On some lower bounds of Kolmogorov widths

Yu.V. Malykhin

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Definitions

Let X be a linear normed space; $K, Q \subset X$. *Deviation* of K from Q:

$$E(K, Q)_X := \sup_{x \in K} \inf_{y \in Q} \|x - y\|_X.$$

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Kolmogorov n-width of a set K in X:

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We can assume that K is convex and symmetric. By duality, Kolmogorov widths are equivalent to Gelfand widths.
Lower bound $d_n(K,X) \geqslant r$ means that for any system $\{\varphi_j\}_{j=1}^n \subset X$ the set K cannot be approximated by "polynomials": there exists $x \in K$, s.t.

$$\|x - \sum_{j=1}^n c_j\varphi_j\| \geqslant r, \quad \text{for any } c_1,\ldots,c_n \in \mathbb{R}.$$

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$$\|x\|_p := (|x_1|^p + \ldots + |x_N|_p)^{1/p}, \quad 1 \leqslant p < \infty,$$

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- we write $A_n \asymp B_n$, if $cA_n \leqslant B_n \leqslant CA_n$ for some absolute constants $c, C > 0$.

Often it is too difficult to find $d_n(K,X)$ precisely and we are interested in the order of decay of d_n as $n \to \infty$.

Width of octahedra in Euclid space

Theorem (S.B. Stechkin, 1954)

The following equality holds for any $n \leqslant N$:

$$d_n(B_1^N, \ell_2^N) = \sqrt{1 - n/N}.$$

Corollary

- $d_n(W_1^r, L_2) \asymp n^{-r+1/2}$,
- $d_n(W_\infty^r, L_2) \asymp n^{-r}$.

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To prove the lower bound in the corollary, one has to place a $2n$-dimensional octahedron $r \cdot B_1^{2n}$ (i.e., orthogonal system $\varphi_1, \dots, \varphi_{2n}$ with $\|\varphi_j\| = r$) in a functional class K and conclude that

$$d_n(K, L_2) \geqslant r d_n(B_1^{2n}, \ell_2^{2n}) = r/\sqrt{2}.$$

For example,

$$\{n^{-1} \sin kx,\ n^{-1} \cos kx\}_{k=1}^n \subset W_\infty^1 \quad \Longrightarrow \quad d_n(W_\infty^1, L_2) \geqslant c/n.$$

Discretization

Later the method of discretization of widths (reducing widths of functional classes to widths of finite-dimensional sets) appeared in the papers of E.D. Gluskin, R.S. Ismagilov, V.E. Maiorov and others.

Theorem

For any $n \leqslant N$, $1 \leqslant p, q \leqslant \infty$, and $r > 0$,

$$d_n(W_p^r, L_q) \geqslant CN^{-r+1/p-1/q} d_n(B_p^N, \ell_q^N).$$

This gives us optimal lower bounds in many cases (often one can take $N = 2n$).

Widths of Sobolev Classes

The order of decay of $d_n(W^r_p, L_q)$ as $n \to \infty$, is known for all $r \in \mathbb{N}$ and $1 \leqslant p, q \leqslant \infty$ (V.M. Tikhomirov, B.S. Kashin, E.D. Gluskin, R.S. Ismagilov and others), except $r = 1$, $p = 1$, $q \in (2, \infty)$.

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The answer: let $\mathcal{T}_n = \mathrm{span}\{\exp(ikx) \colon |k| \leqslant n\}$,

$$d_n(W^r_p, L_q) \asymp \begin{cases} E(W^r_p, \mathcal{T}_n)_q \asymp n^{-r+(1/p-1/q)_+}, & p \geqslant q \text{ or } q \leqslant 2, \\ n^{-r}, & q \geqslant p \geqslant 2, \\ n^{-r+(1/p-1/2)}, & p \leqslant 2 \leqslant q,\ rp > 1. \end{cases}$$

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- first it was found by Ismagilov, $d_n(W^2_1, L_\infty)$ (he disproved the common belief that trigonometric polynomials always give optimal approximation);
- the key role plays the upper bound for the case $p = 2$, $q = \infty$, done by Kashin (1977); the optimal approximation subspaces are random (related to RIP).

The W_1^1 case.

In the case of W_1^1 it was known that (E.D. Kulanin):

$$c(q,\varepsilon)n^{-1/2}\log^{1/2-\varepsilon} n \leqslant d_n(W_1^1, L_q) \leqslant C(q)n^{-1/2}\log n.$$

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The asymptotics is known for the special class of n-dimensional approximation subspaces $Q_n = \mathrm{span}\{\exp(im_jx)\}_{j=1}^n$ (Ed. Belinsky, 1987):

$$d_n^T(W_1^1, L_q) \asymp n^{-1/2}\log n, \quad q \in (2,\infty).$$

Besov classes

Sobolev spaces are strongly connected with Besov spaces. Let us give a definition.
Parameters: $r > 0$, $p, \theta \in [1, \infty]$; $I \subset \mathbb{R}$. Take $s = 1 + \lfloor r \rfloor$. Let $f\colon I \to \mathbb{R}$.

$$|f|_{B^r_{p,\theta}(I)} := \begin{cases} \int_0^\infty (t^{-r}\omega_s(f,t)_p^\theta)^{1/\theta}\frac{dt}{t}, & 1 \leqslant \theta < \infty, \\ \sup_{t>0} t^{-r}\omega_s(f,t)_p, & \theta = \infty. \end{cases}$$

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Norm in the Besov space $\mathcal{B}^r_{p,\theta}(I)$: $\|f\|_{L_p} + |f|_{B^r_{p,\theta}(I)}$.
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Besov class $B^r_{p,\theta}(I)$ is a unit ball in that space.
The following embeddings hold:

$$\mathcal{B}^r_{p,\min(p,2)} \hookrightarrow \mathcal{W}^r_p \hookrightarrow \mathcal{B}^r_{p,\max(p,2)}, \quad 1 < p < \infty,$$

$$\mathcal{B}^r_{p,1} \hookrightarrow \mathcal{W}^r_p \hookrightarrow \mathcal{B}^r_{p,\infty}, \quad p = 1 \text{ or } p = \infty.$$

Widths of $B^1_{1,\theta}$

The space embeddings imply the following inclusions:

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Let $q \in (2,\infty)$, $1 \leqslant \theta \leqslant \infty$; set $\gamma := \frac{1}{2} + (\frac{1}{2} - \frac{1}{\theta})_+$. Then for any $\varepsilon > 0$ there exist $c_{q,\theta,\varepsilon} > 0$, $C_{q,\theta} > 0$, such that for $n = 2, 3, \ldots$ the following inequalities hold:

$$c_{q,\theta,\varepsilon} n^{-\frac{1}{2}} \log^{\gamma-\varepsilon} n \leqslant d_n(B^1_{1,\theta}[0,1], L_q[0,1]) \leqslant C_{q,\theta} n^{-\frac{1}{2}} \log^{\gamma} n.$$

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This gives us no new information about widths of W^1_1 :(One of the problems is that the Sobolev space $\mathcal{W}^1_1$, unlike Besov spaces, does not admit an unconditional basis!

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Let ψ be a wavelet with scaling function φ, such that ψ and φ have compact support and $\psi, \varphi \in C^{r_0}(\mathbb{R})$ for some $r_0 \in \mathbb{N}$. Denote $\psi_{k,j}(x) := 2^{k/2}\psi(2^k x - j)$, $\varphi_{0,j}(x) := \varphi(x - j)$ for $k, j \in \mathbb{Z}$.

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Theorem (See Y. Meyer, H. Triebel, ...)

Let $0 < r < r_0$; $p, \theta \in [1, \infty]$. Then $f \in \mathcal{B}^r_{p,\theta}(\mathbb{R})$ if and only if

$$\|(\langle f, \varphi_{0,j}\rangle)_{j\in\mathbb{Z}}\|_{\ell_p} + \|(\langle f, \psi_{k,j}\rangle)_{k\geqslant 0, j\in\mathbb{Z}}\|_{b^\sigma_{p,\theta}} < \infty, \quad \sigma := r + \frac{1}{2} - \frac{1}{p}.$$

Here the $b^\sigma_{p,\theta}$ is the discrete norm

$$\|(\lambda_{k,j})_{k\geqslant 0, j\in\mathbb{Z}}\|_{b^\sigma_{p,\theta}} = \begin{cases} \left\{\sum_{k\geqslant 0} 2^{k\sigma\theta}\|(\lambda_{k,j})_{j\in\mathbb{Z}}\|^\theta_{\ell_p}\right\}^{1/\theta}, & \theta < \infty, \\ \sup_{k\geqslant 0} 2^{k\sigma}\|(\lambda_{k,j})_{j\in\mathbb{Z}}\|_{\ell_p}, & \theta = \infty. \end{cases}$$

This allows us to discretize the sets $B^1_{1,\theta}$ using the coordinates

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$$d_n(B^1_{1,\theta}[0,1], L_q) \gg d_n(B_\theta\{\ell_1^1, \ell_1^2, \ldots, \ell_1^{2^k}, \ldots\}, \ell_{q,w}),$$

where

$$B_\theta\{\ldots\} = \{x\colon \sum_{k=0}^{\infty}(\sum_{j=1}^{2^k}|x_{k,j}|)^\theta \leqslant 1\},$$

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For $\theta = \infty$ we get the product of octahedra:

$$B_\infty\{\ldots\} = B_1^1 \times B_1^2 \times \ldots \times B_1^{2^k} \times \ldots$$

Widths of a product of octahedra, $q > 2$

Consider a weighted ℓ_q-norm in $\mathbb{R}^N$: $\|x\|_{\ell_{q,w}^N} := \left\{\sum_{i=1}^N w_i|x_i|^q\right\}^{1/q}$.

Suppose that the coordinates in $\mathbb{R}^N$ are split into k blocks: $\{1,\ldots,N\} = \sqcup_{s=1}^k \Delta_s$, $|\Delta_s| = N_s$.

In each block we take the octahedron $B_1^{N_s} = \{x \in \mathbb{R}^{N_s} \colon \sum_{i\in\Delta_s} |x_i| \leqslant 1\}$.

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Theorem (M.)

Let $q \in (2, \infty)$, $n \in \mathbb{N}$, and the conditions hold:

1) *the sums of weights in each block are equal,*
2) $\max\limits_{1 \leqslant i \leqslant N} w_i \leqslant (4n)^{-1} \sum_{i=1}^N w_i$,
3) $n > Ck \ln(k+1)$ *with an absolute constant* $C > 0$.

Then: $d_n(\prod_{s=1}^k B_1^{N_s}, \ell_{q,w}^N) \geqslant \min(c_q \sqrt{\frac{k}{n}} (\sum_{i=1}^N w_i)^{1/q}, \frac{1}{2} \min_i w_i^{1/q})$.

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The theorem generalizes the result of Kashin (1980):

$$d_n(B_1^N, \ell_q^N) \asymp \min(n^{-1/2}N^{1/q}, 1), \quad n \leqslant N/2,\ q \in (2,\infty).$$

Idea of the proof: take random point $x \in \mathrm{extr} B_1^N$ and it's approximation $y(x) \in L_n$. Project the difference $x - y(x)$ on the set Ω of random $2n$ coordinates, bound ℓ_q-norm in terms of ℓ_2-norm and use that B_1^{2n} is badly approximated in average:

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For products of octahedra, the principle is the same. One should:

- make the approximation $y(x)$ satisfy the additional property: $y(x)_i = x_i$ for $x \in \mathrm{extr}(B_1^{N_1} \times \ldots \times B_k^{N_k})$ and $i \in \mathrm{supp}(x)$,
- take each $i \in \Omega$ with probability, proportional to the weight w_i,
- take care about intersections $\Omega \cap \Delta_s$ for each block: it is required that $|\Omega \cap \Delta_s| \approx 2n/k$.

Mixed norms

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Consider the space $\mathbb{R}^N$ with $N = mk$. The set of coordinates $\{1, \ldots, N\}$ is split into k blocks $\Delta_1, \ldots, \Delta_k$ of cardinality n.
We denote the restriction of a vector $x \in \mathbb{R}^N$ to the s-th block as $x[s]$.
We equip $\mathbb{R}^N$ with mixed norms

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Let $B_p^k(\ell_q^m)$ be the unit ball of the space $\ell_p^k(\ell_q^m)$.

So, $B_\infty^k(\ell_1^m) = B_1^m \times \ldots \times B_1^m$ (product of k octahedra).

Widths of a product of octahedra, $q \leqslant 2$

Galeev had proved the following inequality:

$$d_{N/2}(B_\infty^k(\ell_1^m)), \ell_q^k(\ell_2^m)) \geqslant c_q k^{1/q}, \quad 1 < q \leqslant 2.$$

(It means lack of non-trivial approximation, because $d_n(B_\infty^k(\ell_1^m), \ell_q^k(\ell_2^m)) \leqslant k^{1/q}$ for all n and q.)

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For all $k, m \in \mathbb{N}$ we have: $d_{N/2}(B^k_\infty(\ell^m_1), \ell^k_1(\ell^m_2)) \geqslant ck.$

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This theorem may be used, e.g., to get lower bounds for linear widths of Hölder–Nikolskii and Besov classes of several variables.
There are some other results on widths with mixed norms (A.A. Vasilyeva, T. Ullrich and others).

Thank you for your attention!