Ball Average Characterizations of Function Spaces

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(Joint work)

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§III **Ball average** characterizations of **second order** Besov and Triebel–Lizorkin spaces

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Main Motivation

- ► Since there exists **no differential structure** on a general metric measure space, it is still an incompletely solved question how to introduce **function spaces with smoothness** on such a setting.

- ► Find **new characterizations** of well-known function spaces so that these new characterizations can be used as the definitions of the corresponding function spaces on **metric measure spaces**.

§I. **Pointwise** characterizations of **Besov** and **Triebel–Lizorkin** spaces with smoothness not greater than 1

Homogeneous Sobolev Spaces $\dot{W}^{m,p}(\mathbb{R}^n)$ / §I

- Let $m \in \mathbb{N} := \{1, 2, \ldots\}$, $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$ & $p \in (1, \infty)$.

  - $f \in \dot{W}^{m,p}(\mathbb{R}^n)$ (**homogeneous Sobolev space**) $\Longleftrightarrow$ $f \in \mathcal{S}'(\mathbb{R}^n)$ (**Schwartz distribution**) and $\partial^\gamma f \in L^p(\mathbb{R}^n)$ for all $|\gamma| = m$; moreover,

$$\|f\|_{\dot{W}^{m,p}(\mathbb{R}^n)} := \sum_{|\gamma|=m} \|\partial^\gamma f\|_{L^p(\mathbb{R}^n)}.$$

  - **Homogeneous**: for any $\lambda \in (0, \infty)$ and $f \in \dot{W}^{m,p}(\mathbb{R}^n)$,

$$\|f(\lambda \cdot)\|_{\dot{W}^{m,p}(\mathbb{R}^n)} = \lambda^{m-n/p} \|f\|_{\dot{W}^{m,p}(\mathbb{R}^n)}.$$

Inhomogeneous Sobolev Spaces $W^{m,p}(\mathbb{R}^n)$ / §I

- Let $m \in \mathbb{N} := \{1, 2, \ldots\}$, $\mathbb{Z}_+ := \mathbb{N} \cup \{0\}$ & $p \in (1, \infty)$.
  - $f \in W^{m,p}(\mathbb{R}^n) \Longleftrightarrow f \in L^p(\mathbb{R}^n)$ and $\partial^\gamma f \in L^p(\mathbb{R}^n)$ for all $|\gamma| \le m$; moreover,

$$\|f\|_{W^{m,p}(\mathbb{R}^n)} := \sum_{|\gamma|\le m} \|\partial^\gamma f\|_{L^p(\mathbb{R}^n)}.$$

$$\left[\|f\|_{W^{m,p}(\mathbb{R}^n)} \sim \|f\|_{L^p(\mathbb{R}^n)} + \|f\|_{\dot{W}^{m,p}(\mathbb{R}^n)}\right]$$

Besov Spaces $\dot{B}^{\alpha}_{p,\,q}(\mathbb{R}^n)$ / §I

Let $\varphi_0 \in \mathcal{S}(\mathbb{R}^n)$ (**Schwartz functions**) satisfy

$$(1.1) \qquad \varphi_0(x) = 1 \ \text{ if } \ |x| \le 1 \quad \text{and} \quad \varphi_0(y) = 0 \ \text{ if } \ |y| \ge 3/2,$$

and let

$$(1.2) \quad \varphi^{(j)}(x) := \varphi_0(2^{-j}x) - \varphi_0(2^{-j+1}x), \quad \forall\, x \in \mathbb{R}^n, \ \ \forall\, j \in \mathbb{Z}.$$

Then

$$\sum_{j\in\mathbb{Z}} \varphi^{(j)}(x) = 1, \qquad \forall\, x \in \mathbb{R}^n \setminus \{\vec{0}_n\}.$$

(**homogeneous partition of unity**). Let (**Triebel, 83 book**)

$$\mathcal{S}_\infty(\mathbb{R}^n) := \left\{ f \in \mathcal{S}(\mathbb{R}^n) : \int_{\mathbb{R}^n} f(x) x^\alpha \, dx = 0, \ \forall \alpha \in \mathbb{Z}^n_+ \right\}.$$

Besov Spaces $\dot{B}^{\alpha}_{p,\,q}(\mathbb{R}^n)$ / §I

▶ If $\alpha \in (0,\,\infty)$ & $p,\ q \in (0,\,\infty]$, then $f \in \dot{B}^{\alpha}_{p,\,q}(\mathbb{R}^n)$ (**homogeneous Besov space**) $\Longleftrightarrow f \in \mathcal{S}'_\infty(\mathbb{R}^n)$ (**dual** of $\mathcal{S}_\infty(\mathbb{R}^n)$) such that

$$\|f\|_{\dot{B}^{\alpha}_{p,\,q}(\mathbb{R}^n)} := \left\{\sum_{j=-\infty}^{\infty} 2^{j\alpha q} \left\|\left(\varphi^{(j)}\widehat{f}\right)^{\vee}\right\|^q_{L^p(\mathbb{R}^n)}\right\}^{1/q}$$

$$=: \left\|\left\{2^{j\alpha}\left\|\left(\varphi^{(j)}\widehat{f}\right)^{\vee}\right\|_{L^p(\mathbb{R}^n)}\right\}_{j\in\mathbb{Z}}\right\|_{\ell^q} < \infty.$$

▶ In what follows, for any function or distribution g, $\widehat{g}$ (or $g^{\vee}$) denotes its Fourier (or **inverse Fourier**) transform.

Triebel–Lizorkin Spaces $\dot{F}^{\alpha}_{p,\,q}(\mathbb{R}^n)$ / §I

▶ If $\alpha \in (0,\, \infty)$ & $p,\, q \in (0,\, \infty]$, then $f \in \dot{F}^{\alpha}_{p,\,q}(\mathbb{R}^n)$ (**homogeneous Triebel–Lizorkin space**) $\Longleftrightarrow f \in \mathcal{S}'_{\infty}(\mathbb{R}^n)$ such that $\|f\|_{\dot{F}^{\alpha}_{p,\,q}(\mathbb{R}^n)} < \infty$, where

$$\|f\|_{\dot{F}^{\alpha}_{p,\,q}(\mathbb{R}^n)} := \left\|\left\|\{2^{j\alpha}(\varphi^{(j)}\widehat{f})^{\vee}\}_{j\in\mathbb{Z}}\right\|_{\ell^q}\right\|_{L^p(\mathbb{R}^n)}, \; p < \infty$$

and

$$\|f\|_{\dot{F}^{\alpha}_{\infty,\,q}(\mathbb{R}^n)} := \sup_{\substack{x\in\mathbb{R}^n \\ m\in\mathbb{Z}}} \left\{2^{mn}\int_{B(x,\,2^{-m})}\sum_{j=m}^{\infty}|2^{j\alpha}(\varphi^{(j)}\widehat{f})^{\vee}(y)|^q\,dy\right\}^{\frac{1}{q}}.$$

▶ $\dot{F}^{0}_{\infty,2}(\mathbb{R}^n) = \mathrm{BMO}(\mathbb{R}^n)$, $\dot{F}^{0}_{p,2}(\mathbb{R}^n) = H^p(\mathbb{R}^n),\; p \in (0,\infty)$ & $\dot{F}^{m}_{p,2}(\mathbb{R}^n) = \dot{W}^{m,p}(\mathbb{R}^n),\; m \in \mathbb{Z}_+,\; p \in (1,\infty)$.

$B^{\alpha}_{p,\,q}(\mathbb{R}^n)$ & $F^{\alpha}_{p,\,q}(\mathbb{R}^n)$ / §I

Let $\varphi_0 \in \mathcal{S}(\mathbb{R}^n)$ be as in (1.1) and let $\widetilde{\varphi}^{(0)} := \varphi_0$ and $\widetilde{\varphi}^{(j)} := \varphi^{(j)}$ for any $j \in \mathbb{N}$, where $\varphi^{(j)}$ is as in (1.2). Then

$$\sum_{j=0}^{\infty} \widetilde{\varphi}^{(j)}(x) = 1,\ \forall\, x \in \mathbb{R}^n$$ (**inhomogeneous partition of unity**).

► The **inhomogeneous** Besov space $B^{\alpha}_{p,\,q}(\mathbb{R}^n)$ & Triebel–Lizorkin space $F^{\alpha}_{p,\,q}(\mathbb{R}^n)$ are defined via replacing $\mathcal{S}'_{\infty}(\mathbb{R}^n)$ and $\{\varphi^{(j)}\}_{j\in\mathbb{Z}}$ in $\dot{B}^{\alpha}_{p,\,q}(\mathbb{R}^n)$ & $\dot{F}^{\alpha}_{p,\,q}(\mathbb{R}^n)$, respectively, by $\mathcal{S}'(\mathbb{R}^n)$ (**Schwartz distributions**) and $\{\widetilde{\varphi}^{(j)}\}_{j\in\mathbb{Z}_+}$. Moreover, $\|f\|_{F^{\alpha}_{\infty,\,q}(\mathbb{R}^n)}$

$$:= \sup_{\substack{x\in\mathbb{R}^n\\ m\in\mathbb{Z}}} \left\{ 2^{mn} \int_{B(x,2^{-m})} \sum_{j=\max\{m,0\}}^{\infty} |2^{j\alpha}(\widetilde{\varphi}^{(j)}\widehat{f})^{\vee}(y)|^q \, dy \right\}^{\frac{1}{q}}.$$

Hajłasz–Sobolev Spaces (1) / §I

- $(\mathcal{X}, d, \mu)$: $\mathcal{X}$ — nonempty set;
  d — quasi metric, namely, $d(x, y) \le A[d(x, z) + d(z, y)]$;
  μ — regular Borel measure

- $p \in (1, \infty)$, $s \in (0, 1]$

- The **homogeneous fractional Hajłasz–Sobolev space** $\dot{M}^{s,p}(\mathcal{X})$ is defined to be the set of all measurable functions $f \in L^p_{\mathrm{loc}}(\mathcal{X})$ for which there exist a $0 \le g \in L^p(\mathcal{X})$ and a set $E \subset \mathcal{X}$ of measure zero such that, for any $x,\ y \in \mathcal{X} \setminus E$,

$$|f(x) - f(y)| \le [d(x, y)]^s [g(x) + g(y)]. \tag{1.3}$$

- Denote by $\mathcal{D}(f)$ the class of all non-negative Borel

Hajłasz–Sobolev Spaces (2) / §I

measurable functions g satisfying (1.3). Moreover, define

$$\|f\|_{\dot{M}^{s,p}(\mathcal{X})} := \inf_{g\in\mathcal{D}(f)}\{\|g\|_{L^p(\mathcal{X})}\}.$$

Let $M^{s,p}(\mathcal{X}) := L^p(\mathcal{X}) \cap \dot{M}^{s,p}(\mathcal{X})$ and, for any $f \in M^{s,p}(\mathcal{X})$, let

$$\|f\|_{M^{s,p}(\mathcal{X})} := \|f\|_{L^p(\mathcal{X})} + \|f\|_{\dot{M}^{s,p}(\mathcal{X})}.$$

Remarks:

- $\dot{M}^{1,p}(\mathcal{X})$ & $M^{1,p}(\mathcal{X})$ were introduced by Hajłasz [H96].
- $\dot{M}^{s,p}(\mathcal{X})$ & $M^{s,p}(\mathcal{X})$ when $s \in (0,1)$ were introduced by Hu [Hu03] for subsets (fractals) of $\mathbb{R}^n$ and Yang [Y03] for metric measure spaces.

Hajłasz–Sobolev Spaces (3) / §I

▶ It was proved in [H96] that

$$\dot{M}^{1,p}(\mathbb{R}^n) = \dot{W}^{1,p}(\mathbb{R}^n) = \dot{F}^1_{p,2}(\mathbb{R}^n)$$

and in [Y03] that, when $s \in (0,1)$,

$$\dot{M}^{s,p}(\mathbb{R}^n) = \dot{F}^s_{p,\infty}(\mathbb{R}^n) \supsetneqq \dot{F}^s_{p,2}(\mathbb{R}^n).$$

(There exists a **gap** for **Triebel–Lizorkin spaces**.)

• [H96] **P. Hajłasz**, Sobolev spaces on an arbitrary metric space, **Potential Anal. 5 (1996), 403-415**.

• [Hu03] **J. Hu**, A note on Hajłasz–Sobolev spaces on fractals, **J. Math. Anal. Appl. 280 (2003), 91-101**.

• [Y03] **D. Yang**, New characterizations of Hajłasz–Sobolev spaces on metric spaces, **Sci. China Ser. A 46 (2003), 675-689**.

Fractional s–Hajłasz Gradient / §I

• [KYZ11] **P. Koskela, D. Yang & Y. Zhou**, Pointwise characterizations of Besov and Triebel–Lizorkin spaces and quasiconformal mappings, **Adv. Math. 226 (2011), 3579-3621**.

► **Definition**. Let $s \in (0, \infty)$ and u be a measurable function on $\mathcal{X}$. A sequence of nonnegative measurable functions, $\vec{g} := \{g_k\}_{k\in\mathbb{Z}}$, is called a **fractional s-Hajłasz gradient** of u if there exists $E \subset \mathcal{X}$ with $\mu(E) = 0$ such that, for any $k \in \mathbb{Z}$ and $x, y \in \mathcal{X} \setminus E$ satisfying $2^{-k-1} \le d(x, y) < 2^{-k}$,

$$|u(x) - u(y)| \le [d(x, y)]^s [g_k(x) + g_k(y)].$$

Denote by $\mathbb{D}^s(u)$ the collection of all fractional s-Hajłasz gradients of u.

$\dot{M}^s_{p,\,q}(\mathcal{X})$ & $\dot{N}^s_{p,\,q}(\mathcal{X})$ / §I

- The **homogeneous Hajłasz–Triebel–Lizorkin space** $\dot{M}^s_{p,\,q}(\mathcal{X})$ is defined to be the space of all measurable functions u such that

$$\|u\|_{\dot{M}^s_{p,\,q}(\mathcal{X})} := \inf_{\vec{g}\in\mathbb{D}^s(u)} \left\|\left\|\{g_j\}_{j\in\mathbb{Z}}\right\|_{\ell^q}\right\|_{L^p(\mathcal{X})} < \infty.$$

- The **homogeneous Hajłasz–Besov space** $\dot{N}^s_{p,\,q}(\mathcal{X})$ is defined to be the space of all measurable functions u such that

$$\|u\|_{\dot{N}^s_{p,\,q}(\mathcal{X})} := \inf_{\vec{g}\in\mathbb{D}^s(u)} \left\|\left\{\|g_j\|_{L^p(\mathcal{X})}\right\}_{j\in\mathbb{Z}}\right\|_{\ell^q} < \infty.$$

RD-Spaces (1) / §I

• A triple $(\mathcal{X}, d, \mu)$: $\mathcal{X}$ is a non-empty set, d a quasi-metric (usually, for simplicity, metric), and μ a regular Borel measure.

▶ A space of homogenous type of Coifman-Weiss: if μ-**doubling** ($\mu(B(x, 2r)) \lesssim \mu(B(x, r))$).

▶ An RD-space if μ is both doubling and **reverse-doubling** [$\mu(B(x, 2r)) \geq C_0 \mu(B(x, r))$ and $C_0 > 1$].

▶ There exist many examples of RD-spaces. Especially, all **connected** spaces of homogeneous type are RD-spaces.

RD-Spaces (2) / §I

• [HMY06] **Y. Han, D. Müller & D. Yang**, Littlewood–Paley characterizations for Hardy spaces on spaces of homogeneous type, **Math. Nachr. 279 (2006), 1505-1537**.

• [HMY08] **Y. Han, D. Müller & D. Yang**, A theory of Besov and Triebel–Lizorkin spaces on metric measure spaces modeled on Carnot–Carathéodory spaces, **Abstr. Appl. Anal. 2008 Art. ID 893409, 250 pp**.

• [MY09] **D. Müller & D. Yang**, A difference characterization of Besov and Triebel–Lizorkin spaces on RD-spaces, **Forum Math. 21 (2009), 259-298**.

• **D. Yang & Y. Zhou**, New properties of Besov and Triebel–Lizorkin spaces on RD-spaces, **Manuscripta Math. 134 (2011), 59-90**.

Theorem 1.5 / §I

▶ [KYZ11] **Theorem**. Let $\mathcal{X}$ be an RD-space with the upper dimension n.

(i) If $s \in (0, 1)$, $p \in (n/(n+s), \infty)$ and $q \in (n/(n+s), \infty]$, then $\dot{M}^s_{p,q}(\mathcal{X}) = \dot{F}^s_{p,q}(\mathcal{X})$.

(ii) If $s \in (0, 1)$, $p \in (n/(n+s), \infty)$ and $q \in (0, \infty]$, then $\dot{N}^s_{p,q}(\mathcal{X}) = \dot{B}^s_{p,q}(\mathcal{X})$.

- $\dot{F}^s_{p,q}(\mathcal{X})$ & $\dot{B}^s_{p,q}(\mathcal{X})$ were studied in [HMY08] & [MY09].

- Applications to the **invariance** under **quasiconformal mappings** of the certain function spaces.

▶ More recent related papers: **H. Koch, P. Koskela, E. Saksman & T. Soto** [JFA 266 (2014)], **M. Bonk, E. Saksman & T. Soto** [IUMJ 67 (2018) & **D. Yang, W. Yuan & Y. Zhou** [JGA 27 (2017)].

More Papers on $\dot{M}^s_{p,\,q}(\mathcal{X})$ & $\dot{N}^s_{p,\,q}(\mathcal{X})$ / §I

- **A. Gogatishvili, P. Koskela & Y. Zhou**, Characterizations of Besov and Triebel–Lizorkin spaces on metric measure spaces, **Forum Math. 25 (2013), 787-819**.

- **T. Heikkinen & H. Tuominen**, Approximation by Hölder functions in Besov and Triebel–Lizorkin spaces, **Constr. Approx. 44 (2016), 455-482**.

- **T. Heikkinen, L. Ihnatsyeva & H. Tuominen**, Measure density and extension of Besov and Triebel–Lizorkin functions, **J. Fourier Anal Appl. 22 (2016), 334-382**.

- **T. Heikkinen, P. Koskela & H. Tuominen**, Approximation and quasicontinuity of Besov and Triebel–Lizorkin functions, **Trans. Amer. Math. Soc. 369 (2017), 3547-3573**.

§II. **Ball average** characterizations of **second order** Sobolev spaces

Theorem of [AMV12] / §II

- For any $t \in (0,\infty)$, $g \in L^1_{\rm loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, let

$$B_t(g)(x) := \frac{1}{|B(x,t)|}\int_{B(x,t)} g(y)\,dy.$$

- ([AMV12]) Let $p \in (1,\infty)$. Then $f \in W^{2,p}(\mathbb{R}^n)$ if and only if $f \in L^p(\mathbb{R}^n)$ and there exists $g \in L^p(\mathbb{R}^n)$ such that

$$\mathcal{G}(f,g)(\cdot) := \left\{\int_0^\infty \left|\frac{B_t(f)(\cdot) - f(\cdot)}{t^2} - B_t(g)(\cdot)\right|^2 \frac{dt}{t}\right\}^{\frac{1}{2}} \in L^p(\mathbb{R}^n).$$

- theory of **Vector-valued C-Z operators**, subtle estimates

- [AMV12] **R. Alabern, J. Mateu & J. Verdera**, A new characterization of Sobolev spaces on $\mathbb{R}^n$, **Math. Ann. 354 (2012), 589-626**.

Lusin-Area Funct. Charact. (1) / §II

▶ ([HYY15]) (i) If $p \in [2, \infty)$, then $f \in W^{2,p}(\mathbb{R}^n)$ if and only if $f \in L^p(\mathbb{R}^n)$ and there exists $g \in L^p(\mathbb{R}^n)$ such that

$$\mathcal{S}(f,g)(\cdot) := \left\{ \int_0^\infty \int_{B(\cdot,t)} \left| \frac{B_t(f)(y) - f(y)}{t^2} - B_t(g)(y) \right|^2 dy \, \frac{dt}{t^{n+1}} \right\}^{\frac{1}{2}} \in L^p(\mathbb{R}^n).$$

(ii) If $p \in (1, 2)$ and $n \in \{1, 2, 3\}$, then $f \in W^{2,p}(\mathbb{R}^n)$ if and only if $f \in L^p(\mathbb{R}^n)$ and there exists $g \in L^p(\mathbb{R}^n)$ such that $\mathcal{S}(f, g) \in L^p(\mathbb{R}^n)$.

• [HYY15] **Z. He, D. Yang & W. Yuan**, Littlewood–Paley characterizations of second-order Sobolev spaces via averages on balls, **Canadian Math. Bull. 59 (2016), 104-108**.

Lusin-Area Funct. Charact. (2) / §II

▶ ([DLYY]) (i) Let $n \in [4, \infty) \cap \mathbb{N}$ and $p \in (\frac{2n}{4+n}, 2)$. Then $f \in W^{2,p}(\mathbb{R}^n)$ if and only if $f \in L^p(\mathbb{R}^n)$ and there exists $g \in L^p(\mathbb{R}^n)$ such that $\mathcal{S}(f, g) \in L^p(\mathbb{R}^n)$.

(ii) Let $n \in [5, \infty) \cap \mathbb{N}$ and $p \in (1, \frac{2n}{4+n})$. Then the conclusion of (i) does not hold true.

▶ In (i), if $n = 4$, then $p \in (1, 2)$. The conclusion of (i) is **near sharp**.

▶ Instead of **a** vector-valued C-Z operator, we use **a series of** vector-valued C-Z operators

• [DLYY] **F. Dai, J. Liu, D. Yang & W. Yuan**, Littlewood–Paley characterizations of fractional Sobolev spaces via averages on balls, **Proc. Roy. Soc. Edinburgh Sect. A. 148 (2018), 1135-1163**.

$\mathcal{G}_\lambda^*$ Characterization (1) / §II

- ([HYY15]) (i) If $p \in [2, \infty)$ and $\lambda \in (1, \infty)$, then $f \in W^{2,p}(\mathbb{R}^n) \iff f \in L^p(\mathbb{R}^n)$ and $\exists\, g \in L^p(\mathbb{R}^n)$ such that

$$\mathcal{G}_\lambda^*(f,g)(\cdot) := \left\{ \int_0^\infty \int_{\mathbb{R}^n} \left| \frac{B_t(f)(y) - f(y)}{t^2} - B_t(g)(y) \right|^2 \times \left(\frac{t}{t + |\cdot - y|} \right)^{\lambda n} dy \, \frac{dt}{t^{n+1}} \right\}^{\frac{1}{2}} \in L^p(\mathbb{R}^n).$$

(ii) If $p \in (1, 2)$, $\lambda \in (2/p, \infty)$ and $n \in \{1, 2, 3\}$, then $f \in W^{2,p}(\mathbb{R}^n) \iff f \in L^p(\mathbb{R}^n)$ and $\exists\, g \in L^p(\mathbb{R}^n)$ such that $\mathcal{G}_\lambda^*(f,g) \in L^p(\mathbb{R}^n)$.

$\mathcal{G}_\lambda^*$ Characterization (2) / §II

▶ ([DLYY]) (i) Let $n \in [4, \infty) \cap \mathbb{N}$, $p \in (\frac{2n}{4+n}, 2)$ and $\lambda \in (2/p, \infty)$. Then $f \in W^{2,p}(\mathbb{R}^n)$ if and only if $f \in L^p(\mathbb{R}^n)$ and there exists $g \in L^p(\mathbb{R}^n)$ such that $\mathcal{G}_\lambda^*(f, g) \in L^p(\mathbb{R}^n)$.

(ii) Let $n \in [5, \infty) \cap \mathbb{N}$, $p \in (1, \frac{2n}{4+n})$ and $\lambda \in (2/p, \infty)$. Then the conclusion of (i) does not hold true.

▶ In (i), if $n = 4$, then $p \in (1, 2)$. The conclusion of (i) is **near sharp**.

▶ It is still unclear on the endpoint case $p = \frac{2n}{4+n}$.

▶ The **Lusin-area** function and $\mathcal{G}_\lambda^*$ function characterizations for **all** Sobolev spaces:

- **Z. He, D. Yang & W. Yuan**, Littlewood–Paley characterizations of higher-order Sobolev spaces via averages on balls, **Math. Nachr. 291 (2018), 284-325**.

Pointwise Characterization (1) / §II

▶ ([DGYY15]) Let $p \in (1, \infty)$. Then $f \in W^{2,p}(\mathbb{R}^n)$

$\Longleftrightarrow$ $f \in L^p(\mathbb{R}^n)$ and $\exists\, 0 \le g \in L^p(\mathbb{R}^n)$ and $C_0 > 0$ such that, for any $t \in (0, \infty)$ and almost every $x \in \mathbb{R}^n$,

$$|f(x) - B_t(f)(x)| \le C_0 t^2 g(x)$$

$\Longleftrightarrow$ $f \in L^p(\mathbb{R}^n)$ and

$$\sup_{t \in (0,\infty)} \frac{\|f - B_t(f)\|_{L^p(\mathbb{R}^n)}}{t^2} =: C_1 < \infty.$$

- **Not known** for spaces of homogenous type.

- [DGYY15] **F. Dai, A. Gogatishvili, D. Yang & W. Yuan**, Characterizations of Sobolev spaces via averages on balls, **Nonlinear Anal. 128 (2015), 86-99**.

Pointwise Characterization (2) / §II

▶ ([DGYY15]) Let $p \in (1, \infty)$. Then $f \in W^{2,p}(\mathbb{R}^n)$

$\Longleftrightarrow$ $f \in L^p(\mathbb{R}^n)$ and $\exists\, 0 \le g \in L^p(\mathbb{R}^n)$ and C, $\widetilde{C} > 0$ such that, for any $t \in (0, \infty)$ and almost every $x \in \mathbb{R}^n$,

$$\left|B_t\left(f - B_{\widetilde{C}t}(f)\right)(x)\right| \le Ct^2 g(x)$$

$\Longleftrightarrow$ $f \in L^p(\mathbb{R}^n)$ and $\exists\, 0 \le g \in L^p(\mathbb{R}^n)$ and c, C, $\widetilde{C} > 0$ such that, for any $t \in (0, \infty)$ and almost every $x \in \mathbb{R}^n$,

$$B_t\left(|f - B_{\widetilde{C}t}(f)|\right)(x) \le Ct^2 B_{ct}(g)(x).$$

▶ The second equivalence also holds true on **spaces of homogeneous type**.

Pointwise Characterization (3) / §II

▶ ([DGYY15]) Let $p \in (1, \infty)$, $q \in [1, p)$, $c \in (0, \infty)$ and $K \in (0, \infty]$. Then $f \in W^{2,p}(\mathbb{R}^n)$

$\Longleftrightarrow$ $f \in L^p(\mathbb{R}^n)$ and

$$f^{\sharp,K}_{c,q}(\cdot) := \sup_{t\in(0,K)} t^{-2} \left\{ B_t \left(|f - B_{ct}(f)|^q \right)(\cdot) \right\}^{1/q} \in L^p(\mathbb{R}^n).$$

A Key Lemma / §II

Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$ and $\widetilde{C} \in (0, \infty)$ be a constant. Then

$$\lim_{t \to 0^+} \frac{\varphi - B_t(\varphi)}{t^2} = -\frac{1}{2(n+2)} \Delta \varphi$$

and

$$\lim_{t \to 0^+} B_t \left(\frac{\varphi - B_{\widetilde{C}t}(\varphi)}{t^2} \right) (\cdot) = -\frac{\widetilde{C}^2}{2(n+2)} \Delta \varphi(\cdot)$$

with convergence in $\mathcal{S}(\mathbb{R}^n)$.

Further Results / §II

▶ For any $\ell \in \mathbb{N}$, $t \in (0, \infty)$, $f \in L^1_{\rm loc}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$, let

$$B_{\ell,t}(f)(x) := -\frac{2}{\binom{2\ell}{\ell}} \sum_{j=1}^{\ell} (-1)^j \binom{2\ell}{\ell - j} B_{jt}(f)(x).$$

[Binomial coefficients; Observe that $B_{1,t}(f) = B_t(f)$.]

▶ All aforementioned characterizations of $W^{2,p}(\mathbb{R}^n)$ via pointwise inequalities remain true for $W^{2\ell,p}(\mathbb{R}^n)$, with $\ell \in \mathbb{N}$ and $p \in (1, \infty)$, if we replace $B_t(f)$ by $B_{\ell,t}(f)$ therein.

▶ ([CYYZ]) Also true for Morrey–Sobolev spaces.

• [CYYZ] **D.-C. Chang, D. Yang, W. Yuan & J. Zhang**, Some recent developments of high order Sobolev-type spaces, **J. Nonlinear Convex Anal. 17 (2016), 1831-1865**.

Open Questions / §II

▶ On spaces of homogeneous type (or even **smooth domains** of $\mathbb{R}^n$), whether or not these Sobolev spaces **coincide**? (We now have **several different** definitions.)

▶ On spaces of homogeneous type, whether or not fractional Sobolev spaces **contain** the known Hajłasz–Sobolev spaces or the known Newton–Sobolev spaces?

▶ For analysis on metric measure spaces, any **applications**?

§III. **Ball average** characterizations of **second order** Besov and Triebel–Lizorkin spaces

Littlewood–Paley Characterization (1) / §III

▶ ([YYZ13, DGYY15]) Let $\alpha \in (0,\, 2)$ and $q \in (1,\, \infty]$.

(i) If $p \in (1, \infty)$, then $f \in F^{\alpha}_{p,\,q}(\mathbb{R}^n)$ if and only if $f \in L^p(\mathbb{R}^n)$ and

$$\left\|\left\|\left\{2^{k\alpha}\left|f - B_{2^{-k}}(f)\right|\right\}_{k\in\mathbb{Z}_+}\right\|_{\ell^q}\right\|_{L^p(\mathbb{R}^n)} < \infty.$$

(ii) If $p = \infty$, then $f \in F^{\alpha}_{\infty,\,q}(\mathbb{R}^n)$ if and only if $f \in C(\mathbb{R}^n)$ and

$$\sup_{\substack{\ell\in\mathbb{Z}\\ x\in\mathbb{R}^n}} \left[B_{2^{-\ell}}\left(\sum_{k\geq\max\{\ell,0\}} 2^{k\alpha q}\left|f - B_{2^{-k}}(f)\right|^q\right)(x)\right]^{\frac{1}{q}} < \infty.$$

▶ $C(\mathbb{R}^n)$: the space of all **uniformly continuous bounded** functions

Littlewood–Paley Characterization (2) / §Ⅲ

▶ ([YYZ13, DGYY15]) Let $\alpha \in (0,\, 2)$, $p \in (1, \infty]$ and $q \in (0,\, \infty]$. Then $f \in B^{\alpha}_{p,\,q}(\mathbb{R}^n)$ if and only if $f \in L^p(\mathbb{R}^n)$ when $p < \infty$, or $f \in C(\mathbb{R}^n)$ when $p = \infty$, and

$$\left\{\sum_{j=0}^{\infty} 2^{j\alpha q} \left\| f - B_{2^{-j}}(f)\right\|^q_{L^p(\mathbb{R}^n)}\right\}^{\frac{1}{q}} < \infty.$$

• [YYZ13] **D. Yang, W. Yuan & Y. Zhou**, A new characterization of Triebel–Lizorkin spaces on $\mathbb{R}^n$, **Publ. Mat. 57 (2013), 57-82**.

• [DGYY15] **F. Dai, A. Gogatishvili, D. Yang & W. Yuan**, Characterizations of Besov and Triebel–Lizorkin spaces via averages on balls, **J. Math. Anal. Appl. 433 (2016), 1350-1368**.

Littlewood–Paley Characterization (3) / §III

- Lusin-area type function: For any $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$\mathcal{A}_r(f)(x) := \left\{ \sum_{k=1}^{\infty} 2^{k\alpha q} \left[B_{2^{-k}} \left(|f - B_{2^{-k}}(f)|^r \right)(x) \right]^{\frac{q}{r}} \right\}^{\frac{1}{q}}.$$

- ([CLYY15]) Let $\alpha \in (0,2)$, $p \in (1,\infty)$, $q \in (1,\infty]$ and $r \in [1,q)$. Then $f \in F^{\alpha}_{p,q}(\mathbb{R}^n)$ if and only if $f \in L^p(\mathbb{R}^n)$ and $\mathcal{A}_r(f) \in L^p(\mathbb{R}^n)$. (A similar result for **Littlewood–Paley g^*_λ-type functions** is missing.)

- [CLYY15] **D.-C. Chang, J. Liu, D. Yang & W. Yuan**, Littlewood–Paley characterizations of Hajłasz–Sobolev and Triebel–Lizorkin spaces via averages on balls, **Potential Anal. 46 (2017), 227-259**.

Littlewood–Paley Characterization (4) / §III

- ([CLYY15]) Let $\alpha \in (0,2)$ and $p \in (1,\infty)$, $q \in (1,\infty]$.
  - (i) If $f \in L^p(\mathbb{R}^n)$ and $\mathcal{A}_q(f) \in L^p(\mathbb{R}^n)$, then $f \in F^\alpha_{p,q}(\mathbb{R}^n)$.
  - (ii) If $p \in [q,\infty)$ and $\alpha \in (0,2)$, or $p \in (1,q)$ and $\alpha \in (n(1/p - 1/q), 2)$, then $f \in F^\alpha_{p,q}(\mathbb{R}^n)$ implies that $f \in L^p(\mathbb{R}^n)$ and $\mathcal{A}_q(f) \in L^p(\mathbb{R}^n)$.
  - (ii) Similar results via **Littlewood–Paley g^*_λ-type functions also hold true**.
- In case when $q = 2$ and $\alpha < 1$, i. e., $F^\alpha_{p,2}(\mathbb{R}^n) = W^{\alpha,p}(\mathbb{R}^n)$, then (ii) is not true when $\alpha < n(1/p - 1/2)$. But, what happens when $\alpha = n(1/p - 1/q)$?

Littlewood–Paley Characterization (5) / §III

- Part of aforementioned results are also true for $B^{\alpha,\tau}_{p,q}(\mathbb{R}^n)$ & $F^{\alpha,\tau}_{p,q}(\mathbb{R}^n)$ (see **[ZSYY]**) and Triebel–Lizorkin–Morrey spaces $\mathcal{E}^{\alpha}_{u,q,p}(\mathbb{R}^n)$ (see **[ZZYH]**).

- [ZSYY] **C. Zhuo, W. Sickel, D. Yang & W. Yuan**, Characterizations of Besov-type and Triebel–Lizorkin-type spaces via averages on balls, **Canad. Math. Bull. 60 (2017), 655-672**.

- [ZZYH] **J. Zhang, C. Zhuo, D. Yang and Z. He**, Littlewood–Paley characterizations of Triebel–Lizorkin–Morrey spaces via ball averages, **Nonlinear Anal. 150 (2017), 76-103**.

  - A new idea is to introduce the **local** Hardy–Littlewood maximal function.

Pointwise Characterization (1) / §III

▶ ([YY15]) Let $\alpha \in (0,\infty)$ and $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. A sequence $\vec{g} := \{g_j\}_{j\geq 0}$ of non-negative measurable functions is called an α-order Hajłasz type gradient sequence of f if, for each j, there exists a set $E_j \subset \mathbb{R}^n$ with measure zero such that

$$|f(x) - B_{2^{-j}} f(x)| \leq 2^{-j\alpha} g_j(x), \quad \forall\, x \in \mathbb{R}^n \setminus E_j.$$

Each g_j satisfying the above is called an α-order Hajłasz type gradient of f at level j.

▶ [YY15] **D. Yang & W. Yuan**, Pointwise characterizations of Besov and Triebel–Lizorkin spaces in terms of averages on balls, **Trans. Amer. Math. Soc. 369 (2017), 7631-7655**.

Pointwise Characterization (2) / §III

▶ ([YY15]) Let $\alpha \in (0,2)$ and $p,\ q \in (1,\infty]$. Then $f \in F^{\alpha}_{p,q}(\mathbb{R}^n)$ if and only if $f \in L^p(\mathbb{R}^n)$ when $p \in (0,\infty)$ or $f \in C(\mathbb{R}^n)$ when $p=\infty$, and there exists an **α-order Hajłasz type gradient sequence** $\vec{g} := \{g_k\}_{k=0}^{\infty}$ of f such that

$$\left\| \left\{ \sum_{k=0}^{\infty} 2^{k\alpha q} |g_k|^q \right\}^{1/q} \right\|_{L^p(\mathbb{R}^n)} < \infty, \quad p < \infty$$

and

$$\sup_{\substack{\ell \in \mathbb{Z} \\ x \in \mathbb{R}^n}} \left[B_{2^{-\ell}} \left(\sum_{k \geq \max\{\ell,0\}} 2^{k\alpha q} |g_k|^q \right)(x) \right]^{1/q} < \infty, \quad p = \infty.$$

Pointwise Characterization (3) / §III

- ([YY15]) Let $\alpha \in (0,2)$, $p \in (1,\infty]$ and $q \in (0,\infty]$. Then $f \in B^{\alpha}_{p,q}(\mathbb{R}^n)$ if and only if $f \in L^p(\mathbb{R}^n)$ when $p \in (0,\infty)$, or $f \in C(\mathbb{R}^n)$ when $p = \infty$, and there exists an **α-order Hajłasz type gradient sequence** $\vec{g} := \{g_k\}_{k=0}^{\infty}$ of f such that

$$\left\{\sum_{k=0}^{\infty} 2^{k\alpha q} \|g_k\|^q_{L^p(\mathbb{R}^n)}\right\}^{1/q} < \infty.$$

- Let $\ell \in \mathbb{N}$. All aforementioned pointwise characterizations of $B^{\alpha}_{p,q}(\mathbb{R}^n)$ and $F^{\alpha}_{p,q}(\mathbb{R}^n)$ remain true when $\alpha \in (0, 2\ell)$ if we replace $\{f - B_{2^{-j}}(f)\}_j$ by $\{f - B_{\ell,2^{-j}}(f)\}_j$.

- Applications?

§V. **Further remarks**

Variable Besov- & Triebel–Lizorkin-Type Spaces / §IV

- **C. Zhuo, D.-C. Chang, D. Yang & W. Yuan**, Characterizations of variable Triebel–Lizorkin-type spaces via ball averages, **J. Nonlinear Convex Anal. 19 (2018), 19-40**.
- **C. Zhuo, D.-C. Chang & D. Yang**, Ball average characterizations of variable Besov-type spaces, **Taiwanese J. Math.** (to appear).
- **G. Xie, D. Yang & W. Yuan**, Pointwise Characterizations of even order Sobolev spaces via **derivatives of ball averages**, **Canad. Math. Bull.** (to appear).

[$\frac{1}{t}\frac{\partial}{\partial t}\left(\frac{f-B_t f}{t^2}\right): W^{4,p}(\mathbb{R}^n)$; no $\frac{1}{t}\frac{\partial}{\partial t}: W^{2,p}(\mathbb{R}^n)$.]

Newton Spaces (1) / §IV

▶ **N. Shanmugalingam**, Newtonian spaces: An extension of Sobolev spaces to metric measure spaces, **Rev. Mat. Iberoamericana 16 (2000), 243-279**.

▶ Defined via **upper gradients** (J. Heinonen & P. Koskela [Acta Math., 1998]; P. Koskela & P. MacManus [Studia Math., 1998]). Instead of straight lines by **curves** γ: $|f(\gamma(a)) - f(\gamma(b))| \leq \int_\gamma g\,ds$.

▶ Advantage: **Strong locality** (If a function is **constant on a measurable set**, then we can take **upper gradient to be zero almost everywhere on that set**; however, we cannot take the **Hajłasz gradient to be zero almost everywhere on that set**.)

▶ **N. Shanmugalingam, D. Yang & W. Yuan**, Newton–Besov spaces and Newton–Triebel–Lizorkin spaces, **Positivity 19 (2015), 177-220**.

Newton Spaces (2) / §IV

- **Newtonian-type Orlicz–Sobolev** spaces on metric measure spaces
  - **H. Tuominen**, Orlicz–Sobolev spaces on metric measure spaces, **Dissertation, University of Jyväskylä, Jyväskylä, 2004. Ann. Acad. Sci. Fenn. Math. Diss. No. 135 (2004), 86 pp**.
- **Hajłasz-type Orlicz–Sobolev** spaces and **Newtonian-type Orlicz–Sobolev** spaces
  - **T. Ohno & T. Shimomura**, Musielak–Orlicz–Sobolev spaces on metric measure spaces, **Czechoslovak Math. J. 65 (140) (2015), 435-474**.

Sphere Average Charact. / §IV

- **P. Hajłasz & Z. Liu**, A Marcinkiewicz integral type characterization of the Sobolev space, **Publ. Mat. 61 (2017), 83-104**.

  - Let $p \in (1, \infty)$. Then $f \in W^{1,p}(\mathbb{R}^n)$ if and only if $f \in L^p(\mathbb{R}^n)$ and

$$\left[\int_0^\infty \left| f(\cdot) - \frac{1}{|S(\cdot,t)|} \int_{S(\cdot,t)} f(y)\, d\sigma(y) \right|^2 \frac{dt}{t^3}\right]^{1/2} \in L^p(\mathbb{R}^n),$$

    where $S(x,t)$ denotes the sphere centered at x with the radius t. (**Not so useful for metric measure spaces**)

Weighted Sobolev Spaces / §IV

- A new and simplified proof of the characterization of $W^{1,p}(\mathbb{R}^n)$ with $p \in (1, \infty)$ ([Theorem 1, AMV12])

▶ **S. Sato**, Littlewood–Paley operators and Sobolev spaces, **Illinois J. Math. 58 (2014), 1025-1039**.

- Generalize [Theorem 1, AMV12] to the weighted case: $W^{\alpha,p}_w(\mathbb{R}^n)$, $\alpha \in (0, 2)$, $p \in (1, \infty)$ and $w \in A_p(\mathbb{R}^n)$

▶ **S. Sato**, Littlewood–Paley equivalence and homogeneous Fourier multipliers, **Integral Equations Operator Theory 87 (2017), 15-44**.

▶ **S. Sato**, Spherical square functions of Marcinkiewicz type with Riesz potentials, **Arch. Math. (Basel) 108 (2017), 415-426**.

Generalized Means / §IV

- Let Φ be a **bounded radial** function on $\mathbb{R}^n$ with **compact support** satisfying $\int_{\mathbb{R}^n} \Phi(x)\, dx = 1$.
- Generalized means:

$$G_t(g)(x) := \int_{\mathbb{R}^n} \frac{1}{t^n} \Phi\left(\frac{x-y}{t}\right) g(y)\, dy.$$

If let $\Phi := \frac{1}{|B(\vec{0}_n,1)|} \chi_{B(\vec{0}_n,1)}$, then $G_t(g) = B_t(g)$.

- **S. Sato, F. Wang, D. Yang & W. Yuan**, Generalized Littlewood–Paley characterizations of fractional Sobolev spaces, **Commun. Contemp. Math.** 20 (2018), 1750077, 48 pp. [only $\alpha \in (0, 2]$]
- **Y. Zhang, D.-C. Chang & D. Yang**, Generalized Littlewood–Paley characterizations of Triebel–Lizorkin spaces, **J. Nonlinear Convex Anal.** 18 (2017), 1171-1190. [only $\alpha \in (0, 2)$]

Morrey–Sobolev Spaces / §IV

- **Morrey**–Sobolev Spaces on Metric Measure Spaces

  - Let $0 < p \le q \le \infty$. Recall that the **Morrey space** $\mathcal{M}_p^q(\mathcal{X})$ is defined to be the space of all measurable functions f on $\mathcal{X}$ such that

$$\|f\|_{\mathcal{M}_p^q(\mathcal{X})} := \sup_{B\subset\mathcal{X}} [\mu(B)]^{1/q-1/p} \left[\int_B |f(x)|^p \, d\mu(x)\right]^{1/p} < \infty,$$

    where the supremum is taken over all balls B in $\mathcal{X}$.

  - Replace $L^p(\mathcal{X})$ by $\mathcal{M}_p^q(\mathcal{X})$

  - **Y. Lu, D. Yang & W. Yuan**, Morrey–Sobolev spaces on metric measure spaces, **Potential Anal. 41 (2014), 215-243**.

Haroske and Triebel / §IV

- Triebel [T10] introduced the **higher** version of Hajłasz–Sobolev spaces on $\mathbb{R}^n$ via **higher differences**, and some very interesting applications are given in [T11] and [HT11]:

- [HT11] **D. D. Haroske & H. Triebel**, Embeddings of function spaces: a criterion in terms of differences, **Complex Var. Elliptic Equ. 56 (2011), 931-944.**
- [T10] **H. Triebel**, Sobolev–Besov spaces of measurable functions, **Studia Math. 201 (2010), 69-86.**
- [T11] **H. Triebel**, Limits of Besov norms, **Arch. Math. 96 (2011), 169-175.**

(Not suitable for metric measure spaces.)

Sobolev Spaces Associated with Operators / §IV

- **L. Yan & D. Yang**, New Sobolev spaces via generalized Poincaré inequalities on metric measure spaces, **Math. Z. 255 (2007), 133-159**.
- **S. Hofmann, S. Mayboroda & A. McIntosh**, Second order elliptic operators with complex bounded measurable coefficients in L^p, Sobolev and Hardy spaces, **Ann. Sci. École Norm. Sup. (4) 44 (2011), 723-800**.
- **F. Bernicot, T. Coulhon & F. Dorothee**, Sobolev algebras through heat kernel estimates, **J. Éc. polytech. Math. 3 (2016), 99-161**.
- **J. Zhang, D.-C. Chang & D. Yang**, Characterizations of Sobolev spaces associated to operators satisfying off-diagonal estimates on balls, **Math. Methods Appl. Sci. 40 (2017), 2907-2929**.

Thank you for your attention.