Exponential Tractability for Weighted Tensor Product Problems

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Based on

Peter Kritzer + H.W.

“Simple characterization of exponential tractability for linear multivariate problems, J. Complexity, 2018

Peter Kritzer, Friedrich Pillichshammer +H.W.

“Exponential tractability for linear weighted tensor products “ (work in progress).

Tractability

Studies complexity of d-variate problems solved to within ε

- Algebraic tractability:

  studied mostly so far and relates complexity to d and ε^{-1}

- Exponential tractability:

  studied more recently and relates complexity to d and $\ln \varepsilon^{-1}$

- Different settings:

  such as worst case, average case,...

- Different types of information:

  arbitrary linear functionals, only function values

Multivariate Problems

$$\mathcal{S} = \{S_d : \mathcal{H}_d \to \mathcal{G}_d\}_{d\in\mathbb{N}}.$$

Here, $\mathcal{H}_d$, $\mathcal{G}_d$ Hilbert spaces, S_d compact linear, $\|S_d\| = 1$,

$$S_d(f) \approx A_{d,n}(f) = \phi_{d,n}(L_1(f), L_2(f), \dots, L_n(f))$$

with $\phi_n : \mathbb{C}^n \to \mathcal{G}_d$, $L_j \in \mathcal{H}_d^*$ can be chosen adaptively.

The worst case setting:

$$e(A_{d,n}) = \sup_{\substack{f\in\mathcal{H}_d \\ \|f\|_{\mathcal{H}_d}\le 1}} \|S_d(f) - A_{d,n}(f)\|_{\mathcal{G}_d}.$$

The nth minimal error

$$e(n, S_d) = \inf_{A_{d,n}} e(A_{d,n}).$$

Information Complexity

Information complexity:

$$n(\varepsilon, S_d) = \min\{n : \ e(n, S_d) \le \varepsilon\},$$

Known:

$$W_d = S_d^* S_d : \mathcal{H}_d \to \mathcal{H}_d.$$

$$W_d \, \eta_{d,j} = \lambda_{d,j} \, \eta_{d,j}$$

$$1 = \lambda_{d,1} \ge \lambda_{d,2} \ge \dots \ge 0 \quad \text{and} \quad \lim_{j \to \infty} \lambda_{d,j} = 0.$$

Then

$$\begin{aligned} n(\varepsilon, S_d) &= \min\{n : \ \lambda_{d,n+1} \le \varepsilon^2\}, \\ A_{d,n}(f) &= \sum_{j=1}^{n} \langle f, \eta_{d,j} \rangle_{\mathcal{H}_d} \, S_d \eta_{d,j} \quad n\text{th optimal}, \end{aligned}$$

To omit the trivial problem $\lambda_{d,2} > 0 \quad \forall d.$

Tractability

We study how $n(\varepsilon, S_d)$ depends on ε and d. We relate:

- Algebraic Tractability (ALG) with respect to (d, ε^{-1}).
- Exponential Tractability (EXP) with respect to $(d, \ln \varepsilon^{-1})$.

ALG : our book

"Tractability of Multivariate Problems" E. Novak and H. W. [2008,2010,2012], and many papers cited there

EXP : many papers after 2012.

Strong Polynomial Tractability

$\mathcal{S}$ is ALG-SPT iff $\exists\ C, p \geq 0$ such that

$$n(\varepsilon, S_d) \leq C(1+\varepsilon^{-1})^p \quad \text{for all} \quad d \in \mathbb{N}, \varepsilon > 0.$$

$$p^{\mathrm{ALG-SPT}} = \text{the infimum of } p \text{ from above}$$

$\mathcal{S}$ is EXP-SPT iff $\exists\ C, p \geq 0$ such that

$$n_{(}\varepsilon, S_d) \leq C(1+\ln(1+\varepsilon^{-1}))^p \quad \text{for all} \quad d \in \mathbb{N}, \varepsilon > 0.$$

$$p^{\mathrm{EXP-SPT}} = \text{the infimum of } p \text{ from above}$$

Table 1: **SPT**

$\mathcal{S}$ ALG-SPT iff

$\exists\ \tau \geq 0$: $\quad \sup_{d\in\mathbb{N}}\ \sum_{j=1}^{\infty} \left(\frac{\lambda_{d,j}}{\lambda_{d,1}}\right)^{\tau} < \infty.$

the exponent $p^{\,\mathrm{ALG-SPT}} = \inf\{2\tau :\ \tau$ from above$\}$.

$\mathcal{S}$ is EXP-SPT iff

$\exists\ \tau \geq 0$: $\quad \sup_{d\in\mathbb{N}}\ \sum_{j=1}^{\infty} \left(\frac{\lambda_{d,j}}{\lambda_{d,1}}\right)^{j^{-\tau}} < \infty.$

the exponent $p^{\,\mathrm{EXP-SPT}} = \inf\{1/\tau :\ \tau$ from above$\}$,

Unweighted Tensor Product Problems

$$S_d = \otimes_{j=1}^d S_1, \qquad F_d = \otimes_{j=1}^d F_1, \qquad G_d = \otimes_{j=1}^d G_1$$

with F_1, G_1 Hilbert spaces, $S_1 : F_1 \to G_1$ linear and compact

$$\{\lambda_j\}_{j=1}^\infty \qquad \textbf{eigenvalues of} \quad S_1^* S_1$$

$$\lambda_1 \ge \lambda_2 \ge \cdots \ge \lambda_j \ge \cdots \ge 0 \qquad \textbf{and} \quad \lambda_1 = 1 \ge \lambda_2 > 0.$$

Then the eigenvalues of $S_d^* S_d$ are

$$\{\lambda_{d,j}\}_{j\in\mathbb{N}} = \{\lambda_{1,j_1}\lambda_{2,j_2}\cdots\lambda_{d,j_d}\}_{(j_1,j_2,\dots,j_d)\in\mathbb{N}^d}\dots$$

Unweighted Tensor Product Problems

ALG-SPT and EXP-SPT never holds
but some weaker notions of tractability do hold.

Why?

It is enough to show that ALG-SPT does not hold. From Table 1 we have

$$\sum_{j=1}^{\infty} \left(\frac{\lambda_{d,j}}{\lambda_{d,1}} \right)^{\tau} \geq (1 + \lambda_2^{\tau})^d$$

and it goes to ∞ exponentially fast in d.

Weighted Tensor Product Problems

So far we had $S_d = \otimes_{j=1}^d S_1, \quad F_d = \otimes_{j=1}^d F_1, \quad G_d = \otimes_{j=1}^d G_1$ **for** F_1, G_1 **Hilbert spaces,** $S_1 : F_1 \to G_1$ **linear, compact and** $S_1^* S_1 \eta_j = \lambda_j \eta_j$ **with orthonormal** η_j**'s, nonincreasing** λ_j **and** $\lambda_1 = 1 \ge \lambda_2 > 0$**.**

Weights:

$$\gamma = \{\gamma_k\} \quad \text{with} \quad 1 \ge \gamma_1 \ge \gamma_2 \ge \cdots \ge 0$$

Weighted Spaces: $F_{1,\gamma_k} \subseteq F_1$ but with a norm

$$f \in F_{1,\gamma_k} \quad \text{iff} \quad \|f\|_{F_{1,\gamma_k}}^2 = |\langle f, \eta_1 \rangle|_{F_1}^2 + \frac{1}{\gamma_k} \sum_{j=2}^{\infty} |\langle f, \eta_j \rangle|^2$$

Note

$$\|f\|_{F_{1,\gamma_k}} \le 1 \quad \text{for small } \gamma_k \quad \Longrightarrow \quad \sum_{j=2}^{\infty} |\langle f, \eta_j \rangle|^2 \quad \text{is small}$$

Hence, γ_k controls the dependence on the kth component.

Weighted Tensor Product Problems

We define $S_{d,\gamma} : \otimes_{k=1}^{d} F_{1,\gamma_k} \to G_d = \otimes_{j=1}^{d} G_1$ as

$$S_{d,\gamma} f = f \quad \forall\, f \in \otimes_{k=1}^{d} F_{1,\gamma_k} \subseteq F_d$$

The eigenvalues of $S_{d,\gamma}^* S_{d,\gamma}$ are now

$$\lambda_{1,j_1} \lambda_{2,j_2} \cdots \lambda_{d,j_d}$$

with

$$\lambda_{k,j} = \begin{cases} \lambda_1 = 1, & \text{if } j = 1, \\ \gamma_k \lambda_j, & \text{if } j \geq 2. \end{cases}$$

For the unweighted case, $\gamma_k = 1$, we have $\lambda_{k,j} = \lambda_j$, as before

Back to SPT

ALG-SPT, i.e., $n(\varepsilon, S_d) \le C(1+\varepsilon^{-1})^p$, holds iff

$$p_\gamma := \inf\{\tau > 0 \,:\, \sum_{k=1}^{\infty} \gamma_k^\tau < \infty\} < \infty$$

$$p_\lambda := \inf\{\tau > 0 \,:\, \sum_{j=1}^{\infty} \lambda_j^\tau < \infty\} < \infty$$

If so then

$$p^{\mathrm{ALG-SPT}} = 2\max(p_\gamma, p_\lambda).$$

see Greg Wasilkowski + H.W. 1999, and Erich Novak + H.W. Theorem 5.7 in Volume 1.

Back to SPT

EXP-SPT, i.e., $n(\varepsilon, S_d) \le C(1+\ln(1+\varepsilon^{-1}))^p$, holds iff

$$\lim_{j\to\infty} \lambda_j = \lim_{k\to\infty} \gamma_k = 0 \quad \text{and} \quad B_{\mathrm{EXP-SPT}} := \limsup_{\varepsilon\to 0} \frac{d(\varepsilon)\ln\, j(\varepsilon)}{\ln\,\ln \varepsilon^{-1}} < \infty,$$

where

$$\begin{aligned} d(\varepsilon) &:= \max\{k\in\mathbb{N}\ :\ \gamma_k > \varepsilon^2\}, \\ j(\varepsilon) &:= \max\{k\in\mathbb{N}\ :\ \lambda_k > \varepsilon^2\}. \end{aligned}$$

If so then

$$p^{\mathrm{EXP-SPT}} = B_{\mathrm{EXP-SPT}}.$$

Examples

For $p_1, p_2 > 0$.

- $\gamma_k = k^{-p_1}, \ \lambda_j = j^{-p_2}$.

  ALG-SPT holds with $p^{\mathrm{ALG-SPT}} = 2\max(p_1^{-1}, p_2^{-1})$.

- $\gamma_k = k^{-p_1}, \ \lambda_j = j^{-p_2}$.

  No EXP-SPT

- $\gamma_k = \exp(-k^{p_1}), \ \lambda_j = \exp(-j^{p_2})$.

  NO EXP-SPT

- $\gamma_k = \exp(-\exp(k^{p_1})), \ \lambda_j = \exp(-\exp(j^{p_2}))$.

  EXP-SPT holds iff $p_1 > 1$. Then $p^{\mathrm{EXP-SPT}} = 1 - p_1^{-1}$.

Further Results for EXP

- EXP-PT means that $n(\varepsilon, S_d) \le C d^{\,q}\,(1+\ln(1+\varepsilon^{-1}))^p$.

  EXP-PT holds iff EXP-SPT holds.

- EXP-QPT means that $n(\varepsilon, S_d) \le C(1+\ln(1+\varepsilon^{-1}))^{t(1+\ln\, d)}$.

  EXP-QPT holds iff

$$\lim_{j\to\infty} \lambda_j = \lim_{k\to\infty} \gamma_k = 0 \;\; B_{\mathrm{EXP-QPT}} := \limsup_{\varepsilon\to 0} \frac{d(\varepsilon)\ln\, j(\varepsilon)}{[\ln\, d(\varepsilon)][\ln\,\ln \varepsilon^{-1}]} < \infty.$$

If so then

$$t^{\mathrm{EXP-QPT}} = \text{the infimum of } t \text{ from above} = B_{\mathrm{EXP-QPT}}.$$

Further Results for EXP

EXP-(s,t)-WT means that $\lim_{d+\varepsilon^{-1}\to\infty} \frac{\ln\, n(\varepsilon,S_d)}{d^{\,t}+(\ln\,\varepsilon^{-1})^s} = 0$

- EXP-$(1,1)$-WT iff

$$\lim_{k\to\infty} \gamma_k = 0 \quad \text{and} \quad \lim_{j\to\infty} \frac{\ln\,\lambda_j^{-1}}{\ln\,j} = \infty.$$

- EXP-$(1,t)$-WT with $t<1$ iff

$$\lim_{k\to\infty} \frac{\ln\,\gamma_k^{-1}}{\ln\,k} = \infty \quad \text{and} \quad \lim_{j\to\infty} \frac{\ln\,\lambda_j^{-1}}{\ln\,j} = \infty.$$

- EXP-$(1,t)$-WT with $t>1$ iff

$$\gamma_k\text{'s arbitrary} \quad \text{and} \lim_{j\to\infty} \frac{\ln\,\lambda_j^{-1}}{\ln\,j} = \infty.$$

That is all folks

Many Thanks for Your Attention