Higher Hochschild homology as a functor

(joint work with Geoffrey POWELL (CNRS - Angers))

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Plan

Higher Hochschild homology

Theorem of Turchin-Willwacher

Exponential functors

Outer functors

Theorem

Examples

Definition of higher Hochschild homology [Pirashvili, 2000]

Δ: category of ordinals (simplex category)
$\mathbf{Set}_*$: category of pointed sets
$\Gamma \subset \mathbf{Set}_*$: subcategory of finite pointed sets

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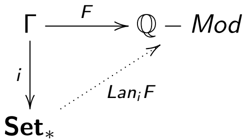

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Definition: [Pirashvili, 2000]

$HH_\bullet(Y; F) :=$ homology of the chain complex associated to $Lan_i F \circ Y$

Loday functor

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where $[n] = \{\underline{0}, 1, \ldots, n\}$

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We are interested in

$$HH_\bullet(\bigvee_n S^1; \mathcal{L}(A, A))$$

for particular A

Theorem of Turchin-Willwacher

$A = \mathbb{Q}[\epsilon]$ dual numbers (i.e. $\mathbb{Q} \oplus \mathbb{Q}$ s.t. $(a, b)(a', b') := (aa', ab' + a'b)$)

Theorem: [Turchin-Willwacher 2018]

$$HH_d(\bigvee_n S^1; \mathcal{L}(\mathbb{Q}[\epsilon], \mathbb{Q}[\epsilon])) = U_d^I \oplus U_d^{II}$$

where U_d^I and U_d^{II} are representations of $Out(F_n)$ which does not factorize through $GL(n, \mathbb{Z})$ (for d big enough)

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Aim:

Extend this theorem to other commutative algebras.
Our theorem: $HH_d(\bigvee_n S^1; \mathcal{L}(A_V, A_V))$ where A_V square zero extension of $\mathbb{Q}$ (i.e. $A_V = \mathbb{Q} \oplus V$ s.t. $(a, v)(a', v') := (aa', av' + a'v)$.)

Our method

Method

To exploit the functorial structure and use functor tools

- exponential functors
- outer functors
- polynomial functors

Higher Hochschild homology as functor on **gr**

gr: category of finitely generated free groups (objects: $\mathbb{Z}^{*r}$)

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$\bigvee_n S^1$ is the classifying space $B\mathbb{Z}^{*r}$

Fact

We have a graded functor

$$HH_\bullet(B(-);F) : \mathbf{gr} \to \mathbb{Q} - Mod$$

given by $HH_\bullet(B(-);F)(\mathbb{Z}^{*n}) := HH_\bullet(B\mathbb{Z}^{*n};F)$

Exponential functors

Definition:

$M : \mathbf{gr} \to \mathbb{Q} - Mod$ is an exponential functor if it is a strong symmetric monoidal functor.
In particular, there are natural isomorphisms

$$M(\mathbb{Z}^{*n} * \mathbb{Z}^{*m}) \simeq M(\mathbb{Z}^{*n}) \otimes M(\mathbb{Z}^{*m})$$

$\mathcal{F}^{exp}(\mathbf{gr}, \mathbb{Q})$: category of exponential functors from $\mathbf{gr}$ to $\mathbb{Q} - Mod$

Exponential functors and Hopf algebras I

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Proposition:

$ev_{\mathbb{Z}} : \mathcal{F}^{exp}(\mathbf{gr}, \mathbb{Q}) \to Hopf_{\mathbb{Q}}^{com}$ given by $ev_{\mathbb{Z}}(F) = F(\mathbb{Z})$ is a functor

- $F(\mathbb{Z}) \otimes F(\mathbb{Z}) \simeq F(\mathbb{Z} * \mathbb{Z}) \xrightarrow{F(f)} F(\mathbb{Z})$ where $f : \mathbb{Z} * \mathbb{Z} \to \mathbb{Z}$ fold map (i.e. $f(x_1) = f(x_2) = x$)
- $F(\mathbb{Z}) \xrightarrow{F(g)} F(\mathbb{Z} * \mathbb{Z}) \simeq F(\mathbb{Z}) \otimes F(\mathbb{Z})$ where $g : \mathbb{Z} \to \mathbb{Z} * \mathbb{Z}$ diagonal map (i.e. $g(x) = x_1 x_2$)
- $F(\mathbb{Z}) \xrightarrow{F(c)} F(\mathbb{Z})$ where $c : \mathbb{Z} \to \mathbb{Z}$ given by $c(x) = x^{-1}$

Exponential functors and Hopf algebras II

$ev_{\mathbb{Z}} : \mathcal{F}^{exp}(\mathbf{gr}, \mathbb{Q}) \to Hopf_{\mathbb{Q}}^{com}$

Theorem:

$ev_{\mathbb{Z}}$ induces an equivalence of categories.
$\psi : Hopf_{\mathbb{Q}}^{com} \to \mathcal{F}^{exp}(\mathbf{gr}, \mathbb{Q})$ given by $\psi(H) : \mathbb{Z}^{*n} \mapsto H^{\otimes n}$ for $H \in Hopf_{\mathbb{Q}}^{com}$

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Example:

For $V \in \mathbb{Q} - mod$

$\mathbb{T}^{\dagger}_{\mathrm{coalg}}(V) = (T^d(V), d \in \mathbb{N})$ is a graded Hopf algebra in $\mathbb{Q}$-*mod*.

Underlying coalgebra: the tensor coalgebra

Product: Koszul-signed shuffle

$\psi(\mathbb{T}^{\dagger}_{\mathrm{coalg}}(V))$ graded object in $\mathcal{F}^{exp}(\mathbf{gr}, \mathbb{Q})$

Outer functors

$$\begin{array}{ccc} Aut(\mathbb{Z}^{*n}) & \twoheadrightarrow & GL(n,\mathbb{Z}) \\ \downarrow\!\!\downarrow & \nearrow\!\!\nearrow & \\ Out(\mathbb{Z}^{*n}) := Aut(\mathbb{Z}^{*n})/Inn(\mathbb{Z}^{*n}) & & \end{array}$$

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$\mathcal{F}(\mathbf{gr},\mathbb{Q})$: functors from **gr** to $\mathbb{Q}-Mod$

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Definition:

$\mathcal{F}^{Out}(\mathbf{gr},\mathbb{Q})$ is the full subcategory of $\mathcal{F}(\mathbf{gr},\mathbb{Q})$ of objects M s.t. $Inn(\mathbb{Z}^{*n})$ acts trivially on $M(\mathbb{Z}^{*n})$ for all $n \in \mathbb{N}$

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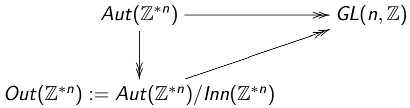

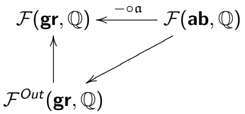

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Example:

$\psi(\mathbb{T}^{\dagger}_{\mathrm{coalg}}(V))\notin\mathcal{F}^{Out}(\mathbf{gr},\mathbb{Q})$ for $V\in\mathbb{Q}-mod$, $V\neq 0$

$\omega(\psi(\mathbb{T}^{\dagger}_{\mathrm{coalg}}(V)))$ graded object in $\mathcal{F}^{Out}(\mathbf{gr},\mathbb{Q})$ for $V\in\mathbb{Q}-mod$

Definition of $\omega_1\psi(\mathbb{T}^{\dagger}_{\mathrm{coalg}}(V))$

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Proposition:

$$\omega_1\psi(\mathbb{T}^{\dagger}_{\mathrm{coalg}}(V)) \in \mathcal{F}^{Out}(\mathbf{gr}, \mathbb{Q})$$

Theorem

Theorem: [Powell-Vespa]

There is a natural isomorphism of graded commutative monoids in $\mathcal{F}(\mathbf{gr}, \mathbb{Q})$

$$HH_\bullet(B(-); \mathcal{L}(A_V, A_V)) \simeq \omega\psi(\mathbb{T}^\dagger_{coalg}(V)) \oplus \omega_1\psi(\mathbb{T}^\dagger_{coalg}(V))[-1]$$

$\omega\psi(\mathbb{T}^\dagger_{coalg}(V))$ and $\omega_1\psi(\mathbb{T}^\dagger_{coalg}(V))$ lie in $\mathcal{F}^{Out}(\mathbf{gr}, \mathbb{Q})$
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Theorem: [Turchin-Willwacher revisited]

$$HH_\bullet(B(-); \mathcal{L}(A_\mathbb{Q}, A_\mathbb{Q})) \simeq \omega\psi(\mathbb{T}^\dagger_{coalg}(\mathbb{Q})) \oplus \omega_1\psi(\mathbb{T}^\dagger_{coalg}(\mathbb{Q}))[-1]$$

Combinatorial coefficients

Simple $\mathfrak{S}_n$-representations indexed by the partitions λ of n

S_λ: simple representation of $\mathfrak{S}_n$ where $\lambda \vdash n$

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$$\begin{array}{rccc} \mathrm{Inj}^{\mathbf{Fin}}: & \Sigma^{op} \times \mathbf{Fin} & \to & \mathbb{Q}-Mod \\ & (m, U) & \mapsto & \mathbb{Q}[\mathrm{Inj}_{\mathbf{Fin}}(m, U)] \end{array}$$

$$\mathrm{Inj}^{\mathbf{Fin}}_\lambda := \mathrm{Inj}^{\mathbf{Fin}} \underset{\Sigma}{\otimes} S_\lambda : \mathbf{Fin} \to \mathbb{Q}-Mod$$

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$\theta : \Gamma \to$ **Fin**: forgetful functor

$$\theta^*(Inj^{\mathbf{Fin}}_\lambda) \in \mathcal{F}(\Gamma, \mathbb{Q})$$

Decomposition of $HH_\bullet(B(-); \mathcal{L}(A_V, A_V))$

Theorem: [Powell-Vespa]

$$HH_\bullet(B(-); \mathcal{L}(A_V, A_V)) \cong \bigoplus_{n\in\mathbb{N}} \bigoplus_{\substack{\lambda\vdash n \\ l(\lambda)\leq \dim V}} HH_\bullet(B(-); \theta^* \mathrm{Inj}_\lambda^{\mathsf{Fin}}) \otimes \mathbf{S}_\lambda(V)$$

where $\mathbf{S}_\lambda$ is the Schur functor associated to the representation S_λ
(i.e. $\mathbf{S}_\lambda(V) = V^{\otimes n} \otimes_{\mathfrak{S}_n} S_\lambda$)

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Computation of $HH_*(B(-); \theta^* \mathrm{Inj}_\lambda^{\mathbf{Fin}})$

$$\mathcal{P}ol_n(\mathbf{gr}; \mathbb{Q}) \xrightarrow[cr_n]{} \mathbb{Q}[\mathfrak{S}_n] - Mod$$

Computation of $HH_*(B(-);\theta^*\mathrm{Inj}_\lambda^{\mathsf{Fin}})$

$$\mathcal{P}ol_n(\mathbf{gr};\mathbb{Q}) \underset{cr_n}{\longrightarrow} \mathbb{Q}[\mathfrak{S}_n]-Mod \quad \text{with } \alpha_n, \beta_n : \mathbb{Q}[\mathfrak{S}_n]-Mod \to \mathcal{P}ol_n(\mathbf{gr};\mathbb{Q})$$

where $\alpha_n(M) = \mathfrak{a}_{\mathbb{Q}}^{\otimes n} \otimes_{\mathfrak{S}_n} M$

Computation of $HH_*(B(-);\theta^*\mathrm{Inj}_\lambda^{\mathsf{Fin}})$

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Theorem: [Powell-Vespa]

$$HH_*(B(-);\theta^*\mathrm{Inj}_\lambda^{\mathsf{Fin}}) \cong \begin{cases} \omega\beta_n S_{\lambda^\dagger} & * = n \\ \omega_1\beta_n S_{\lambda^\dagger} & * = n-1 \\ 0 & \text{otherwise.} \end{cases}$$

where $0 \to \omega\beta_n S_{\lambda^\dagger} \to \beta_n S_{\lambda^\dagger} \overset{\overline{ad}}{\to} \bigoplus_{\substack{\mu\subset\lambda^\dagger \\ |\lambda^\dagger|=|\mu|+1}} \beta_{n-1}S_\mu \to \omega_1\beta_n S_{\lambda^\dagger} \to 0.$

Computation of $HH_*(B(-); \theta^* \mathrm{Inj}_\lambda^{\mathbf{Fin}})$

$$\mathcal{P}ol_n(\mathbf{gr}; \mathbb{Q}) \underset{\beta_n}{\overset{\alpha_n}{\leftrightarrows}} \xrightarrow{cr_n} \mathbb{Q}[\mathfrak{S}_n] - Mod \quad \text{where } \alpha_n(M) = \mathfrak{a}_{\mathbb{Q}}^{\otimes n} \otimes_{\mathfrak{S}_n} M$$

Theorem: [Powell-Vespa]

$$HH_*(B(-); \theta^* \mathrm{Inj}_\lambda^{\mathbf{Fin}}) \cong \begin{cases} \omega\beta_n S_{\lambda^\dagger} & * = n \\ \omega_1\beta_n S_{\lambda^\dagger} & * = n-1 \\ 0 & \text{otherwise.} \end{cases}$$

where $0 \to \omega\beta_n S_{\lambda^\dagger} \to \beta_n S_{\lambda^\dagger} \overset{\overline{ad}}{\to} \bigoplus_{\substack{\mu \subset \lambda^\dagger \\ |\lambda^\dagger| = |\mu|+1}} \beta_{n-1} S_\mu \to \omega_1 \beta_n S_{\lambda^\dagger} \to 0.$

Proposition: [Powell-Vespa]

For $\nu \vdash d$, there is an equality in the Grothendieck group of $\mathcal{F}(\mathbf{gr}, \mathbb{Q})$:

$$[\beta_d S_\nu] = \sum_\rho \sum_{\underline{a}} \sum_{\underline{\lambda} \vdash \underline{a}} \sum_{\underline{\mu}} \left(c^\rho_{\underline{\lambda}} c^\nu_{\underline{\mu}} \prod_{\substack{n \\ a_n \neq 0}} p^{\mu(n)}_{\lambda(n), Lie(n)} \right) [\alpha_{|\rho|} S_\rho].$$

$c^\rho_{\underline{\lambda}}$: Littlewood-Richardson coefficients; $p^{\mu(n)}_{\lambda(n),Lie(n)}$: plethysm coefficients

Example 1: $\lambda = (1^n)$

$$0 \to \omega\beta_n S_{(n)} \to \beta_n S_{(n)} \xrightarrow{\overline{ad}} \beta_{n-1} S_{(n-1)} \to \omega_1 \beta_n S_{(n)} \to 0.$$

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Proposition

$$\overline{ad} = 0$$

$$\omega\beta_n S_{(n)} = \beta_n S_{(n)} \cong \alpha_n S_{(n)} \quad \text{and} \quad \omega_1 \beta_n S_{(n)} \cong \alpha_{n-1} S_{(n-1)}.$$

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Proposition

$$\mathrm{HH}_*(B(-); \theta^* \mathrm{Inj}^{\mathbf{Fin}}_{(1^n)}) \cong \begin{cases} \omega\beta_n S_{(n)} \simeq \alpha_n S_{(n)} & * = n \\ \omega_1 \beta_n S_{(n)} \simeq \alpha_{n-1} S_{(n-1)} & * = n-1 \\ 0 & \text{otherwise.} \end{cases}$$

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Proposition:

$$[\omega\beta_n S_{(1^n)}] \simeq \sum_{a+2b=n} [\alpha S_{((b+1)1^{a-1})}] \simeq \sum_{a+2b=n+1} [\alpha S_{(b1^a)}]$$
$$[\omega_1 \beta_n S_{(1^n)}] \simeq \sum_{a+2b=n-1} [\alpha S_{(b1^a)}]$$

Proposition:

$$\omega_1 \beta_n S_{(1^n)} \simeq \omega\beta_{n-2} S_{(1^{n-2})}$$

Example 2: $\lambda = (n)$

Proposition

$$\mathsf{HH}_*(B(-);\theta^*\mathrm{Inj}^{\mathbf{Fin}}_{(n)}) \cong \begin{cases} \omega\beta_n S_{(1^n)} & * = n \\ \omega_1\beta_n S_{(1^n)} \simeq \omega\beta_{n-2} S_{(1^{n-2})} & * = n-1 \\ 0 & \text{otherwise.} \end{cases}$$

where

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Proposition

$\omega\beta_n S_{(1^n)}$ is the uniserial functor:

$$\begin{array}{c} \alpha S_{(p1^r)} \\ | \\ \alpha S_{((p-1)1^{2+r})} \\ | \\ \cdots \\ | \\ \alpha S_{(31^{n-5})} \\ | \\ \alpha S_{(21^{n-3})} \\ | \\ \alpha S_{(1^n)} \end{array}$$

where $n+1 = 2p + r$ with $r \in \{0, 1\}$.

Example 3: $\lambda = (22)$

$$0 \to \omega\beta_4 S_{(22)} \to \beta_4 S_{(22)} \xrightarrow{\overline{ad}} \beta_3 S_{(21)} \to \omega_1 \beta_4 S_{(22)} \to 0.$$

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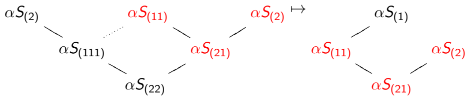

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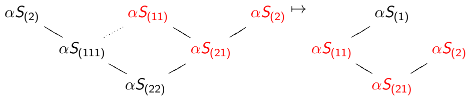

Proposition: [Powell-Vespa]

$$HH_*(B(-);\theta^*\mathrm{Inj}^{\mathbf{Fin}}_{(22)}) \cong \begin{cases} \omega\beta_4 S_{(22)} = \begin{array}{c} \alpha S_{(2)} \\ | \\ \alpha S_{(111)} \\ | \\ \alpha S_{(22)} \end{array} & * = n \\ \omega_1\beta_4 S_{(22)} = \alpha S_{(1)} & * = n-1 \\ 0 & \text{otherwise.} \end{cases}$$

Example 4: $\lambda = (21^{n-1})$

$$0 \to \omega\beta_{n+1}S_{(n1)} \to \beta_{n+1}S_{(n1)} \xrightarrow{\overline{ad}} \beta_n S_{(n)} \oplus \beta_n S_{((n-1)1)} \to \omega_1\beta_{n+1}S_{(n1)} \to 0.$$

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Proposition: [Powell-Vespa]

$$HH_*(B(-);\theta^*\mathrm{Inj}^{\mathbf{Fin}}_{(21^{n-1})}) \cong \begin{cases} \omega\beta_{n+1}S_{(n1)} \simeq \alpha S_{(n1)} & * = n \\ \omega_1\beta_{n+1}S_{(n1)} = 0 & * = n-1 \\ 0 & \text{otherwise.} \end{cases}$$

Theorem of Turchin and Willwacher revisited

$HH_d(B(-); \mathcal{L}(\mathbb{Q}[\epsilon], \mathbb{Q}[\epsilon]))$

Theorem of Turchin and Willwacher revisited

$$HH_d(B(-);\mathcal{L}(\mathbb{Q}[\epsilon],\mathbb{Q}[\epsilon])) \quad \cong \quad \bigoplus_{n\in\mathbb{N}} HH_d(B(-);\theta^*\mathrm{Inj}^{\mathsf{Fin}}_{(n)}) \otimes \mathbf{S}_{(n)}(\mathbb{Q})$$

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&\cong \omega\beta_d S_{(1^d)}\oplus\omega\beta_{d-1}S_{(1^{d-1})}
\end{aligned}
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with

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$HH_d(B(-);\mathcal{L}(A_{\mathbb{Q}^{\oplus 2}}, A_{\mathbb{Q}^{\oplus 2}}))$?

$$HH_*(B(-);\mathcal{L}(A_{\mathbb{Q}^{\oplus 2}}, A_{\mathbb{Q}^{\oplus 2}})) \quad \cong \quad \bigoplus_{n\in\mathbb{N}} \bigoplus_{\substack{\lambda\vdash n \\ l(\lambda)\leq 2}} HH_*(B(-);\theta^*\mathrm{Inj}_\lambda^{\mathbf{Fin}}) \otimes \mathbf{S}_\lambda(V)$$

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$\lambda \vdash n$ such that $l(\lambda) \leq 2$

- $\lambda = (n)$
- $\lambda = ((n-k)k)$ where $2k \leq n$

Work in progress: Representations of $Out(F_n)$

Definition:

A functor $F : \mathbf{gr} \to \mathbb{Q} - Mod$ is generated in cardinality $\leq d$ if it is isomorphic to a quotient of a direct sum of functors of type $P_k = \mathbb{Q}[Hom_{\mathbf{gr}}(\mathbb{Z}^{*k}, -)]$ where $k \leq d$

A polynomial functor of degree d is generated in cardinality $\leq d$.

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If $F \in \mathcal{F}^{Out}(\mathbf{gr}, \mathbb{Q})$ is generated in cardinality $\leq d$ and does not factorize through the abelianization then for $n \geq d + 2$ the representation $Out(\mathbb{Z}^{*n}) \to Aut(F(\mathbb{Z}^{*n}))$ does not factorize through $GL(n, \mathbb{Z})$.

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Example

$Ext^1_{\mathcal{F}^{Out}(\mathbf{gr},\mathbb{Q})}(\alpha S_{(2)}, \alpha S_{(1^3)}) \simeq \mathbb{Q}$
Generator M_3 is generated in cardinality 1
$M_3(\mathbb{Z}^{*3})$ is the representation of $Out(\mathbb{Z}^{*3})$ of dimension 7 of Turchin-Willwacher.

Work in progress: *Ext* computations

Theorem [Vespa 2018]:

$$Ext^*_{\mathcal{F}(\mathbf{gr},\mathbb{Z})}(T^n \circ \mathfrak{a}, T^m \circ \mathfrak{a}) \simeq \begin{cases} \mathbb{Z}[Surj(m,n)] & \text{if } * = m-n \\ 0 & \text{otherwise} \end{cases}$$

$Surj(m,n)$: set of surjections from the set m to the set n
actions of the symmetric groups $\mathfrak{S}_m$ and $\mathfrak{S}_n$ induced by the composition of surjections (up to a sign)

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Open question:

$Ext^*_{\mathcal{F}^{Out}(\mathbf{gr},\mathbb{Q})}(T^n \circ \mathfrak{a}_{\mathbb{Q}}, T^m \circ \mathfrak{a}_{\mathbb{Q}})$ + actions of $\mathfrak{S}_m$ and $\mathfrak{S}_n$?

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Related to Koszul duality (in the sense of Beilinson-Ginzburg-Soergel)

Theorem:

The Koszul dual of $\mathcal{P}ol_{\leq\infty}(\mathbf{gr},\mathbb{Q})$ is $\mathcal{F}(\Omega,\mathbb{Q})$
Ω: category of surjections

$\mathcal{F}(\Omega,\mathbb{Q}) \simeq \mathcal{F}(\Gamma,\mathbb{Q})$ by Dold-Kan type thm of Pirashvili