\begin{aligned}
\mathbb{E}\left[\langle M_N, e_l\rangle, \langle M_N, e_m\rangle\right]_t &= \left(\frac{1}{\log N} \sum_{|k|\leq N} \frac{1}{|k|^2} \right) t\, |l|^2\, \delta_{l,m} \\
&\to t\, |l|^2\, \delta_{l,m} \\
&= \left[\left\langle \nabla^\perp \cdot W, e_l \right\rangle, \left\langle \nabla^\perp \cdot W, e_m \right\rangle \right]_t .
\end{aligned}$$

Discussion: extensions, interpretation, motivations

- There are heuristic reasons to believe that part of the scaling limit discussed above is a more general fact; in particular, the presence of the Laplacian in the limit equation, with a true parabolic character.
- Maybe precisely additive white noise is special, due to the "white noise" solution regime.
- [Private discussions years ago with Weber and Hauray.] A common space-dep noise with very short correlation range acting on particles is similar to independent BMs and should give rise to a Laplacian in the mean field limit.
- With Dejun Luo we spent much time trying to prove that multiplicative noise restores uniqueness in 2D Euler equations and had the heuristic impression that $\gamma = 2$ was relevant (but it is not admissible).
- In a sense we have studied here the case $\gamma = 2$, and the limit equation, stochastic Navier-Stokes, has uniqueness!