The Haar system and Besov-type spaces

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1. The Haar wavelet system

$\varepsilon = (\varepsilon_1, \ldots, \varepsilon_d)$, $\varepsilon_i \in \{0,1\}$,

$$h_\varepsilon(x) := \left(\prod_{\{i:\, \varepsilon_i=0\}} \widetilde{\mathcal{X}}(x_i)\right) \left(\prod_{\{i:\, \varepsilon_i=1\}} \widetilde{h}(x_i)\right), \qquad x = (x_1, \ldots, x_d) \in \mathbb{R}^d .$$

$\varepsilon = (0, \ldots, 0)$: $\quad \mathcal{X} := h_{(0,\ldots,0)}$.
$2^d - 1$ generators $h_1, \ldots, h_{2^d-1}$.

$$\mathcal{X}_{j,k} := 2^{jd/2}\, \mathcal{X}(2^j \cdot -k), \qquad h_{i,j,k} := 2^{jd/2}\, h_i(2^j \cdot -k), \quad j \in \mathbb{N}_0,\ k \in \mathbb{Z}^d.$$

We shall call

$$\left\{\mathcal{X}_{0,k},\, h_{i,j,k} :\ i \in \{1, \ldots, 2^d - 1\},\ j \in \mathbb{N}_0,\ k \in \mathbb{Z}^d\right\}$$

Haar wavelet system in $\mathbb{R}^d$.

2. Nikol'skij-Besov spaces

$0 < p, q \leq \infty$, $s \in \mathbb{R}$: $\quad B^s_{p,q}(\mathbb{R}^d)$
(Fourier analysis, atoms, molecules, wavelets, ...)

$$\sigma_p := d \max\left(0, \frac{1}{p} - 1\right)$$

$\underline{\sigma_p < s < 1}$:

A function $f \in L_{\max(1,p)}(\mathbb{R}^d)$ belongs to $B^s_{p,q}(\mathbb{R}^d)$ if

$$\Bigg(\int_{\substack{h \in \mathbb{R}^d \\ 0<|h|<1}} \Big(|h|^{-s} \, \| \underbrace{f(\,\cdot - h) - f(\,\cdot\,)}_{\Delta^1_h f(\,\cdot\,)} \, | L_p(\mathbb{R}^d) \| \Big)^q \frac{dh}{|h|^d} \Bigg)^{1/q} < \infty \, .$$

One example

$$\mathcal{X} \in B^{1/p}_{p,\infty}(\mathbb{R}^d) \qquad \text{for all} \quad p$$

and

$$\mathcal{X} \notin B^{1/p}_{p,q}(\mathbb{R}^d) \qquad \text{for all} \quad q, \quad 0 < q < \infty\,.$$

Besov spaces with negative smoothness

- Fourier analytic definition;
- characterisation by smooth wavelets or atoms;
- duality.

Let $1 < p, q < \infty$. Then

$$\Big(B^s_{p,q}(\mathbb{R}^d)\Big)' = B^{-s}_{p',q'}(\mathbb{R}^d)\,.$$

3. The Haar wavelet and Besov spaces

- Triebel 1973 ($s > 0$), 1978, 2010, 2013;
- Ciesielski 1975 (Sobolev spaces, splines);
- Ropela 1976 ($s > 0$);
- Oswald 1979, 2018 (limiting cases);
- Garrigos, Seeger, T. Ullrich 2015-2019.

Under which conditions we can characterize Besov spaces in terms of the Fourier-Haar coefficients ?

$$h_{i,-1,m} := \begin{cases} \mathcal{X}_{0,m} & \text{if} \quad i = 1,\ m \in \mathbb{Z}^d\,, \\ 0 & \text{otherwise.} \end{cases}$$

$$\|\mu\|_{b^s_{p,q}(\mathbb{R}^d)} := \left\{ \sum_{j=-1}^{\infty} 2^{j(s+\frac{d}{2}-\frac{d}{p})q} \sum_{i=1}^{2^d-1} \left[\sum_{k\in\mathbb{Z}^d} |\mu_{i,j,k}|^p \right]^{\frac{q}{p}} \right\}^{\frac{1}{q}} .$$

Theorem

Let $0 < p, q \leq \infty$ *and*

$$\max\left(\frac{1}{p} - 1, \frac{d}{p} - d\right) < s < \min\left(1, \frac{1}{p}\right).$$

Let $f \in \mathcal{S}'(\mathbb{R}^d)$. *Then* $f \in B^s_{p,q}(\mathbb{R}^d)$ *if and only if* f *can be represented in* $\mathcal{S}'(\mathbb{R}^d)$ *as*

$$f = \sum_{i=1}^{2^d-1} \sum_{j=-1}^{\infty} \sum_{k \in \mathbb{Z}^d} \langle f, \, h_{i,j,k} \rangle \, h_{i,j,k}$$

(with convergence in $B^{s-\varepsilon}_{p,q}(\mathbb{R}^d)$ *for all* $\varepsilon \in (0, \infty)$*) s. t.*

$$\|\mu(f)\|_{b^s_{p,q}(\mathbb{R}^d)} := \| \{\langle f, \, h_{i,j,k}\rangle\}_{i,j,k}\|_{b^s_{p,q}(\mathbb{R}^d)} < \infty \, .$$

The mapping

$$J : \; f \mapsto \{\langle f, \, h_{i,j,k}\rangle\}_{i,j,k}$$

is an isomorphic map of $B^s_{p,q}(\mathbb{R}^d)$ *onto* $b^s_{p,q}(\mathbb{R}^d)$.

• Triebel 1978 (2010)

Necessary conditions

$$f = \sum_{i=1}^{2^d-1} \sum_{j=-1}^{\infty} \sum_{k\in\mathbb{Z}^d} \langle f, h_{i,j,k}\rangle \, h_{i,j,k}$$

• Convergence in $B^{s-\varepsilon}_{p,q}(\mathbb{R}^d)$ for all $\varepsilon > 0$:

$$\mathcal{X}_{0,k}, h_{i,j,k} \in B^s_{p,q}(\mathbb{R}^d) \quad \Longleftrightarrow \quad \mathcal{X} \in B^s_{p,q}(\mathbb{R}^d)$$

• Oswald (2018), Garrigos, Seeger, T. Ullrich (2019):

Let $s = 1$, $d/(d+1) \le p < 1$ and $0 < q < \infty$.
Then the Fourier-Haar series of the function $f(x) := x_1 + \ldots + x_d$ does not converge to f in the norm of $B^1_{p,q}([0,1]^d)$.

$$s \le \min(1, 1/p)$$

Under which conditions the mappings

$$f \mapsto \langle f, \mathcal{X}_{0,k} \rangle \, , \qquad f \mapsto \langle f, h_{i,j,k} \rangle$$

are meaningful ?

$$B^s_{p,q}(\mathbb{R}^d) \hookrightarrow L_1(\mathbb{R}^d) : \quad |\langle f, \mathcal{X} \rangle| \le \| f | L_1(\mathbb{R}^d) \|$$

The Dirac δ distribution: $\quad \langle \delta, \mathcal{X} \rangle = ?$

Let $\psi \in C_0^\infty(\mathbb{R}^d)$, $\int \psi(x)\, dx = 1$. Then for all continuous functions f we have

$$\lim_{n \to \infty} n^d \int \psi(nx)\, f(x)\, dx = f(0) \, .$$

$$\lim_{n \to \infty} n^d \, \psi(n \, \cdot \,) = \delta$$

Case 1: $\operatorname{supp}\psi \subset \{x : \quad x_i > 0 \ \ \forall\, i\}$.

$$\lim_{n\to\infty} n^d \int \psi(nx)\, \mathcal{X}(x)\, dx = 1\,.$$

Case 2: $\operatorname{supp}\psi \subset \{x : \quad x_i < 0 \ \ \forall\, i\}$.

$$\lim_{n\to\infty} n^d \int \psi(nx)\, \mathcal{X}(x)\, dx = 0\,.$$

If $\delta \in B^s_{p,q}(\mathbb{R}^d)$, then $f \mapsto \langle f, \mathcal{X}\rangle$ can not extended from $B^s_{p,q}(\mathbb{R}^d) \cap L_1(\mathbb{R}^d)$ to $B^s_{p,q}(\mathbb{R}^d)$ as a linear and continuous functional.

- $\delta \in B^{\frac{d}{p}-d}_{p,\infty}(\mathbb{R}^d)$.
- $0 < p \le 1$: $B^{\frac{d}{p}-d}_{p,1}(\mathbb{R}^d) \hookrightarrow L_1(\mathbb{R}^d)$.

Theorem

Let $0 < p \leq 1$, $0 < q \leq \infty$ and $s \in \mathbb{R}$. The mapping $f \mapsto \langle f, \mathcal{X} \rangle$ has an extension from $B^s_{p,q}(\mathbb{R}^d) \cap L_1(\mathbb{R}^d)$ as a linear and continuous functional on $B^s_{p,q}(\mathbb{R}^d)$ if and only if either

$$s > \frac{d}{p} - d \qquad \text{and} \qquad 0 < q \leq \infty$$

or

$$s = \frac{d}{p} - d \qquad \text{and} \qquad 0 < q \leq 1.$$

Theorem

Let $1 \le p \le \infty$, $q \in (0, \infty]$ *and* $s \in \mathbb{R}$.
The mapping $f \mapsto \langle f, \mathcal{X} \rangle$ *has an extension from* $B^s_{p,q}(\mathbb{R}^d) \cap L_1(\mathbb{R}^d)$ *as a linear and continuous functional on* $B^s_{p,q}(\mathbb{R}^d)$ *if and only if either*

$$s = \frac{1}{p} - 1 \qquad \text{and} \qquad 0 < q \le 1$$

or

$$s > \frac{1}{p} - 1 \qquad \text{and} \qquad 0 < q \le \infty\,.$$

- Oswald (2018);
- S., Yang, Yuan (2018);
- Garrigos, Seeger, T. Ullrich (2019).

Summary:

$0 < p, q \le \infty$, $s \in \mathbb{R}$:

$$\max\left(\frac{1}{p} - 1,\ \frac{d}{p} - d\right) \le s \le \min\left(1, \frac{1}{p}\right)$$

$\sigma_p < s < 1$: A function $f \in L_p(\mathbb{R}^d)$ belongs to $B^s_{p,p}(\mathbb{R}^d)$, if

$$\Big(\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{|f(x)-f(y)|^p}{|x-y|^{d+sp}}\,dx\,dy\Big)^{1/p} < \infty\,.$$

$\sigma_p < s < 1$, $0 \le \tau \le 1/p$: A function f belongs to $B^{s,\tau}_{p,p}(\mathbb{R}^d)$, if

$$\sup_{|Q|\ge 1} \frac{1}{|Q|^\tau}\Big(\int_Q |f(x)|^p dx\Big)^{1/p} < \infty$$

and

$$\sup_{Q} \frac{1}{|Q|^\tau}\Big(\int_Q\int_Q \frac{|f(x)-f(y)|^p}{|x-y|^{d+sp}}\,dx\,dy\Big)^{1/p} < \infty\,.$$

The *Besov-type space* $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ is defined as the space of all $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f\|_{B^{s,\tau}_{p,q}(\mathbb{R}^d)} :=$$

$$\sup_{Q\in\mathcal{Q}} \frac{1}{|Q|^\tau} \left\{ \sum_{j=\max\{j_Q,0\}}^{\infty} 2^{jsq} \left[\int_Q |\mathcal{F}^{-1}[\varphi_j\, \mathcal{F}f](x)|^p\, dx \right]^{q/p} \right\}^{1/q} < \infty$$

with the usual modifications made in case $p = \infty$ and/or $q = \infty$.
Q dyadic cube in $\mathbb{R}^d$ with side-length $\ell(Q)$:

$$j_Q := -\log_2 \ell(Q)\,.$$

- El Baraka (2002)
- D. Yang, W. Yuan (2008)

- $\mathcal{N}^s_{u,p,q}(\mathbb{R}^d)$, $\mathcal{E}^s_{u,p,q}(\mathbb{R}^d)$, $F^{s,\tau}_{p,q}(\mathbb{R}^d)$

Why ?

H. Kozono and M. Yamazaki, Semilinear heat equations and the Navier-Stokes equation with distributions in new function spaces as initial data, Comm. Par. Diff. Equ. **19** (1994), 959-1014.

Properties

- $B^{s,0}_{p,q}(\mathbb{R}^d) = B^{s}_{p,q}(\mathbb{R}^d)$;
- $B^{s,1/p}_{p,p}(\mathbb{R}^d) = F^{s}_{\infty,p}(\mathbb{R}^d)$ Frazier, Jawerth (1990).
- If $\tau > 0$, then $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ is nonseparable.

Influence of the Morrey parameter τ: $\tau > 0$

$$B^{s,\tau}_{p,q}(\mathbb{R}^d) \hookrightarrow C_{ub}(\mathbb{R}^d) \qquad \Longleftrightarrow \qquad s + d\tau - \frac{d}{p} > 0$$

Theorem

Let s satisfy

$$\max\left\{d\Big(\frac{1}{p}-1\Big),\ \frac{1}{p}-1\right\} < s < \min\left\{1,\ \frac{1}{p},\ d\Big(\frac{1}{p}-\tau\Big)\right\}.$$

Then $f \in \mathcal{S}'(\mathbb{R}^d)$ belongs to $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ if and only if f has a representation in $\mathcal{S}'(\mathbb{R}^d)$ in the form

$$f = \sum_{i=1}^{2^d-1} \sum_{j=-1}^{\infty} \sum_{k\in\mathbb{Z}^d} \langle f,\, h_{i,j,k}\rangle\, h_{i,j,k}$$

and

$$\|f\| := \Big\| \{\langle f,\, h_{i,j,m}\rangle\}_{i,j,m} \Big| b^{s,\tau}_{p,q} \Big\| < \infty\,.$$

The mapping $f \mapsto \{\langle f,\, h_{i,j,m}\rangle\}_{i,j,m}$ is an isomorphism of $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ onto $b^{s,\tau}_{p,q}$.

- $\|f\| < \infty$ implies convergence in $B^{s-\varepsilon,\tau}_{p,q}(\mathbb{R}^d)$ for all $\varepsilon > 0$.

We put

$$\|\mu\|_{b^{s,\tau}_{p,q}} := \sup_{P\in\mathcal{Q}} \frac{1}{|P|^\tau} \left\{ \sum_{j=\max(j_P,-1)}^{\infty} 2^{j(s+\frac{d}{2}-\frac{d}{p})q} \sum_{i=1}^{2^d-1} \left[\sum_{m:\ Q_{j,m}\subset P} |\mu_{i,j,m}|^p \right]^{\frac{q}{p}} \right\}^{\frac{1}{q}} .$$

Recall

$$\|\mu\|_{b^{s}_{p,q}(\mathbb{R}^d)} := \left\{ \sum_{j=0}^{\infty} 2^{j(s+\frac{d}{2}-\frac{d}{p})q} \sum_{i=1}^{2^d-1} \left[\sum_{k\in\mathbb{Z}^d} |\mu_{i,j,k}|^p \right]^{\frac{q}{p}} \right\}^{\frac{1}{q}} .$$

Necessary conditions

Theorem

Let $0 < p, q \leq \infty$, $0 \leq \tau \leq 1/p$ *and* $s \in \mathbb{R}$. *Then* $\mathcal{X} \in B^{s,\tau}_{p,q}(\mathbb{R}^d)$ *if and only if either*

$$s = \frac{1}{p}, \qquad q = \infty \qquad \text{and} \qquad s \leq d\Big(\frac{1}{p} - \tau\Big)$$

or

$$s < \frac{1}{p}, \qquad 0 < q \leq \infty \qquad \text{and} \qquad s \leq d\Big(\frac{1}{p} - \tau\Big).$$

- Influence of the Morrey parameter τ.

The parameter τ comes into play with respect to the regularity of $\mathcal{X}$ if $\tau \geq \frac{d-1}{dp}$.

Theorem

Let $1 \le p \le \infty$, $0 < q \le \infty$ and $0 \le \tau \le 1/p$.
The mapping $f \mapsto \langle f, \mathcal{X} \rangle$ has an extension from $B^{s,\tau}_{p,q}(\mathbb{R}^d) \cap L_1(\mathbb{R}^d)$ on $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ as a linear and continuous functional if and only if either

$$s = \frac{1}{p} - 1\,, \quad 0 \le \tau \le \frac{d-1}{dp}\,, \quad \textit{und} \quad 0 < q \le 1$$

or

$$s > \frac{1}{p} - 1, \quad 0 \le \tau \le \frac{d-1}{dp}\,, \quad \textit{und} \quad 0 < q \le \infty$$

or

$$s > \frac{d}{p} - d\tau - 1, \quad \tau > \frac{d-1}{dp}, \quad \textit{und} \quad 0 < q \le \infty.$$

- Again $\tau = \frac{d-1}{dp}$ is a critical value.

Theorem

Let $0 < p < 1$ *and* $0 < q \le \infty$.

(a) Let $\frac{d-1}{dp} < \tau \le 1/p$. *The mapping* $f \mapsto \langle f, \mathcal{X} \rangle$ *has an extension from* $B^{s,\tau}_{p,q}(\mathbb{R}^d) \cap L_1(\mathbb{R}^d)$ *on* $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ *as a linear and continuous functional if and only if*

$$s > \frac{d}{p} - d\tau - 1\,.$$

(b) Let $0 \le \tau \le \frac{d-1}{dp}$. *If either*

$$s > (1-\tau p)\Big(\frac{d}{p} - d\Big) \qquad \text{und} \qquad 0 < q \le \infty$$

or

$$s = (1-\tau p)\Big(\frac{d}{p} - d\Big) \qquad \text{und} \qquad 0 < q \le p\,,$$

then the mapping $f \mapsto \langle f, \mathcal{X} \rangle$ *has an extension from* $B^{s,\tau}_{p,q}(\mathbb{R}^d) \cap L_1(\mathbb{R}^d)$ *on* $B^{s,\tau}_{p,q}(\mathbb{R}^d)$ *as a linear and continuous functional.*

Conjecture:

If s is not satisfying

$$\max\left\{d\left(\frac{1}{p}-1\right),\ \frac{1}{p}-1\right\} \le s \le \min\left\{1,\ \frac{1}{p},\ d\left(\frac{1}{p}-\tau\right)\right\},$$

then $B^{s,\tau}_{p,p}(\mathbb{R}^d)$ will not allow a characterization by the Haar system.

Applications: characteristic functions of sets as pointwise multipliers