

A PDE construction of the Euclidean Φ_3^4 quantum field theory

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joint work with Massimiliano Gubinelli

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- **probability measure on $\mathcal{S}'(\mathbb{R}^3)$**

- given formally by

$$\nu(d\varphi) \sim \exp\left\{-2 \int_{\mathbb{R}^3} \frac{1}{2} |\nabla \varphi|^2 + \frac{m^2}{2} \varphi^2 + \frac{\lambda}{4} \varphi^4\right\} \prod_{x \in \mathbb{R}^3} d\varphi(x) \quad m^2 > 0, \lambda > 0$$

- non-Gaussian + Osterwalder–Schrader axioms
 - Euclidean invariance, reflection positivity, regularity, analyticity, ergodicity
- Schwinger-Dyson equations (integration by parts formula)

- success of the constructive quantum field theory program in the 60' – 80':

- Glimm, Jaffe, Feldman, Osterwalder, Brydges, Fröhlich, Sokal, ...

- Parisi–Wu '81 proposed to study the Φ^4 model

$$(\partial_t - \Delta + m^2)\varphi + \lambda\varphi^3 = \xi \quad \text{on } \mathbb{R}_+ \times \mathbb{R}^3$$

- ξ a space-time white noise on $\mathbb{R}_+ \times \mathbb{R}^3$
 - ν is the **invariant measure**
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- ξ space-time white noise on $\mathbb{R}_+ \times \mathbb{R}^3$ - a random distribution of regularity $-5/2 - \kappa$
 - Schauder estimates - gain of 2 degrees of regularity
 - $\Rightarrow \varphi$ is a distribution of regularity $-1/2 - \kappa$ (renormalization needed to define φ^3)

GAIN: we have a dynamics & tools from PDEs, SPDEs & Gaussian noise

- $m^2 > 0$, $\lambda > 0$ arbitrary
- use the dynamics to construct the measure

$$\nu(d\varphi) \sim \exp\left\{-2 \int_{\mathbb{R}^3} \frac{1}{2} |\nabla \varphi|^2 + \frac{m^2}{2} \varphi^2 + \frac{\lambda}{4} \varphi^4\right\} \prod_{x \in \mathbb{R}^3} d\varphi(x) \quad \text{on} \quad \mathcal{S}'(\mathbb{R}^3)$$

- non-Gaussian
- Osterwalder–Schrader axioms
 - translation invariance
 - reflection positivity
- integration by parts formula, Swinger–Dyson equations

Applies to: multicomponent models with $O(N)$ symmetry

Additionally: regularity, analyticity, ergodicity (using correlation inequalities)

Missing: rotation invariance

- let $\Lambda_{M,\varepsilon} = (\varepsilon(\mathbb{Z}/M\mathbb{Z}))^3$ – periodic lattice of mesh size ε and side length M
- consider the **Gibbs measure** on $\mathbb{R}^{\Lambda_{M,\varepsilon}}$

$$\nu_{M,\varepsilon}(d\varphi) \sim \exp \left\{ -2\varepsilon^3 \sum_{\Lambda_{M,\varepsilon}} \left[\frac{1}{2} |\nabla_\varepsilon \varphi|^2 + \frac{m^2 - 3a_{M,\varepsilon} + 3b_{M,\varepsilon}}{2} |\varphi|^2 + \frac{1}{4} |\varphi|^4 \right] \right\} \prod_{x \in \Lambda_{M,\varepsilon}} d\varphi(x)$$

- $a_{M,\varepsilon}, b_{M,\varepsilon}$ renormalization constants
- translation invariant, reflection positive, ...
- study **stationary solutions** of (finite-dimensional) stochastic quantization equation

$$(\partial_t - \Delta_\varepsilon + m^2)\varphi + (-3a_{M,\varepsilon} + 3b_{M,\varepsilon})\varphi + \varphi^3 = \xi_{M,\varepsilon} \quad \text{on } \mathbb{R}_+ \times \Lambda_{M,\varepsilon}$$

- find **energy estimates** uniform in both parameters $M, \varepsilon \Rightarrow$ **tightness** of $\nu_{M,\varepsilon}$

- for a **nice** f consider

$$(\partial_t - \Delta + m^2)u + u^3 = f$$

- test the equation by u (= multiply by u and integrate wrt x)

$$\langle \partial_t u, u \rangle - \langle \Delta u, u \rangle + m^2 \langle u, u \rangle + \langle u^3, u \rangle = \langle f, u \rangle$$

$$\frac{1}{2} \partial_t \|u\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + m^2 \|u\|_{L^2}^2 + \|u\|_{L^4}^4 \leq \|f\|_{L^2} \|u\|_{L^2} \leq C_\delta \|f\|_{L^2}^2 + \delta \|u\|_{L^2}^2$$

- if u **stationary** then

$$\frac{1}{2} \underbrace{\partial_t \mathbb{E} \|u(t)\|_{L^2}^2}_{=0} + \mathbb{E} \|\nabla u(t)\|_{L^2}^2 + (m^2 - \delta) \mathbb{E} \|u(t)\|_{L^2}^2 + \mathbb{E} \|u(t)\|_{L^4}^4 \leq C_\delta \mathbb{E} \|f(t)\|_{L^2}^2$$

- \Rightarrow control of the **moments of the invariant measure** depending on f
- other bounds for $\langle f, u \rangle$ possible

$$\langle f, u \rangle \leq \|f\|_{L^{4/3}} \|u\|_{L^4}, \quad \langle f, u \rangle \leq \|f\|_{H^{-1}} (\|u\|_{L^2} + \|\nabla u\|_{L^2})$$

- product of distributions are generally not well-defined
- Littlewood–Paley decomposition

$$f = \sum_{i \geq -1} \Delta_i f, \quad g = \sum_{j \geq -1} \Delta_j g$$

- leads to

$$\begin{aligned} fg &= \left(\sum_{i \geq -1} \Delta_i f \right) \left(\sum_{j \geq -1} \Delta_j g \right) \\ &= \sum_{i, j: i < j-1} \Delta_i f \Delta_j g + \sum_{i, j: j < i-1} \Delta_i f \Delta_j g + \sum_{i, j: |i-j| \leq 1} \Delta_i f \Delta_j g \\ &=: f \prec g + f \succ g + f \circ g \end{aligned}$$

- the two paraproducts always well-defined
- the resonant term requires positive sum of the regularities

- let φ be a **stationary** solution to

$$(\partial_t - \Delta_\varepsilon + m^2)\varphi + (-3a + 3b)\varphi + \varphi^3 = \xi \quad \text{on } \mathbb{R}_+ \times \Lambda_{M,\varepsilon}$$

- ansatz** $\varphi = X + \eta$ where $(\partial_t - \Delta_\varepsilon + m^2) \underbrace{X}_{-1/2-\kappa} = \underbrace{\xi}_{-5/2-\kappa}$ (stationary) gives

$$(\partial_t - \Delta_\varepsilon + m^2)\eta + 3b\varphi + \underbrace{[[X^3]]}_{-3/2-\kappa} + 3\eta \underbrace{[[X^2]]}_{-1-\kappa} + 3\eta^2 \underbrace{X}_{-1/2-\kappa} + \eta^3 = 0$$

- instead of** removing X^{Ψ} where $(\partial_t - \Delta_\varepsilon + m^2)X^{\Psi} = -[[X^3]]$
- let** Y solve $(\partial_t - \Delta_\varepsilon + m^2)Y = -[[X^3]] - 3(\Delta_{>L}[[X^2]]) \succ Y$ (via fixed point)
- and **define** $\varphi = X + Y + \phi$ which leads to

$$(\partial_t - \Delta_\varepsilon + m^2)\phi + \phi^3 = -3[[X^2]] \succ \phi - 3[[X^2]] \circ \phi + \text{better (after renormalization)}$$

$$\frac{1}{2} \partial_t \|\phi\|_{L^2, \varepsilon}^2 + \|\phi\|_{L^4, \varepsilon}^4 + \langle \phi, (m^2 - \Delta_\varepsilon) \underbrace{\phi}_{1-\kappa} \rangle_\varepsilon$$

$$= \langle \phi, \underbrace{-3[[X^2]] \succ \phi}_{-1-\kappa} \rangle_\varepsilon + \langle \phi, \underbrace{-3[[X^2]] \circ \phi}_{-1-\kappa} \rangle_\varepsilon + \langle \phi, \text{better (after renormalization)} \rangle_\varepsilon$$

- **approximate duality**

$$\langle \phi, -3[[X^2]] \circ \phi \rangle_\varepsilon - \langle -3[[X^2]] \succ \phi, \phi \rangle_\varepsilon =: D(\phi, -3[[X^2]], \phi)$$

bounded if the **sum** of the regularities of $\phi, -3[[X^2]], \phi$ **positive!**

- combine with the Laplace term

$$\langle \phi, (m^2 - \Delta_\varepsilon) \phi + 2 \cdot 3[[X^2]] \succ \phi \rangle_\varepsilon$$

- complete the square using **elliptic paracontrolled ansatz** (ψ is more regular than ϕ)

$$(m^2 - \Delta_\varepsilon) \psi := (m^2 - \Delta_\varepsilon) \phi + 3[[X^2]] \succ \phi$$

- include a polynomial weight $\rho(x) = (1 + |x|^2)^{-\theta/2} \in L^4$ (= test by $\rho^4\phi$ instead of ϕ)
- denote $\mathbb{X}_{M,\varepsilon} = (X_{M,\varepsilon}, \llbracket X_{M,\varepsilon}^2 \rrbracket, X_{M,\varepsilon}^{\Psi}, \dots)$
- we obtain – **uniformly** in M, ε

$$\begin{aligned} \frac{1}{2} \partial_t \|\rho^2 \phi_{M,\varepsilon}\|_{L^{2,\varepsilon}}^2 + \|\rho \phi_{M,\varepsilon}\|_{L^{4,\varepsilon}}^4 + \|\rho^2 \phi_{M,\varepsilon}\|_{H^{1-2\kappa,\varepsilon}}^2 + \|\rho^2 \psi_{M,\varepsilon}\|_{L^{2,\varepsilon}}^2 + \|\rho^2 \nabla_{\varepsilon} \psi_{M,\varepsilon}\|_{L^{2,\varepsilon}}^2 \\ \leq (|\log t| + 1) Q_{\rho}(\mathbb{X}_{M,\varepsilon}). \end{aligned}$$

- the resonant product $\llbracket X^2 \rrbracket \circ \phi$ not controlled; $\llbracket X^2 \rrbracket \circ \psi$ also not
- analogy with PDE **weak solutions** (equation interpreted in a suitable duality sense)
- cf. **strong solutions** approach Gubinelli, H. [arXiv:1804.11253]
- slightly higher regularity possible to recover the continuum equation
- \Rightarrow integration by parts + Schwinger–Dyson equations

- recall
 - $\varphi_{M,\varepsilon} = X_{M,\varepsilon} + Y_{M,\varepsilon} + \phi_{M,\varepsilon}$ is stationary with law $\nu_{M,\varepsilon}$
 - $X_{M,\varepsilon}$ stationary, $Y_{M,\varepsilon}$ **not** stationary $\Rightarrow \phi_{M,\varepsilon}$ **not** stationary
- let $\varphi_{M,\varepsilon} = X_{M,\varepsilon} + X_{M,\varepsilon}^{\Psi} + \zeta_{M,\varepsilon} \Rightarrow$ all components stationary

Theorem

- The family of joint laws of $(\varphi_{M,\varepsilon}, X_{M,\varepsilon})$ evaluated at some $t \geq 0$ is tight.
- Let $\zeta := \varphi - X - X^{\Psi}$. Then any limit measure μ satisfies for all $p \in [1, \infty)$

$$\mathbb{E}_{\mu} \|\varphi\|_{H^{-1/2-2\kappa}(\rho^2)}^{2p} + \mathbb{E}_{\mu} \|\zeta\|_{L^2(\rho^2)}^{2p} + \mathbb{E}_{\mu} \|\zeta\|_{H^{1-2\kappa}(\rho^2)}^2 + \mathbb{E}_{\mu} \|\zeta\|_{B_{4,\infty}^0(\rho)}^4 < \infty.$$

Osterwalder–Schrader axioms and nontriviality

- denote ν the limit of $\nu_{M,\varepsilon}$ (projection of μ on the first component)

Nontriviality = non-Gaussianity Show that the *connected four-point function*

$$U_{4,\nu}(x_1, \dots, x_4) := \langle \varphi(x_1) \cdots \varphi(x_4) \rangle - \langle \varphi(x_1) \varphi(x_2) \rangle \langle \varphi(x_3) \varphi(x_4) \rangle \\ - \langle \varphi(x_1) \varphi(x_3) \rangle \langle \varphi(x_2) \varphi(x_4) \rangle - \langle \varphi(x_1) \varphi(x_4) \rangle \langle \varphi(x_2) \varphi(x_3) \rangle, \quad x_1, \dots, x_4 \in \mathbb{R}^3$$

is nonzero

Idea:

- use the decomposition $\varphi = X + X^\Psi + \zeta$
- together with the moment estimates

- let $K_i * \cdot = \Delta_i \cdot$ be the Littlewood–Paley projector on the scale 2^i
- study $U_{4,\nu}$ convolved with (K_i, K_i, K_i, K_i) and evaluated at $(x_1, x_2, x_3, x_4) = (0, 0, 0, 0)$

$$U_{4,\nu} * (K_i, \dots, K_i)(0, \dots, 0) = \langle (\Delta_i \varphi)^4(0) \rangle - 3 \langle (\Delta_i \varphi)^2(0) \rangle^2 =: L(\varphi, \varphi, \varphi, \varphi)$$

- use the decomposition $\varphi = X + X^{\Psi} + \zeta$ to get

$$L(\varphi, \varphi, \varphi, \varphi) = L(X, X, X, X) + 4L(X, X, X, X^{\Psi}) + R$$

- show that

$$|R| \lesssim 2^{i(1/2+5\kappa)}$$

- whereas explicit computation yields

$$L(X, X, X, X^{\Psi}) \approx -2^i$$

Integration by parts and Schwinger–Dyson equations

$$\nu_{M,\varepsilon}(d\varphi) \sim \exp \left\{ -2\varepsilon^3 \sum_{\Lambda_{M,\varepsilon}} \left[\frac{1}{2} |\nabla_\varepsilon \varphi|^2 + \frac{m^2 - 3a_{M,\varepsilon} + 3b_{M,\varepsilon}}{2} |\varphi|^2 + \frac{1}{4} |\varphi|^4 \right] \right\} \prod_{x \in \Lambda_{M,\varepsilon}} d\varphi(x)$$

- let F be a cylinder functional on $\mathcal{S}'(\Lambda_{M,\varepsilon})$: $F(\varphi) = \Phi(\varphi(f_1), \dots, \varphi(f_n))$
- integration by parts gives

$$\int DF(\varphi) \nu_{M,\varepsilon}(d\varphi) = 2 \int F(\varphi) [\varphi^3 + (-3a_{M,\varepsilon} + 3b_{M,\varepsilon})\varphi + (m^2 - \Delta_\varepsilon)\varphi] \nu_{M,\varepsilon}(d\varphi)$$

In order to pass to the limit:

- use the decomposition $\varphi = X + X^{\Psi} + \zeta$
- problematic:
 - $[[X^2]] \circ \zeta$ – not well-defined based on the energy estimates so far
 - $[[X^3]]$ – only exists as a space-time distribution

- let $h: \mathbb{R} \rightarrow \mathbb{R}$ be smooth with $\text{supp } h \subset \mathbb{R}_+$ and $\int_{\mathbb{R}} h \, dt = 1$
- then letting $[[\varphi^3]] := \varphi^3 + (-3a_{M,\varepsilon} + 3b_{M,\varepsilon})\varphi$ we get

$$\int F(\varphi) [[\varphi^3]] \nu_{M,\varepsilon}(d\varphi) = \mathbb{E}[F(\varphi_{M,\varepsilon}(t)) [[\varphi_{M,\varepsilon}^3(t)]]] = \mathbb{E}\left[\int_{\mathbb{R}} h(t) F(\varphi_{M,\varepsilon}(t)) [[\varphi_{M,\varepsilon}^3(t)]] \, dt\right]$$

- prove tightness of $(\varphi_{M,\varepsilon}, \mathbb{X}_{M,\varepsilon})$ in $C_{\text{loc}}^{2\kappa}(\mathbb{R}_+; B_{1,4/3}^{-1-3\kappa}(\rho^4) \times \mathcal{X})$

Theorem Any limit measure ν satisfies

$$\int \text{D}F(\varphi) \nu(d\varphi) = 2 \int F(\varphi) [(m^2 - \Delta)\varphi] \nu(d\varphi) + 2\mathcal{J}(F),$$

where

$$\mathcal{J}(F) = \mathbb{E}\left[\int_{\mathbb{R}} h(t) F(\varphi(t)) [[\varphi^3]](t) \, dt\right]$$

$$[[\varphi^3]] = [[X^3]] + 3[[X^2]] \succ (-X \Psi + \zeta) + 3[[X^2]] \prec (-X \Psi + \zeta) + \dots$$

- operator product expansion

$$\mathcal{J}(F) = \mathbb{E} \left[\int_{\mathbb{R}} h(t) F(\varphi(t)) [\varphi^3](t) dt \right] = \lim_{N \rightarrow \infty} \int F(\varphi) [(\Delta_{\leq N} \varphi)^3 - c_N(\Delta_{\leq N} \varphi)] \nu(d\varphi)$$

- Schwinger–Dyson equations

$$\sum_{i=1}^n \delta(x - x_i) \langle \varphi(x_1) \cdots \varphi(x_i) \cdots \varphi(x_n) \rangle = 2 \langle [(m^2 - \Delta) \varphi(x)] \varphi(x_1) \cdots \varphi(x_n) \rangle$$

$$+ 2 \lim_{N \rightarrow \infty} \langle \varphi(x_1) \cdots \varphi(x_n) [(\Delta_{\leq N} \varphi(x))^3 - c_N(\Delta_{\leq N} \varphi(x))] \rangle$$

$$\delta(x - y) = 2(m^2 - \Delta_x) S_{2,\nu}(x - y)$$

$$+ 2 \lim_{N \rightarrow \infty} [((\mathbb{I} \otimes \Delta_{\leq N}^{\otimes 3}) S_{4,\nu})(y, x, x, x) - c_N(\Delta_{\leq N} S_{2,\nu})(x - y)]$$

Thank you for your attention!