Order statistics and Mallat–Zeitouni problem

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based on a recent joint work with

K. Tikhomirov

and on a series of earlier works with

Y. Gordon, C. Schütt and E. Werner

INI, Cambridge, 2019

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Given a sequence of real numbers $a_1, \ldots, a_n$ denote the k-th smallest one by

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Given a sequence of random variables $\{\xi_i\}_{i\le n}$ the sequence of of order statistics is

$$\{k\text{-}\min_{1\le i\le n} \xi_i\}_{k\le n}.$$

Order statistics in Asymptotic Geometric Analysis

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The norm $\|x\| = \sum_{j=1}^k j\text{-}\max |x_i|$, and thus, the expectation $\mathbb{E} \sum_{j=1}^k j\text{-}\max |g_i|$,

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In my work with Gordon, Schütt, and Werner, we studied the following norms (for a given fixed sequence $(a_1, ..., a_N)$ in $\mathbb{R}^n$):

$$\|x\|_{kq} = \Big(\sum_{j=1}^k j\text{-}\max|\langle x, a_i\rangle|^q\Big)^{1/q}.$$

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In all such examples **maximal** order statistics appear naturally.

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Conjecture 1 (Mallat-Zeitouni, 2000).

Let $X = (X_1, \ldots, X_n)$ be an n-dimensional random Gaussian vector with independent centered coordinates (with possibly different variances). Let T be an orthogonal transformation of $\mathbb{R}^n$ and $Y = TX$. Then every $k \leq n$,

$$\mathbb{E} \sum_{j=1}^{k} j\text{-}\min_{i \leq n} X_i{}^2 \leq \mathbb{E} \sum_{j=1}^{k} j\text{-}\min_{i \leq n} Y_i{}^2.$$

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Our main result: this conjecture holds up to an absolute positive constant C, namely

$$\mathbb{E} \sum_{j=1}^{k} j\text{-}\min_{i \le n} X_i{}^2 \le C\, \mathbb{E} \sum_{j=1}^{k} j\text{-}\min_{i \le n} Y_i{}^2 .$$

A stronger conjecture

In their work, Mallat-Zeitouni showed that Conjecture 1 would follow from

Conjecture 2 (Mallat-Zeitouni, 2000).

Let $\{g_i\}_{i\le n}$, $\{h_i\}_{i\le n}$ be sequences of $\mathcal{N}(0,1)$ random variables such that g_i's are independent. Then for every $x\in\mathbb{R}^n$ and every $k\le n$,

$$\mathbb{E}\sum_{j=1}^{k} j\text{-}\min_{1\le i\le n}|x_i g_i|^2 \le \mathbb{E}\sum_{j=1}^{k} j\text{-}\min_{1\le i\le n}|x_i h_i|^2.$$

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Šidák (1967), Gluskin (1989): $\forall p > 0 \quad \mathbb{E}\max_{1\le i\le n} |x_i g_i|^p \ge \mathbb{E}\max_{1\le i\le n} |x_i h_i|^p.$

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Thus, both conjectures hold for $k = n-1$ (then $m = 1$).

Known results

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Theorem 3 (Gordon, L, Schütt, Werner, 2002).

Let $f_1, \dots, f_n$ be independent copies of a random variable f with $\mathbb{E}|f| < \infty$. Let $h_1, \dots, h_n$ be copies of f. Let $x \in \mathbb{R}^n$. Then

$$26\,\mathbb{E}\sum_{j=1}^{m} j\text{-}\max_{1\le i\le n} |x_i f_i| \ge \mathbb{E}\sum_{j=1}^{m} j\text{-}\max_{1\le i\le n} |x_i h_i|.$$

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What to do with smallest order statistics?

It turns out that it is easier to work with individual statistics than with sums.

Known results

We say that a random variable f satisfies (α, β)-condition if every $t > 0$

$$\mathbb{P}\left(|f| \le t\right) \le \alpha t \quad \text{and} \quad \mathbb{P}\left(|f| \ge t\right) \le \exp(-\beta t).$$

Theorem 4 (Gordon, L, Schütt, Werner, 2005, 2006).

Let f_i's be independent copies of a random variable f satisfying (α, β)-condition. Let $p > 0$. Then for every $0 < x_1 \le x_2 \le \cdots \le x_n$,

$$\frac{1}{6\alpha}\left(\frac{6}{7}\right)^{1/p} \max_{1\le j\le k} \frac{k+1-j}{\sum_{i=j}^n 1/x_i} \le \left(\mathbb{E}\ k\text{-}\min_{1\le i\le n} |x_i f_i|^p\right)^{1/p}$$

$$\le \frac{6}{\beta}\ \max\{p, \ln(k+1)\}\ \max_{1\le j\le k} \frac{k+1-j}{\sum_{i=j}^n 1/x_i}.$$

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New results, sums

Using GLSW technique, one can prove

Theorem 5 (LT 2017).

Let f_i's be independent copies of a random variable f satisfying (α, β)-condition. Let $0 < x_1 \leq x_2 \leq \cdots \leq x_n$. Denote

$$b_j := \sum_{i=j}^{n} 1/x_i \quad \text{and} \quad G(x) := \sum_{j=1}^{k} \frac{(k-j+1)^p}{b_j^p}.$$

Then

$$\frac{1}{2}\left(\frac{1}{16\alpha}\right)^p G(x) \leq \mathbb{E} \sum_{j=1}^{k} j\text{-}\min_{1\leq i\leq n} |x_i f_i|^p \leq 3\left(\frac{4}{\beta}\right)^p \Gamma(1+p)\; G(x).$$

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Thus, it remains to obtain a lower bound in the case of dependent random variables.

Low bound

We need another condition. We say that the distribution F of a non-negative r.v. satisfies (A, δ)-condition for $A > 1$, $\delta \in (0, 1)$ if

$$F(t) \geq 2F(t/A) \qquad \text{whenever} \qquad F(t) \leq \delta.$$

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Theorem 6 (LT 2017).

Let $\alpha > 0$, $\delta \in (0,1)$, $A > 1$, $1 \leq k \leq n$ and $0 < x_1 \leq \ldots \leq x_n$. For $j \leq n$, set $b_j := \sum_{i=j}^{n} 1/x_i$. Further, let ξ_i, $i \leq n$, be (possibly dependent) random variables satisfying the α-condition and (A, δ)-condition. Then

$$\operatorname{Med}\left(k\text{-}\min_{1\leq i\leq n} |x_i \xi_i|\right) \geq \frac{\delta}{2A\alpha} \max_{1\leq j\leq k} \frac{k-j+1}{b_j}.$$

Low bound

We need another condition. We say that the distribution F of a non-negative r.v. satisfies (A, δ)-condition for $A > 1$, $\delta \in (0,1)$ if

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Recall that Theorem 5 says that if f_i's are independent copies of a random variable f satisfying (α, β)-condition Then

$$\mathbb{E} \sum_{j=1}^{k} j\text{-}\min_{1 \leq i \leq n} |x_i f_i|^p \approx \sum_{j=1}^{k} \frac{(k-j+1)^p}{b_j^p}.$$

Main results (comparison for k-minima)

Theorem 7 (LT 2017).

If f satisfies (α, β)-condition and (A, δ)-condition, f_i's are independent copies of f, h_i's are (dependent) copies of a random variable f then for all $k \le n$ and $p > 0$,

$$\mathbb{E} \sum_{j=1}^{k} j\text{-}\min_{1 \le i \le n} |x_i f_i|^p \le 6 \left(\frac{32 A \alpha}{\delta \beta} \right)^p \Gamma(1+p)\ \mathbb{E} \sum_{j=1}^{k} j\text{-}\min_{1 \le i \le n} |x_i h_i|^p$$

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Theorem 8 (LT 2017).

Under assumptions of Theorem 7,

$$\left(\mathbb{E}\, k\text{-}\min_{1 \le i \le n} |x_i f_i|^p \right)^{1/p} \le C(A, \delta, \alpha, \beta)\ \max\{p, \ln(k+1)\} \left(\mathbb{E}\, k\text{-}\min_{1 \le i \le n} |x_i h_i|^p \right)^{1/p}.$$

Logarithmic term

Do we need the logarithmic term in

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Example. Let

$$x_1 = x_2 = \cdots = x_k = 1 \qquad \text{and} \qquad x_{k+1} = x_{k+2} = \cdots = x_n = n^2.$$

Let f be the exponential random variable, that is

$$f \ge 0 \qquad \text{and} \qquad \mathbb{P}(f > t) = e^{-t}.$$

Let $f_1, f_2, \dots, f_n$ be the independent copies of f and $h_1 = h_2 = \cdots = h_n = f$. Then direct calculations show

$$\mathbb{E}\, k\text{-}\min_{1\le i\le n} |x_i f_i| \approx \mathbb{E} \max_{1\le i\le k} |f_i| \approx \ln(k+1)$$

while

$$\mathbb{E}\, k\text{-}\min_{1\le i\le n} |x_i h_i| = \mathbb{E} f = 1.$$

Remark: Gaussian case

Let $(\xi_1, ..., \xi_n)$ be a Gaussian vector with $\xi_i \sim \mathcal{N}(0,1)$. We want to minimize

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It is natural to conjecture that the minimum attains when for all $i \neq j$,

$$\mathbb{E}\xi_i\xi_j = -\frac{1}{n-1},$$

that is, when $\xi_1, ..., \xi_n$ form a vertex set of the regular simplex in L_2.

Some ideas used in the lower bound

The proof is partially modelled on the case of uniformly distributed on $[0, 1]$ random variables.

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Let ξ be a r.v. with distribution $F(t) = F_\xi(t) = \mathbb{P}(\xi \leq t)$. The quantile of order $r \in [0, 1]$ is a number $q(r) = q_F(r) = q_\xi(r)$ satisfying

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The following claim provides simple lower bounds on quantiles.

Claim. *Let $k \le n$ and $0 < x_1 \le \ldots \le x_n$. For $j \le n$, set $b_j := \sum_{i=j}^n 1/x_i$. Let ξ_i, $i \le n$, be (possibly dependent) random variables satisfying the α-condition for some $\alpha > 0$, and let F_i, $i \le n$, be the distributions of $|x_i \xi_i|$. Denote*

$$F := \frac{1}{n} \sum_{i=1}^n F_i \qquad \textit{and} \qquad q := q_F\left(\frac{k - 1/2}{n}\right).$$

Then

$$q \ge \frac{1}{2\alpha} \max_{1 \le j \le k} \frac{k - j + 1}{b_j}.$$

Some ideas used in the lower bound

We show that under (A, δ)-condition, denoting

$$t_0 := \min_{i \le n} \sup\{t > 0 \,:\, F_{|\xi_i|}(t) \le \delta\}, \quad \eta_i := \min\{|\xi_i|, t_0\},\ i \le n,$$

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$$F = \frac{1}{n} \sum_{i=1}^{n} F_{x_i \eta_i},$$

one has

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Truncation is needed:

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while

$$q_G\left(\frac{k-1/2}{n}\right) \approx \text{Med}\left(k\text{-}\min_{1\le i\le n}|x_i g_i|\right) \approx \text{Med}\left(\max_{1\le i\le k}|g_i|\right) \approx \sqrt{\ln k}.$$

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