Space-time localisation for the dynamic Φ^4_3 model

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Joint with A. Moinat

The Φ^4 equation

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$\Rightarrow$ φ no better than $C^{1-\frac{d}{2}-\varepsilon}$.

$\Rightarrow$ φ^3 not defined in $d \geq 2$. Renormalisation necessary.

Recent history (d=3)

Local solutions: $\sim$ '13

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- Gubinelli-Hofmanová: Global solutions on $\mathbb{R}^3$, construction of invariant measure.

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Inspiration

- Otto-W., Otto-Sauer-Smith-W.: Extend theory of regularity structures to Quasilinear equations.

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For $\tau = \mathrm{tree}_1, \mathrm{tree}_2, \mathrm{tree}_3, \mathrm{tree}_4, \mathrm{tree}_5, \mathrm{tree}_6$ assume

$$\sup_{0 \leq t \leq 1} \|\tau(t)\|_{\mathcal{B}_\infty^{\alpha_\tau}} \leq K, \qquad \sup_{0 \leq s < t \leq 1} \frac{\|\mathrm{tree}_3(t) - \mathrm{tree}_3(s)\|_{\mathcal{B}_\infty^{\frac{1}{4}-\varepsilon}}}{|t-s|^{\frac{1}{8}}} \leq K.$$

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$$\|\varphi(t)\|_{\mathcal{B}_{\infty,\infty}^{-\frac{1}{2}-\varepsilon}} \lesssim \max\left\{\frac{1}{\sqrt{t}}, K^\kappa\right\}.$$

The coming down from ∞ property cont'd

Taking moments w.r.t. K

$$\mathbb{E}\Big[\sup_{0<t\le 1}\sup_{X_0}\Big(\sqrt{t}\,\|X(t)\|_{\mathcal{B}_\infty^{-\frac{1}{2}-\varepsilon}}\Big)^p\Big] < \infty.$$

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$\Rightarrow$ uniform-in-t bound on moments.

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$\Rightarrow$ uniform-in-t bound on moments.

$\Rightarrow$ Tightness for Krylov-Bogoliubov approximations of invariant measure

$$\mu_T(A) = \frac{1}{T}\int_0^T \mathbb{P}(X(t)\in A)\,\mathrm{d}t.$$

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- $\|u\|_{P_R}$ = supremum norm over smaller cylinder

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- Bound is uniform over all space-time boundary conditions.

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- As maximum of ηu is bounded, it is bounded everywhere.

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Then

$$\|v\|_{P_R} \leqslant C \max\left\{\frac{1}{R}, [\tau]_{|\tau|}^{\frac{1}{n_\tau(\frac{1}{2}-\epsilon)}}, \tau \in L\right\},$$

- $L = \{\mathord{\uparrow}, \vee, \vee_x, \text{[tree]}, \text{[tree]}, \Psi, \text{[tree]}, \text{[tree]}, \text{[tree]}\}$
- $n_\tau =$ "degree" of Gaussian polynomial,
  e.g. $n_{\uparrow} = 1$, $n_{\vee} = 2$, $n_{\text{[tree]}} = 3$, $n_{\text{[tree]}} = 4 \ldots$.

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- Bounds only depend on realisation of ξ over compact space-time set.
- Allows to construct solutions on full space by compactnes.
- Using exponential integrability of τ we get:

$$\mathbb{E}\exp(\|v\|_{P_{\frac{1}{2}}}^{1-\varepsilon}) < \infty.$$

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More regular situation

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- **Example:** $\xi =$ space-time white noise over $\mathbb{R}_x \times \mathbb{R}_t$. Then $\alpha = \frac{1}{2} - \varepsilon$.

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- Interplay between regularity of ξ and exponent. Ultimately corresponds to stochastic integrability of u.

Optimal integrability

If $[\xi]_\alpha$ has Gaussian tails, then

$$\mathbb{E}\Big(\exp\Big(\frac{1}{C}\|u\|_{P_R}^{2+(m-1)\alpha}\Big)\Big) < \infty. \quad (1)$$

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Special case: $\xi =$ space-time white noise (say over $\mathbb{T}^1$).
(RD) describes reversible Markov process w.r.t. measure

$$\mu(du) \propto \exp\Big(-\frac{1}{2}\int_{\mathbb{T}^1} u^4 dx\Big)\nu(du)$$

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Under μ

$$\mathbb{E}\exp\Big(\frac{1}{C}\|u\|_{L^4}^4\Big) < \infty \qquad \text{and} \qquad \mathbb{E}\exp\Big(\frac{1}{C}[u]_\alpha^2\Big) < \infty \quad \alpha < \frac{1}{2}.$$

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Interpolating $\|u\|_{L^4}$ and $[u]_\alpha$ leads to same result as (1).

Philosophy of proof

Scaling argument (general dimension d)

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Rescaling $\hat{t} = \lambda^2 t$, $\hat{x} = \lambda x$, $\hat{\xi} = \lambda^{\frac{d+2}{2}} \xi$, $\hat{\varphi} = \lambda^{\frac{2-d}{2}} \varphi$ yields

$$(\partial_{\hat{t}} - \Delta)\hat{\varphi} = -\lambda^{4-d}\hat{\varphi}^3 + \hat{\xi}.$$

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Scaling argument (general dimension d)

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Rescaling $\hat{t} = \lambda^2 t$, $\hat{x} = \lambda x$, $\hat{\xi} = \lambda^{\frac{d+2}{2}}\xi$, $\hat{\varphi} = \lambda^{\frac{2-d}{2}}\varphi$ yields

$$(\partial_{\hat{t}} - \Delta)\hat{\varphi} = -\lambda^{4-d}\hat{\varphi}^3 + \hat{\xi}.$$

- **Small scales** $\lambda \to 0$**:** cubic term disappears $\Rightarrow$ Subcriticality of equation.
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- Fix appropriate scale T.
- For large scales $\gg T$ use estimate from smooth case.
- For small scales $\ll T$ use Schauder theory (a.k.a. Regularity Structures).

Large scale bound: Reducing to the smooth case

The equation for v

$v = X - \<1>$.

$$(\partial_t - \Delta) v \;=\; -\big(v^3 + 3 v^2 \<1> + 3 v \<2> + \<3> \big).$$

Large scale bound: Reducing to the smooth case

The equation for v

$v = X - \Psi$.

$$(\partial_t - \Delta)v \;=\; -\big(v^3 + 3v^2\Psi + 3v\Psi^{:2:} + \Psi^{:3:}\big).$$

Regularise

$(\cdot)_T =$ smoothing at scale T.

$$(\partial_t - \Delta)v_T = -(v_T)^3 + \underbrace{\mathrm{Error}(T)}_{I} + \underbrace{3(\Psi\, v^2)_T}_{II} + 3\underbrace{(\Psi^{:2:}\, v)_T}_{III} + \underbrace{\Psi^{:3:}_T}_{IV},$$

where $\mathrm{Error}(T) = [(v_T)^3 - (v^3)_T]$ commutator.

Large scale bound: Reducing to the smooth case

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Regularise

$(\cdot)_T =$ smoothing at scale T.

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where $\mathrm{Error}(T) = [(v_T)^3 - (v^3)_T]$ commutator.

Apply smooth bound

$$\|v_T\|_{P_R} \lesssim \max\left\{\frac{1}{R}, \|I\|_{P_0}^{\frac{1}{3}}, \|II\|_{P_0}^{\frac{1}{3}}, \|III\|_{P_0}^{\frac{1}{3}}, \|IV\|_{P_0}^{\frac{1}{3}}\right\}.$$

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Schauder theory

$$(\partial_t - \Delta) v = -\Big(\underbrace{v^3}_{\in C^{\frac{1}{2}-}} + 3 \underbrace{\dot{|}\, v^2}_{\in C^{-\frac{1}{2}-}} + 3 \underbrace{\ddot{\vee}\, v}_{\in C^{-1-}} + \underbrace{\dddot{\Psi}}_{\in C^{-\frac{3}{2}-}} \Big).$$

Suggests $\|v\|_{C^{\frac{1}{2}-}} \lesssim \|\mathrm{RHS}\|_{C^{-\frac{3}{2}-}}$. Not enough!

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Next order

$$(\partial_t - \Delta)[v {\color{orange} + \Psi}] = -\Big(\underbrace{v^3}_{\in C^{\frac{1}{2}-}} + 3 \underbrace{\mathsf{i}\, v^2}_{\in C^{-\frac{1}{2}-}} + 3 \underbrace{\vee\, v}_{\in C^{-1-}} \Big).$$

Suggests $\|v + \Psi\|_{C^{1-}} \lesssim \|\mathrm{RHS}\|_{C^{-1-}}$. Still not enough!

Freezing coefficients

Fix space-time point x.

$$(\partial_t - \Delta)\underbrace{\left[v + \Psi + 3v(x)\Upsilon\right]}_{=:U(x,\cdot)} = -\big(\underbrace{v^3}_{\in C^{\frac{1}{2}-}} + 3\underbrace{v^2 \,\mathsf{l}}_{\in C^{-\frac{1}{2}-}} + 3(v - v(x))\underbrace{\vee}_{\in C^{-1-}}\big).$$

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Suggests on-diagonal regularity:

$$\sup_{x,y} \frac{|U(x,y) - U(x,x) - \nu(x)(y-x)|}{d(x,y)^{\frac{3}{2}-}} \lesssim \|RHS\|.$$

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- Variant of Hairer's Integration Theorem.
- We use (localised version of) Otto, Sauer, Smith, W.

Bounding the products

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- We write

$$
\begin{aligned}
&(v(y) - v(x)) \vee (y) \\
&= (U(x, y) - U(x, x) - \nu(x)(y - x)) \vee (y)
\end{aligned}
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&- \underbrace{[\Psi(y) - \Psi(x)]\vee(y)}_{=:\Psi\vee} - 3v(x)\underbrace{[\Upsilon(y) - \Upsilon(x)]\vee(y)}_{=:\Upsilon\vee} + \nu(x)(y-x)\vee(y)
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&= (U(x,y) - U(x,x) - \nu(x)(y-x))\curlyvee(y) \\
&- \underbrace{[\Upsilon\!\!\!\!\mid(y) - \Upsilon\!\!\!\!\mid(x)]\curlyvee(y)}_{=:\curlyvee\!\!\!\Upsilon} - 3v(x)\underbrace{[\Upsilon(y) - \Upsilon(x)]\curlyvee(y)}_{=:\dot{\curlyvee}} + \nu(x)(y-x)\curlyvee(y)
\end{aligned}
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- $\curlyvee\!\!\!\Upsilon$, $\dot{\curlyvee}$ and $(y-x)\curlyvee$ are controlled by assumption.

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$$

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- We control $(U(x,y) - U(x,x) - \nu(x)(y-x)$ to order $\frac{3}{2}-$ and $\vee(y)$ to order $-1-$.
- Control of such products is precisely the statement of Hairer's reconstruction theorem.
- We use a variant in spirit of Otto-W. (but without semi-group).

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- Plugging everything in, gives a bound that can be iterated.

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