Bochner-Riesz Means for the Dunkl Transforms

Wenrui Ye

University of International Business and Economics

Cambridge, February 19, 2019

Joint work with Dr. Feng Dai .

Weight Functions

Root system R: finite, nonzero vectors, $u\sigma_v \in R$ if $u, v \in R$.

Positive roots: $R_+ = \{v \in R : \langle v, v_0 \rangle \geqslant 0\}$.

Reflection group G: generated by σ_v, $v \in R$.

Multiplicity function: $\kappa_u = \kappa_v$ if $u\sigma_w = v$ for some $w \in R$.

Weight function h_κ: invariant under reflection group G.

$$h_\kappa(x) = \prod_{v \in R_+} |\langle x, v \rangle|^{\kappa_v}.$$

In the case $G = \mathbb{Z}_2^d := \{\pm 1\}^d$, $R_+ = \{e_j : 1 \leqslant j \leqslant d\}$,

$$h_\kappa(x) = \prod_{j=1}^{d} |x_j|^{\kappa_j}, \quad x = (x_1, x_2, \cdots, x_d) \in \mathbb{R}^d,$$

Weight Functions

Root system R: finite, nonzero vectors, $u\sigma_v \in R$ if $u, v \in R$.

Positive roots: $R_+ = \{v \in R : \langle v, v_0 \rangle \geqslant 0\}$.

Reflection group G: generated by σ_v, $v \in R$.

Multiplicity function: $\kappa_u = \kappa_v$ if $u\sigma_w = v$ for some $w \in R$.

Weight function h_κ: invariant under reflection group G.

$$h_\kappa(x) = \prod_{v \in R_+} |\langle x, v \rangle|^{\kappa_v}.$$

In the case $G = \mathbb{Z}_2^d := \{\pm 1\}^d$, $R_+ = \{e_j : 1 \leqslant j \leqslant d\}$,

$$h_\kappa(x) = \prod_{j=1}^{d} |x_j|^{\kappa_j}, \quad x = (x_1, x_2, \cdots, x_d) \in \mathbb{R}^d,$$

Weight Functions

Root system R: finite, nonzero vectors, $u\sigma_v \in R$ if $u, v \in R$.

Positive roots: $R_+ = \{v \in R : \langle v, v_0 \rangle \geqslant 0\}$.

Reflection group G: generated by σ_v, $v \in R$.

Multiplicity function: $\kappa_u = \kappa_v$ if $u\sigma_w = v$ for some $w \in R$.

Weight function h_κ: invariant under reflection group G.

$$h_\kappa(x) = \prod_{v \in R_+} |\langle x, v \rangle|^{\kappa_v} .$$

In the case $G = \mathbb{Z}_2^d := \{\pm 1\}^d$, $R_+ = \{e_j : 1 \leqslant j \leqslant d\}$,

$$h_\kappa(x) = \prod_{j=1}^{d} |x_j|^{\kappa_j}, \quad x = (x_1, x_2, \cdots, x_d) \in \mathbb{R}^d,$$

Weight Functions

Root system R: finite, nonzero vectors, $u\sigma_v \in R$ if $u, v \in R$.

Positive roots: $R_+ = \{v \in R : \langle v, v_0 \rangle \geqslant 0\}$.

Reflection group G: generated by σ_v, $v \in R$.

Multiplicity function: $\kappa_u = \kappa_v$ if $u\sigma_w = v$ for some $w \in R$.

Weight function h_κ: invariant under reflection group G.

$$h_\kappa(x) = \prod_{v \in R_+} |\langle x, v \rangle|^{\kappa_v}.$$

In the case $G = \mathbb{Z}_2^d := \{\pm 1\}^d$, $R_+ = \{e_j : 1 \leqslant j \leqslant d\}$,

$$h_\kappa(x) = \prod_{j=1}^{d} |x_j|^{\kappa_j}, \quad x = (x_1, x_2, \cdots, x_d) \in \mathbb{R}^d,$$

Weight Functions

Root system R: finite, nonzero vectors, $u\sigma_v \in R$ if $u, v \in R$.

Positive roots: $R_+ = \{v \in R : \langle v, v_0 \rangle \geqslant 0\}$.

Reflection group G: generated by σ_v, $v \in R$.

Multiplicity function: $\kappa_u = \kappa_v$ if $u\sigma_w = v$ for some $w \in R$.

Weight function h_κ: invariant under reflection group G.

$$h_\kappa(x) = \prod_{v \in R_+} |\langle x, v \rangle|^{\kappa_v}.$$

In the case $G = \mathbb{Z}_2^d := \{\pm 1\}^d$, $R_+ = \{e_j : 1 \leqslant j \leqslant d\}$,

$$h_\kappa(x) = \prod_{j=1}^{d} |x_j|^{\kappa_j}, \quad x = (x_1, x_2, \cdots, x_d) \in \mathbb{R}^d,$$

Weight Functions

Root system R: finite, nonzero vectors, $u\sigma_v \in R$ if $u, v \in R$.

Positive roots: $R_+ = \{v \in R : \langle v, v_0 \rangle \geqslant 0\}$.

Reflection group G: generated by σ_v, $v \in R$.

Multiplicity function: $\kappa_u = \kappa_v$ if $u\sigma_w = v$ for some $w \in R$.

Weight function h_κ: invariant under reflection group G.

$$h_\kappa(x) = \prod_{v \in R_+} |\langle x, v \rangle|^{\kappa_v}.$$

In the case $G = \mathbb{Z}_2^d := \{\pm 1\}^d$, $R_+ = \{e_j : 1 \leqslant j \leqslant d\}$,

$$h_\kappa(x) = \prod_{j=1}^{d} |x_j|^{\kappa_j}, \quad x = (x_1, x_2, \cdots, x_d) \in \mathbb{R}^d,$$

Weight Functions

Root system R: finite, nonzero vectors, $u\sigma_v \in R$ if $u, v \in R$.

Positive roots: $R_+ = \{v \in R : \langle v, v_0 \rangle \geqslant 0\}$.

Reflection group G: generated by σ_v, $v \in R$.

Multiplicity function: $\kappa_u = \kappa_v$ if $u\sigma_w = v$ for some $w \in R$.

Weight function h_κ: invariant under reflection group G.

$$h_\kappa(x) = \prod_{v \in R_+} |\langle x, v \rangle|^{\kappa_v} .$$

In the case $G = \mathbb{Z}_2^d := \{\pm 1\}^d$, $R_+ = \{e_j : 1 \leqslant j \leqslant d\}$,

$$h_\kappa(x) = \prod_{j=1}^{d} |x_j|^{\kappa_j}, \quad x = (x_1, x_2, \cdots, x_d) \in \mathbb{R}^d,$$

The weighted L^p-space

For $1 \leqslant p < \infty$, let $L^p(\mathbb{R}^d; h_\kappa^2)$ be the space of all functions on $\mathbb{R}^d$ such that

$$\|f\|_{p,\kappa} := \left(\int_{\mathbb{R}^d} |f(x)|^p h_\kappa^2(x) dx \right)^{\frac{1}{p}} < \infty,$$

For $p = \infty$, we use

$$\|f\|_\infty := \operatorname*{ess\,sup}_{x \in \mathbb{R}^d} |f(x)|.$$

The weighted L^p-space

For $1 \leqslant p < \infty$, let $L^p(\mathbb{R}^d; h_\kappa^2)$ be the space of all functions on $\mathbb{R}^d$ such that

$$\|f\|_{p,\kappa} := \left(\int_{\mathbb{R}^d} |f(x)|^p h_\kappa^2(x) dx \right)^{\frac{1}{p}} < \infty,$$

For $p = \infty$, we use

$$\|f\|_\infty := \operatorname*{ess\,sup}_{x \in \mathbb{R}^d} |f(x)|.$$

Dunkl intertwining operator

Let $v \mapsto \kappa_v$ be a multiplicity function associated with a reflection group G. The Dunkl operator is defined by

$$\mathcal{D}_j f(x) = \partial_j f(x) + \sum_{v \in R_+} \kappa_v \frac{f(x) - f(x\sigma_v)}{\langle x, v \rangle} \langle v, e_j \rangle,$$

where $e_1 = (1, 0, \cdots, 0), \cdots, e_d = (0, \cdots, 0, 1)$, $1 \leqslant j \leqslant d$.

$\mathscr{P}_n^d$: d-variable homogeneous polynomials of degree n.

$\Pi^d(\mathbb{R}^d)$: the space of algebraic polynomials on $\mathbb{R}^d$.

Dunkl intertwining operator: a linear operator $V_\kappa : \ \Pi^d \to \Pi^d$ determined uniquely by

$$\mathcal{D}_j V_\kappa = V_\kappa \partial_j, \quad V_\kappa 1 = 1, \quad V_\kappa \mathscr{P}_n^d \subset \mathscr{P}_n^d, \quad 1 \leqslant j \leqslant d,\ n \in \mathbb{N}_0.$$

Dunkl intertwining operator

Let $v \mapsto \kappa_v$ be a multiplicity function associated with a reflection group G. The Dunkl operator is defined by

$$\mathcal{D}_j f(x) = \partial_j f(x) + \sum_{v \in R_+} \kappa_v \frac{f(x) - f(x\sigma_v)}{\langle x, v\rangle} \langle v, e_j\rangle,$$

where $e_1 = (1, 0, \cdots, 0), \cdots, e_d = (0, \cdots, 0, 1)$, $1 \leqslant j \leqslant d$.

$\mathscr{P}_n^d$: d-variable homogeneous polynomials of degree n.

$\Pi^d(\mathbb{R}^d)$: the space of algebraic polynomials on $\mathbb{R}^d$.

Dunkl intertwining operator: a linear operator $V_\kappa : \Pi^d \to \Pi^d$ determined uniquely by

$$\mathcal{D}_j V_\kappa = V_\kappa \partial_j, \quad V_\kappa 1 = 1, \quad V_\kappa \mathscr{P}_n^d \subset \mathscr{P}_n^d, \quad 1 \leqslant j \leqslant d,\ n \in \mathbb{N}_0.$$

Dunkl intertwining operator

Let $v \mapsto \kappa_v$ be a multiplicity function associated with a reflection group G. The Dunkl operator is defined by

$$\mathcal{D}_j f(x) = \partial_j f(x) + \sum_{v \in R_+} \kappa_v \frac{f(x) - f(x\sigma_v)}{\langle x, v \rangle} \langle v, e_j \rangle,$$

where $e_1 = (1, 0, \cdots, 0), \cdots, e_d = (0, \cdots, 0, 1)$, $1 \leqslant j \leqslant d$.

$\mathscr{P}_n^d$: d-variable homogeneous polynomials of degree n.

$\Pi^d(\mathbb{R}^d)$: the space of algebraic polynomials on $\mathbb{R}^d$.

Dunkl intertwining operator: a linear operator $V_\kappa : \Pi^d \to \Pi^d$ determined uniquely by

$$\mathcal{D}_j V_\kappa = V_\kappa \partial_j, \quad V_\kappa 1 = 1, \quad V_\kappa \mathscr{P}_n^d \subset \mathscr{P}_n^d, \quad 1 \leqslant j \leqslant d, \, n \in \mathbb{N}_0.$$

Dunkl intertwining operator

Let $v \mapsto \kappa_v$ be a multiplicity function associated with a reflection group G. The Dunkl operator is defined by

$$\mathcal{D}_j f(x) = \partial_j f(x) + \sum_{v \in R_+} \kappa_v \frac{f(x) - f(x\sigma_v)}{\langle x, v\rangle} \langle v, e_j\rangle,$$

where $e_1 = (1, 0, \cdots, 0), \cdots, e_d = (0, \cdots, 0, 1)$, $1 \leqslant j \leqslant d$.

$\mathscr{P}_n^d$: d-variable homogeneous polynomials of degree n.

$\Pi^d(\mathbb{R}^d)$: the space of algebraic polynomials on $\mathbb{R}^d$.

Dunkl intertwining operator: a linear operator $V_\kappa :\ \Pi^d \to \Pi^d$ determined uniquely by

$$\mathcal{D}_j V_\kappa = V_\kappa \partial_j, \quad V_\kappa 1 = 1, \quad V_\kappa \mathscr{P}_n^d \subset \mathscr{P}_n^d, \quad 1 \leqslant j \leqslant d,\ n \in \mathbb{N}_0.$$

Dunkl intertwining operator

Let $v \mapsto \kappa_v$ be a multiplicity function associated with a reflection group G. The Dunkl operator is defined by

$$\mathcal{D}_j f(x) = \partial_j f(x) + \sum_{v \in R_+} \kappa_v \frac{f(x) - f(x\sigma_v)}{\langle x, v \rangle} \langle v, e_j \rangle,$$

where $e_1 = (1, 0, \cdots, 0), \cdots, e_d = (0, \cdots, 0, 1)$, $1 \leqslant j \leqslant d$.

$\mathscr{P}_n^d$: d-variable homogeneous polynomials of degree n.

$\Pi^d(\mathbb{R}^d)$: the space of algebraic polynomials on $\mathbb{R}^d$.

Dunkl intertwining operator: a linear operator $V_\kappa : \ \Pi^d \to \Pi^d$ determined uniquely by

$$\mathcal{D}_j V_\kappa = V_\kappa \partial_j, \quad V_\kappa 1 = 1, \quad V_\kappa \mathscr{P}_n^d \subset \mathscr{P}_n^d, \quad 1 \leqslant j \leqslant d,\ n \in \mathbb{N}_0.$$

Dunkl transform

The Dunkl transform $\mathcal{F}_\kappa f$ of $f \in L^1(\mathbb{R}^d; h_\kappa^2)$ is defined by

$$\mathcal{F}_\kappa f(x) \equiv \widehat{f}(\xi) := \int_{\mathbb{R}^d} f(y) E_\kappa(x, -iy) h_\kappa^2(y) dy, \quad x, y \in \mathbb{C}^d,$$

where $E_\kappa(x, y) = V_\kappa[\exp(\langle \cdot, y\rangle)](x)$.

We identify $f \in L^p(\mathbb{R}^d; h_\kappa^2)$, $1 \leqslant p \leqslant \infty$ with a tempered distribution in $\mathcal{S}'(\mathbb{R}^d)$ given by

$$(f, \varphi) := \int_{\mathbb{R}^d} f(x)\varphi(x) h_\kappa^2(x)\, dx, \ \ \forall \varphi \in \mathcal{S}(\mathbb{R}^d).$$

For $f \in L^p(\mathbb{R}^d; h_\kappa^2)$ with $2 < p \leqslant \infty$, we define its distributional Dunkl transform $\mathcal{F}_\kappa f$ via

$$(\mathcal{F}_\kappa f, \varphi) := (f, \mathcal{F}_\kappa \varphi) \equiv \int_{\mathbb{R}^d} f(x) \mathcal{F}_\kappa \varphi(x) h_\kappa^2(x)\, dx, \ \forall \varphi \in \mathcal{S}(\mathbb{R}^d).$$

Dunkl transform

The Dunkl transform $\mathcal{F}_\kappa f$ of $f \in L^1(\mathbb{R}^d; h_\kappa^2)$ is defined by

$$\mathcal{F}_\kappa f(x) \equiv \widehat{f}(\xi) := \int_{\mathbb{R}^d} f(y) E_\kappa(x, -iy) h_\kappa^2(y) dy, \quad x, y \in \mathbb{C}^d,$$

where $E_\kappa(x, y) = V_\kappa[\exp(\langle \cdot, y \rangle)](x)$.

We identify $f \in L^p(\mathbb{R}^d; h_\kappa^2)$, $1 \leqslant p \leqslant \infty$ with a tempered distribution in $\mathcal{S}'(\mathbb{R}^d)$ given by

$$(f, \varphi) := \int_{\mathbb{R}^d} f(x) \varphi(x) h_\kappa^2(x)\, dx, \ \ \forall \varphi \in \mathcal{S}(\mathbb{R}^d).$$

For $f \in L^p(\mathbb{R}^d; h_\kappa^2)$ with $2 < p \leqslant \infty$, we define its distributional Dunkl transform $\mathcal{F}_\kappa f$ via

$$(\mathcal{F}_\kappa f, \varphi) := (f, \mathcal{F}_\kappa \varphi) \equiv \int_{\mathbb{R}^d} f(x) \mathcal{F}_\kappa \varphi(x) h_\kappa^2(x)\, dx, \ \ \forall \varphi \in \mathcal{S}(\mathbb{R}^d).$$

Dunkl transform

The Dunkl transform $\mathcal{F}_\kappa f$ of $f \in L^1(\mathbb{R}^d; h_\kappa^2)$ is defined by

$$\mathcal{F}_\kappa f(x) \equiv \widehat{f}(\xi) := \int_{\mathbb{R}^d} f(y) E_\kappa(x, -iy) h_\kappa^2(y) dy, \quad x, y \in \mathbb{C}^d,$$

where $E_\kappa(x, y) = V_\kappa[\exp(\langle \cdot, y \rangle)](x)$.

We identify $f \in L^p(\mathbb{R}^d; h_\kappa^2)$, $1 \leqslant p \leqslant \infty$ with a tempered distribution in $\mathcal{S}'(\mathbb{R}^d)$ given by

$$(f, \varphi) := \int_{\mathbb{R}^d} f(x) \varphi(x) h_\kappa^2(x) \, dx, \ \ \forall \varphi \in \mathcal{S}(\mathbb{R}^d).$$

For $f \in L^p(\mathbb{R}^d; h_\kappa^2)$ with $2 < p \leqslant \infty$, we define its distributional Dunkl transform $\mathcal{F}_\kappa f$ via

$$(\mathcal{F}_\kappa f, \varphi) := (f, \mathcal{F}_\kappa \varphi) \equiv \int_{\mathbb{R}^d} f(x) \mathcal{F}_\kappa \varphi(x) h_\kappa^2(x) \, dx, \ \ \forall \varphi \in \mathcal{S}(\mathbb{R}^d).$$

Properties of Dunkl transform

Many properties of Fourier transforms carry over to the Dunkl transforms.

1. If $f \in L^1(\mathbb{R}^d; h_\kappa^2)$, then $\widehat{f} \in C(\mathbb{R}^d)$ and $\lim_{\|\xi\| \to \infty} \widehat{f}(\xi) = 0$.

2. If $f \in \mathcal{S}(\mathbb{R}^d)$, then

$$\mathcal{F}_\kappa(\mathcal{D}_\kappa^\alpha f)(x) = (-ix)^\alpha \mathcal{F}_\kappa f(x),$$

where $\mathcal{D}_\kappa^\alpha = \mathcal{D}_{\kappa,1}^{\alpha_1} \cdots \mathcal{D}_{\kappa,d}^{\alpha_d}$ and $\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{Z}_+^d$.

3. If $f \in L^2(\mathbb{R}^d; h_\kappa^2)$, $\|f\|_{\kappa,2} = \|\mathcal{F}_\kappa f\|_{\kappa,2}$.

4. If f and $\mathcal{F}_\kappa f$ are both in $L^1(\mathbb{R}^d; h_\kappa^2)$, then

$$f(x) = c_\kappa \int_{\mathbb{R}^d} \mathcal{F}_\kappa f(y) E_\kappa(ix, y) h_\kappa^2(y)\, dy, \quad \text{a.e. } x \in \mathbb{R}^d.$$

Properties of Dunkl transform

Many properties of Fourier transforms carry over to the Dunkl transforms.

1. If $f \in L^1(\mathbb{R}^d; h_\kappa^2)$, then $\widehat{f} \in C(\mathbb{R}^d)$ and $\lim_{\|\xi\| \to \infty} \widehat{f}(\xi) = 0$.

2. If $f \in \mathcal{S}(\mathbb{R}^d)$, then

$$\mathcal{F}_\kappa(\mathcal{D}_\kappa^\alpha f)(x) = (-ix)^\alpha \mathcal{F}_\kappa f(x),$$

where $\mathcal{D}_\kappa^\alpha = \mathcal{D}_{\kappa,1}^{\alpha_1} \cdots \mathcal{D}_{\kappa,d}^{\alpha_d}$ and $\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{Z}_+^d$.

3. If $f \in L^2(\mathbb{R}^d; h_\kappa^2)$, $\|f\|_{\kappa,2} = \|\mathcal{F}_\kappa f\|_{\kappa,2}$.

4. If f and $\mathcal{F}_\kappa f$ are both in $L^1(\mathbb{R}^d; h_\kappa^2)$, then

$$f(x) = c_\kappa \int_{\mathbb{R}^d} \mathcal{F}_\kappa f(y) E_\kappa(ix, y) h_\kappa^2(y)\, dy, \quad \text{a.e. } x \in \mathbb{R}^d.$$

Properties of Dunkl transform

Many properties of Fourier transforms carry over to the Dunkl transforms.

1. If $f \in L^1(\mathbb{R}^d; h_\kappa^2)$, then $\widehat{f} \in C(\mathbb{R}^d)$ and $\lim_{\|\xi\| \to \infty} \widehat{f}(\xi) = 0$.

2. If $f \in \mathcal{S}(\mathbb{R}^d)$, then

$$\mathcal{F}_\kappa(\mathcal{D}_\kappa^\alpha f)(x) = (-ix)^\alpha \mathcal{F}_\kappa f(x),$$

where $\mathcal{D}_\kappa^\alpha = \mathcal{D}_{\kappa,1}^{\alpha_1} \cdots \mathcal{D}_{\kappa,d}^{\alpha_d}$ and $\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{Z}_+^d$.

3. If $f \in L^2(\mathbb{R}^d; h_\kappa^2)$, $\|f\|_{\kappa,2} = \|\mathcal{F}_\kappa f\|_{\kappa,2}$.

4. If f and $\mathcal{F}_\kappa f$ are both in $L^1(\mathbb{R}^d; h_\kappa^2)$, then

$$f(x) = c_\kappa \int_{\mathbb{R}^d} \mathcal{F}_\kappa f(y) E_\kappa(ix, y) h_\kappa^2(y)\, dy, \quad \text{a.e. } x \in \mathbb{R}^d.$$

Properties of Dunkl transform

Many properties of Fourier transforms carry over to the Dunkl transforms.

1. If $f \in L^1(\mathbb{R}^d; h_\kappa^2)$, then $\widehat{f} \in C(\mathbb{R}^d)$ and $\lim_{\|\xi\|\to\infty} \widehat{f}(\xi) = 0$.

2. If $f \in \mathcal{S}(\mathbb{R}^d)$, then

$$\mathcal{F}_\kappa(\mathcal{D}_\kappa^\alpha f)(x) = (-ix)^\alpha \mathcal{F}_\kappa f(x),$$

where $\mathcal{D}_\kappa^\alpha = \mathcal{D}_{\kappa,1}^{\alpha_1} \cdots \mathcal{D}_{\kappa,d}^{\alpha_d}$ and $\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{Z}_+^d$.

3. If $f \in L^2(\mathbb{R}^d; h_\kappa^2)$, $\|f\|_{\kappa,2} = \|\mathcal{F}_\kappa f\|_{\kappa,2}$.

4. If f and $\mathcal{F}_\kappa f$ are both in $L^1(\mathbb{R}^d; h_\kappa^2)$, then

$$f(x) = c_\kappa \int_{\mathbb{R}^d} \mathcal{F}_\kappa f(y) E_\kappa(ix, y) h_\kappa^2(y)\, dy, \quad \text{a.e. } x \in \mathbb{R}^d.$$

Properties of Dunkl transform

Many properties of Fourier transforms carry over to the Dunkl transforms.

1. If $f \in L^1(\mathbb{R}^d; h_\kappa^2)$, then $\widehat{f} \in C(\mathbb{R}^d)$ and $\lim_{\|\xi\| \to \infty} \widehat{f}(\xi) = 0$.

2. If $f \in \mathcal{S}(\mathbb{R}^d)$, then

$$\mathcal{F}_\kappa(\mathcal{D}_\kappa^\alpha f)(x) = (-ix)^\alpha \mathcal{F}_\kappa f(x),$$

where $\mathcal{D}_\kappa^\alpha = \mathcal{D}_{\kappa,1}^{\alpha_1} \cdots \mathcal{D}_{\kappa,d}^{\alpha_d}$ and $\alpha = (\alpha_1, \cdots, \alpha_d) \in \mathbb{Z}_+^d$.

3. If $f \in L^2(\mathbb{R}^d; h_\kappa^2)$, $\|f\|_{\kappa,2} = \|\mathcal{F}_\kappa f\|_{\kappa,2}$.

4. If f and $\mathcal{F}_\kappa f$ are both in $L^1(\mathbb{R}^d; h_\kappa^2)$, then

$$f(x) = c_\kappa \int_{\mathbb{R}^d} \mathcal{F}_\kappa f(y) E_\kappa(ix, y) h_\kappa^2(y)\, dy, \quad \text{a.e. } x \in \mathbb{R}^d.$$

Bochner-Riesz means

Setting

$$\Phi^\delta(x) := (1-\|x\|^2)^\delta \cdot \chi_{\{x:\ \|x\|\leqslant 1\}},$$

For $\delta > -1$, the Bochner-Riesz means of order δ associated with the Dunkl transform are defined by

$$B_R^\delta(h_\kappa^2; f)(x) = c_\kappa \int_{\mathbb{R}^d} \Phi^\delta\left(\frac{y}{R}\right) \mathcal{F}_\kappa f(y) E_\kappa(ix, y) h_\kappa^2(y) dy, \quad R > 0.$$

Our work is to study the minimal index of the almost everywhere convergence of the Bochner-Riesz means.

Bochner-Riesz means

Setting

$$\Phi^{\delta}(x) := (1 - \|x\|^2)^{\delta} \cdot \chi_{\{x:\ \|x\| \leqslant 1\}},$$

For $\delta > -1$, the Bochner-Riesz means of order δ associated with the Dunkl transform are defined by

$$B_R^{\delta}(h_\kappa^2; f)(x) = c_\kappa \int_{\mathbb{R}^d} \Phi^{\delta}\Big(\frac{y}{R}\Big) \mathcal{F}_\kappa f(y) E_\kappa(ix, y) h_\kappa^2(y) dy, \ \ R > 0.$$

Our work is to study the minimal index of the almost everywhere convergence of the Bochner-Riesz means.

Bochner-Riesz means

Setting

$$\Phi^{\delta}(x) := (1 - \|x\|^2)^{\delta} \cdot \chi_{\{x:\ \|x\| \leqslant 1\}},$$

For $\delta > -1$, the Bochner-Riesz means of order δ associated with the Dunkl transform are defined by

$$B_R^{\delta}(h_\kappa^2; f)(x) = c_\kappa \int_{\mathbb{R}^d} \Phi^{\delta}\left(\frac{y}{R}\right) \mathcal{F}_\kappa f(y) E_\kappa(ix, y) h_\kappa^2(y) dy, \ \ R > 0.$$

Our work is to study the minimal index of the almost everywhere convergence of the Bochner-Riesz means.

Background and result

1. **Thangavelu and Y. Xu, 2005:**

$$\lim_{R\to\infty} \|B_R^\delta(h_\kappa^2; f) - f\|_{\kappa,1} = 0, \quad \forall f \in L^1(\mathbb{R}^d; h_\kappa^2),$$

if and only if $\delta > \lambda_\kappa = \frac{d-1}{2} + |\kappa|$.

2. **Stein:**

Under the case $\kappa = 0$.

$$\lim_{R\to\infty} B_R^\delta(f)(x) = f(x), \quad \text{a.e. } x \in \mathbb{R}^d,$$

if and only if $\delta > \lambda_0 = \frac{d-1}{2}$.

Theorem 3.1 (F. Dai and W. Ye, 2016)

If $\kappa \neq 0$ and $\delta \geqslant \lambda_\kappa$, then for all $f \in L^1(h_\kappa^2; \mathbb{R}^d)$,

$$\lim_{R\to\infty} B_R^\delta(h_\kappa^2; f, x) = f(x), \quad a.e.\ x \in \mathbb{R}^d.$$

Background and result

1. **Thangavelu and Y. Xu, 2005:**

$$\lim_{R\to\infty} \|B_R^\delta(h_\kappa^2; f) - f\|_{\kappa,1} = 0, \quad \forall f \in L^1(\mathbb{R}^d; h_\kappa^2),$$

if and only if $\delta > \lambda_\kappa = \frac{d-1}{2} + |\kappa|$.

2. **Stein:**

Under the case $\kappa = 0$.

$$\lim_{R\to\infty} B_R^\delta(f)(x) = f(x), \quad \text{a.e. } x \in \mathbb{R}^d,$$

if and only if $\delta > \lambda_0 = \frac{d-1}{2}$.

Theorem 3.1 (F. Dai and W. Ye, 2016)

If $\kappa \neq 0$ and $\delta \geqslant \lambda_\kappa$, then for all $f \in L^1(h_\kappa^2; \mathbb{R}^d)$,

$$\lim_{R\to\infty} B_R^\delta(h_\kappa^2; f, x) = f(x), \quad a.e.\ x \in \mathbb{R}^d.$$

Background and result

1. **Thangavelu and Y. Xu, 2005:**

$$\lim_{R\to\infty} \|B_R^\delta(h_\kappa^2; f) - f\|_{\kappa,1} = 0, \quad \forall f \in L^1(\mathbb{R}^d; h_\kappa^2),$$

if and only if $\delta > \lambda_\kappa = \frac{d-1}{2} + |\kappa|$.

2. **Stein:**

Under the case $\kappa = 0$.

$$\lim_{R\to\infty} B_R^\delta(f)(x) = f(x), \quad \text{a.e. } x \in \mathbb{R}^d,$$

if and only if $\delta > \lambda_0 = \frac{d-1}{2}$.

Theorem 3.1 (F. Dai and W. Ye, 2016)

If $\kappa \neq 0$ and $\delta \geqslant \lambda_\kappa$, then for all $f \in L^1(h_\kappa^2; \mathbb{R}^d)$,

$$\lim_{R\to\infty} B_R^\delta(h_\kappa^2; f, x) = f(x), \quad a.e.\ x \in \mathbb{R}^d.$$

Weak-type estimate

We denote

$$B_*^\delta(h_\kappa^2; f, x) = \sup_{R>0} |B_R^\delta(h_\kappa^2; f, x)|$$

the Bochner-Riesz means maximal operator.

Theorem 3.2 (F. Dai and W. Ye, 2016)

1. If $\kappa_{\min} > 0$ and $\delta \geqslant \lambda_\kappa$, then for all $f \in L^1(h_\kappa^2; \mathbb{R}^d)$,

$$\text{meas}_\kappa\Big\{x \in \mathbb{R}^d : \ B_*^\delta(h_\kappa^2; f, x) \geqslant \alpha\Big\} \leqslant c_\kappa \cdot \frac{\|f\|_{\kappa,1}}{\alpha}, \quad \forall \alpha > 0.$$

2. If $|\kappa| > 0$, $\kappa_{\min} = 0$ and $\delta = \frac{d-1}{2} + |\kappa|$, then for all $f \in L^1(h_\kappa^2; \mathbb{R}^d)$ and $\alpha > 0$,

$$\text{meas}_\kappa\Big\{x \in \mathbb{R}^d : \ B_*^\delta(h_\kappa^2; f, x) \geqslant \alpha\Big\} \leqslant c_\kappa \left|\ln \frac{\|f\|_{\kappa,1}}{\alpha}\right| \cdot \frac{\|f\|_{\kappa,1}}{\alpha}.$$

Weak-type estimate

We denote

$$B_*^\delta(h_\kappa^2; f, x) = \sup_{R>0} |B_R^\delta(h_\kappa^2; f, x)|$$

the Bochner-Riesz means maximal operator.

Theorem 3.2 (F. Dai and W. Ye, 2016)

1. *If* $\kappa_{\min} > 0$ *and* $\delta \geqslant \lambda_\kappa$*, then for all* $f \in L^1(h_\kappa^2; \mathbb{R}^d)$,

$$\text{meas}_\kappa \Big\{ x \in \mathbb{R}^d : \ B_*^\delta(h_\kappa^2; f, x) \geqslant \alpha \Big\} \leqslant c_\kappa \cdot \frac{\|f\|_{\kappa,1}}{\alpha}, \quad \forall \alpha > 0.$$

2. *If* $|\kappa| > 0$, $\kappa_{\min} = 0$ *and* $\delta = \frac{d-1}{2} + |\kappa|$*, then for all* $f \in L^1(h_\kappa^2; \mathbb{R}^d)$ *and* $\alpha > 0$,

$$\text{meas}_\kappa \Big\{ x \in \mathbb{R}^d : \ B_*^\delta(h_\kappa^2; f, x) \geqslant \alpha \Big\} \leqslant c_\kappa \left| \ln \frac{\|f\|_{\kappa,1}}{\alpha} \right| \cdot \frac{\|f\|_{\kappa,1}}{\alpha}.$$

Strong-type (p,p)-estimate

Corollary 3.3 (Strong-type estimate)

If $1 < p < \infty$ *and* $\delta > 2\lambda_\kappa|\frac{1}{2} - \frac{1}{p}|$, *then* $\forall f \in L^p(\mathbb{R}^d; h_\kappa^2)$,

$$\|B_*^\delta(h_\kappa^2; f)\|_{\kappa,p} \leqslant C\|f\|_{\kappa,p}.$$

Theorem 3.4 (Michael Christ, 1984)

If $2 \leqslant p < \frac{2d}{d-1-2\delta}$, $\delta > \frac{d-1}{2(d+1)}$ *and* $d \geqslant 3$, *then* $\forall f \in L^p(\mathbb{R}^d)$,

$$\|B_*^\delta(f)\|_p \leqslant C\|f\|_p.$$

The conditions in the Theorem 3.5 are equivalent to

$$p \geqslant 2 + \frac{4}{d-1} \quad \text{and} \quad \delta > \max\{0,\ d(\frac{1}{2} - \frac{1}{p}) - \frac{1}{2}\}.$$

Define $\delta_\kappa(p) = (2\lambda_\kappa + 1)(\frac{1}{2} - \frac{1}{p}) - \frac{1}{2}$.

Strong-type (p,p)-estimate

Corollary 3.3 (Strong-type estimate)

If $1 < p < \infty$ *and* $\delta > 2\lambda_\kappa|\frac{1}{2} - \frac{1}{p}|$, *then* $\forall f \in L^p(\mathbb{R}^d; h_\kappa^2)$,

$$\|B_*^\delta(h_\kappa^2; f)\|_{\kappa,p} \leqslant C\|f\|_{\kappa,p}.$$

Theorem 3.4 (Michael Christ, 1984)

If $2 \leqslant p < \frac{2d}{d-1-2\delta}$, $\delta > \frac{d-1}{2(d+1)}$ *and* $d \geqslant 3$, *then* $\forall f \in L^p(\mathbb{R}^d)$,

$$\|B_*^\delta(f)\|_p \leqslant C\|f\|_p.$$

The conditions in the Theorem 3.5 are equivalent to

$$p \geqslant 2 + \frac{4}{d-1} \quad \text{and} \quad \delta > \max\{0,\ d(\frac{1}{2} - \frac{1}{p}) - \frac{1}{2}\}.$$

Define $\delta_\kappa(p) = (2\lambda_\kappa + 1)(\frac{1}{2} - \frac{1}{p}) - \frac{1}{2}$.

Strong-type (p, p)-estimate

Corollary 3.3 (Strong-type estimate)

If $1 < p < \infty$ *and* $\delta > 2\lambda_\kappa|\frac{1}{2} - \frac{1}{p}|$, *then* $\forall f \in L^p(\mathbb{R}^d; h_\kappa^2)$,

$$\|B_*^\delta(h_\kappa^2; f)\|_{\kappa,p} \leqslant C\|f\|_{\kappa,p}.$$

Theorem 3.4 (Michael Christ, 1984)

If $2 \leqslant p < \frac{2d}{d-1-2\delta}$, $\delta > \frac{d-1}{2(d+1)}$ *and* $d \geqslant 3$, *then* $\forall f \in L^p(\mathbb{R}^d)$,

$$\|B_*^\delta(f)\|_p \leqslant C\|f\|_p.$$

The conditions in the Theorem 3.5 are equivalent to

$$p \geqslant 2 + \frac{4}{d-1} \quad \text{and} \quad \delta > \max\{0,\ d(\frac{1}{2} - \frac{1}{p}) - \frac{1}{2}\}.$$

Define $\delta_\kappa(p) = (2\lambda_\kappa + 1)(\frac{1}{2} - \frac{1}{p}) - \frac{1}{2}$.

Strong-type (p,p)-estimate

Corollary 3.3 (Strong-type estimate)

If $1 < p < \infty$ *and* $\delta > 2\lambda_\kappa|\frac{1}{2} - \frac{1}{p}|$, *then* $\forall f \in L^p(\mathbb{R}^d; h_\kappa^2)$,

$$\|B_*^\delta(h_\kappa^2; f)\|_{\kappa,p} \leqslant C\|f\|_{\kappa,p}.$$

Theorem 3.4 (Michael Christ, 1984)

If $2 \leqslant p < \frac{2d}{d-1-2\delta}$, $\delta > \frac{d-1}{2(d+1)}$ *and* $d \geqslant 3$, *then* $\forall f \in L^p(\mathbb{R}^d)$,

$$\|B_*^\delta(f)\|_p \leqslant C\|f\|_p.$$

The conditions in the Theorem 3.5 are equivalent to

$$p \geqslant 2 + \frac{4}{d-1} \quad \text{and} \quad \delta > \max\{0,\ d(\frac{1}{2} - \frac{1}{p}) - \frac{1}{2}\}.$$

Define $\delta_\kappa(p) = (2\lambda_\kappa + 1)(\frac{1}{2} - \frac{1}{p}) - \frac{1}{2}$.

Strong-type (p,p)-estimate

Theorem 3.5 (F. Dai and W. Ye, 2018)

If $p \geqslant 2 + \frac{2}{\lambda_\kappa}$ *and* $\delta > \max\{0, \delta_\kappa(p)\}$, *then* $\forall f \in L^p(\mathbb{R}^d; h_\kappa^2)$,

$$\|B_*^\delta(h_\kappa^2; f)\|_{\kappa,p} \leqslant C\|f\|_{\kappa,p}.$$

This inequality is no longer true for $\delta \leqslant \delta_\kappa(p)$.

Theorem 3.6 (F. Dai and H. Wang, 2010)

For $p \geqslant 2 + \frac{2}{\lambda_\kappa}$,

$$\sup_{R>0} \|B_R^\delta(h_\kappa^2; f)\|_{\kappa,p} \leqslant C\|f\|_{\kappa,p}$$

holds $\forall f \in L^p(\mathbb{R}^d; h_\kappa^2)$ *if and only if* $\delta > \max\{0, \delta_\kappa(p)\}$.

Strong-type (p,p)-estimate

Theorem 3.5 (F. Dai and W. Ye, 2018)

If $p \geqslant 2 + \frac{2}{\lambda_\kappa}$ *and* $\delta > \max\{0, \delta_\kappa(p)\}$, *then* $\forall f \in L^p(\mathbb{R}^d; h_\kappa^2)$,

$$\|B_*^\delta(h_\kappa^2; f)\|_{\kappa,p} \leqslant C\|f\|_{\kappa,p}.$$

This inequality is no longer true for $\delta \leqslant \delta_\kappa(p)$.

Theorem 3.6 (F. Dai and H. Wang, 2010)

For $p \geqslant 2 + \frac{2}{\lambda_\kappa}$,

$$\sup_{R>0} \|B_R^\delta(h_\kappa^2; f)\|_{\kappa,p} \leqslant C\|f\|_{\kappa,p}$$

holds $\forall f \in L^p(\mathbb{R}^d; h_\kappa^2)$ *if and only if* $\delta > \max\{0, \delta_\kappa(p)\}$.

Strong-type (p,p)-estimate

Theorem 3.5 (F. Dai and W. Ye, 2018)

If $p \geqslant 2 + \frac{2}{\lambda_\kappa}$ *and* $\delta > \max\{0, \delta_\kappa(p)\}$, *then* $\forall f \in L^p(\mathbb{R}^d; h_\kappa^2)$,

$$\|B_*^\delta(h_\kappa^2; f)\|_{\kappa,p} \leqslant C \|f\|_{\kappa,p}.$$

This inequality is no longer true for $\delta \leqslant \delta_\kappa(p)$.

Theorem 3.6 (F. Dai and H. Wang, 2010)

For $p \geqslant 2 + \frac{2}{\lambda_\kappa}$,

$$\sup_{R>0} \|B_R^\delta(h_\kappa^2; f)\|_{\kappa,p} \leqslant C \|f\|_{\kappa,p}$$

holds $\forall f \in L^p(\mathbb{R}^d; h_\kappa^2)$ *if and only if* $\delta > \max\{0, \delta_\kappa(p)\}$.

Bochner-Riesz Conjecture

Long-standing open question (Bochner-Riesz Conjecture):

For $2 \leqslant p < 2 + \frac{2}{\lambda_0}$,

$$\sup_{R>0} \|B_R^\delta(f)\|_p \leqslant C\|f\|_p$$

holds $\forall f \in L^p(\mathbb{R}^d)$ if and only if $\delta > \max\{0, \delta_0(p)\}$.

1. **C. Herz, 1954:**
For $p \geqslant 2$ and $d \geqslant 2$, $\delta > \max\{0, \delta_0(p)\}$ is a necessary conditions for the Bochner-Riesz Conjecture.

2. **L. Carleson and P. Sjölin, 1972:**
For $p \geqslant 2$ and $d = 2$, the Bochner-Riesz Conjecture is true.

Bochner-Riesz Conjecture

Long-standing open question (Bochner-Riesz Conjecture):

For $2 \leqslant p < 2 + \frac{2}{\lambda_0}$,

$$\sup_{R>0} \|B_R^\delta(f)\|_p \leqslant C\|f\|_p$$

holds $\forall f \in L^p(\mathbb{R}^d)$ if and only if $\delta > \max\{0, \delta_0(p)\}$.

1. **C. Herz, 1954:**
For $p \geqslant 2$ and $d \geqslant 2$, $\delta > \max\{0, \delta_0(p)\}$ is a necessary conditions for the Bochner-Riesz Conjecture.

2. **L. Carleson and P. Sjölin, 1972:**
For $p \geqslant 2$ and $d = 2$, the Bochner-Riesz Conjecture is true.

Bochner-Riesz Conjecture

Long-standing open question (Bochner-Riesz Conjecture):

For $2 \leqslant p < 2 + \frac{2}{\lambda_0}$,

$$\sup_{R>0} \|B_R^\delta(f)\|_p \leqslant C\|f\|_p$$

holds $\forall f \in L^p(\mathbb{R}^d)$ if and only if $\delta > \max\{0, \delta_0(p)\}$.

1. **C. Herz, 1954:**
For $p \geqslant 2$ and $d \geqslant 2$, $\delta > \max\{0, \delta_0(p)\}$ is a necessary conditions for the Bochner-Riesz Conjecture.

2. **L. Carleson and P. Sjölin, 1972:**
For $p \geqslant 2$ and $d = 2$, the Bochner-Riesz Conjecture is true.

Generalized translation operators

Given $y \in \mathbb{R}^d$ and $f \in \mathcal{S}(\mathbb{R}^d)$, we define

$$T^y f(x) := c_\kappa \int_{\mathbb{R}^d} \widehat{f}(\xi) E_\kappa(-iy, \xi) E_\kappa(i\xi, x) h_\kappa^2(\xi)\, d\xi, \quad x \in \mathbb{R}^d$$

By the inverse formula for Dunkl transforms, $\forall f \in \mathcal{S}(\mathbb{R}^d)$,

$$\mathcal{F}_\kappa(T^y f)(x) = E_\kappa(-ix, y)\mathcal{F}_\kappa f(x).$$

We extended the integral definition of $T^y f$ to $f \in L^p(\mathbb{R}^d; h_\kappa^2)$

Theorem 4.1 (F. Dai and W. Ye, 2018)

The integral representation of T^y can be extended to a bounded operator on $L^p(\mathbb{R}^d; h_\kappa^2)$ for all $1 \leqslant p \leqslant \infty$ with

$$\sup_{y \in \mathbb{R}^d} \|T^y\|_{\kappa,p} \leqslant C_d \|f\|_{\kappa,p}.$$

Generalized translation operators

Given $y \in \mathbb{R}^d$ and $f \in \mathcal{S}(\mathbb{R}^d)$, we define

$$T^y f(x) := c_\kappa \int_{\mathbb{R}^d} \widehat{f}(\xi) E_\kappa(-iy, \xi) E_\kappa(i\xi, x) h_\kappa^2(\xi)\, d\xi, \quad x \in \mathbb{R}^d$$

By the inverse formula for Dunkl transforms, $\forall f \in \mathcal{S}(\mathbb{R}^d)$,

$$\mathcal{F}_\kappa(T^y f)(x) = E_\kappa(-ix, y) \mathcal{F}_\kappa f(x).$$

We extended the integral definition of $T^y f$ to $f \in L^p(\mathbb{R}^d; h_\kappa^2)$

Theorem 4.1 (F. Dai and W. Ye, 2018)

The integral representation of T^y can be extended to a bounded operator on $L^p(\mathbb{R}^d; h_\kappa^2)$ for all $1 \leqslant p \leqslant \infty$ with

$$\sup_{y \in \mathbb{R}^d} \|T^y\|_{\kappa,p} \leqslant C_d \|f\|_{\kappa,p}.$$

Generalized translation operators

Given $y \in \mathbb{R}^d$ and $f \in \mathcal{S}(\mathbb{R}^d)$, we define

$$T^y f(x) := c_\kappa \int_{\mathbb{R}^d} \widehat{f}(\xi) E_\kappa(-iy, \xi) E_\kappa(i\xi, x) h_\kappa^2(\xi)\, d\xi, \quad x \in \mathbb{R}^d$$

By the inverse formula for Dunkl transforms, $\forall f \in \mathcal{S}(\mathbb{R}^d)$,

$$\mathcal{F}_\kappa(T^y f)(x) = E_\kappa(-ix, y)\mathcal{F}_\kappa f(x).$$

We extended the integral definition of $T^y f$ to $f \in L^p(\mathbb{R}^d; h_\kappa^2)$

Theorem 4.1 (F. Dai and W. Ye, 2018)

The integral representation of T^y can be extended to a bounded operator on $L^p(\mathbb{R}^d; h_\kappa^2)$ for all $1 \leqslant p \leqslant \infty$ with

$$\sup_{y \in \mathbb{R}^d} \|T^y\|_{\kappa,p} \leqslant C_d \|f\|_{\kappa,p}.$$

Global restriction theorem

For $f \in L^p(\mathbb{R}^d; h_\kappa^2)$ and $g \in L^q(\mathbb{R}^d; h_\kappa^2)$ with $1 \leqslant p, q \leqslant \infty$ and $\frac{1}{p} + \frac{1}{q} \geqslant 1$, we define

$$f *_\kappa g(x) = \int_{\mathbb{R}^d} f(y) T^y g(x) h_\kappa^2(y)\, dy, \quad x \in \mathbb{R}^d.$$

$d\sigma$: spherical Lebesgue measure.

Define $Rf = \mathcal{F}_\kappa f\Big|_{\mathbb{S}^{d-1}}$ and $R^* g = \mathcal{F}_\kappa(g d\sigma)$. Then $\forall f \in \mathcal{S}(\mathbb{R}^d),\ g \in C(\mathbb{S}^{d-1})$,

$$\langle Rf, g \rangle_{L^2(\mathbb{S}^{d-1}, h_\kappa^2)} = \langle f, R^* g \rangle_{L^2(\mathbb{R}^d, h_\kappa^2)},$$

and

$$R^* R f(x) = c_\kappa f *_\kappa \widehat{d\sigma}(x), \quad \forall f \in \mathcal{S}(\mathbb{R}^d),$$

where

$$\mathcal{F}_\kappa(g d\sigma)(\xi) \equiv \widehat{g d\sigma}(\xi) := \int_{\mathbb{S}^{d-1}} g(y) E_\kappa(-i\xi, y) h_\kappa^2(y)\, d\sigma(y).$$

Global restriction theorem

For $f \in L^p(\mathbb{R}^d; h_\kappa^2)$ and $g \in L^q(\mathbb{R}^d; h_\kappa^2)$ with $1 \leqslant p, q \leqslant \infty$ and $\frac{1}{p} + \frac{1}{q} \geqslant 1$, we define

$$f *_\kappa g(x) = \int_{\mathbb{R}^d} f(y) T^y g(x) h_\kappa^2(y)\, dy, \quad x \in \mathbb{R}^d.$$

$d\sigma$: spherical Lebesgue measure.

Define $Rf = \mathcal{F}_\kappa f\Big|_{\mathbb{S}^{d-1}}$ and $R^* g = \mathcal{F}_\kappa(g d\sigma)$. Then $\forall f \in \mathcal{S}(\mathbb{R}^d),\ g \in C(\mathbb{S}^{d-1})$,

$$\langle Rf, g \rangle_{L^2(\mathbb{S}^{d-1}, h_\kappa^2)} = \langle f, R^* g \rangle_{L^2(\mathbb{R}^d, h_\kappa^2)},$$

and

$$R^* R f(x) = c_\kappa f *_\kappa \widehat{d\sigma}(x), \quad \forall f \in \mathcal{S}(\mathbb{R}^d),$$

where

$$\mathcal{F}_\kappa(g d\sigma)(\xi) \equiv \widehat{g d\sigma}(\xi) := \int_{\mathbb{S}^{d-1}} g(y) E_\kappa(-i\xi, y) h_\kappa^2(y)\, d\sigma(y).$$

Global restriction theorem

For $f \in L^p(\mathbb{R}^d; h_\kappa^2)$ and $g \in L^q(\mathbb{R}^d; h_\kappa^2)$ with $1 \leqslant p, q \leqslant \infty$ and $\frac{1}{p} + \frac{1}{q} \geqslant 1$, we define

$$f *_\kappa g(x) = \int_{\mathbb{R}^d} f(y) T^y g(x) h_\kappa^2(y)\, dy, \quad x \in \mathbb{R}^d.$$

$d\sigma$: spherical Lebesgue measure.

Define $Rf = \mathcal{F}_\kappa f\Big|_{\mathbb{S}^{d-1}}$ and $R^* g = \mathcal{F}_\kappa(g d\sigma)$. Then $\forall f \in \mathcal{S}(\mathbb{R}^d),\ g \in C(\mathbb{S}^{d-1})$,

$$\langle Rf, g\rangle_{L^2(\mathbb{S}^{d-1}, h_\kappa^2)} = \langle f, R^* g\rangle_{L^2(\mathbb{R}^d, h_\kappa^2)},$$

and

$$R^* R f(x) = c_\kappa f *_\kappa \widehat{d\sigma}(x), \quad \forall f \in \mathcal{S}(\mathbb{R}^d),$$

where

$$\mathcal{F}_\kappa(g d\sigma)(\xi) \equiv \widehat{g d\sigma}(\xi) := \int_{\mathbb{S}^{d-1}} g(y) E_\kappa(-i\xi, y) h_\kappa^2(y)\, d\sigma(y).$$

Global restriction theorem

For $f \in L^p(\mathbb{R}^d; h_\kappa^2)$ and $g \in L^q(\mathbb{R}^d; h_\kappa^2)$ with $1 \leqslant p, q \leqslant \infty$ and $\frac{1}{p} + \frac{1}{q} \geqslant 1$, we define

$$f *_\kappa g(x) = \int_{\mathbb{R}^d} f(y) T^y g(x) h_\kappa^2(y)\, dy, \quad x \in \mathbb{R}^d.$$

$d\sigma$: spherical Lebesgue measure.

Define $Rf = \mathcal{F}_\kappa f\Big|_{\mathbb{S}^{d-1}}$ and $R^* g = \mathcal{F}_\kappa(g d\sigma)$. Then $\forall f \in \mathcal{S}(\mathbb{R}^d),\ g \in C(\mathbb{S}^{d-1})$,

$$\langle Rf, g\rangle_{L^2(\mathbb{S}^{d-1}, h_\kappa^2)} = \langle f, R^* g\rangle_{L^2(\mathbb{R}^d, h_\kappa^2)},$$

and

$$R^* R f(x) = c_\kappa f *_\kappa \widehat{d\sigma}(x), \quad \forall f \in \mathcal{S}(\mathbb{R}^d),$$

where

$$\mathcal{F}_\kappa(g d\sigma)(\xi) \equiv \widehat{g d\sigma}(\xi) := \int_{\mathbb{S}^{d-1}} g(y) E_\kappa(-i\xi, y) h_\kappa^2(y)\, d\sigma(y).$$

Global restriction theorem

Theorem 4.2 (F. Dai and W. Ye, 2018)

If $1 \leqslant p \leqslant \frac{2\lambda_\kappa+2}{\lambda_\kappa+2}$, $\|R^*Rf\|_{\kappa,p'} = \|f *_\kappa \widehat{d\sigma}\|_{\kappa,p'} \leqslant C\|f\|_{\kappa,p}$.

Note that

$$\|R\|^2_{L^p(\mathbb{R}^d,h_\kappa^2)\to L^2(\mathbb{S}^{d-1},h_\kappa^2)} = \|R^*\|^2_{L^2(\mathbb{S}^{d-1},h_\kappa^2)\to L^{p'}(\mathbb{R}^d,h_\kappa^2)} = \|R^*R\|_{L^p(\mathbb{R}^d,h_\kappa^2)\to L^{p'}(\mathbb{R}^d,h_\kappa^2)}.$$

Corollary 4.3 (Global Restriction Theorem)

If $1 \leqslant p \leqslant p_\kappa$, R extends to a bounded operator from $L^p(\mathbb{R}^d,h_\kappa^2)$ to $L^2(\mathbb{S}^{d-1},h_\kappa^2)$, and R^* extends to a bounded operator from $L^2(\mathbb{S}^{d-1},h_\kappa^2)$ to $L^{p'}(\mathbb{R}^d,h_\kappa^2)$.

Global restriction theorem

Theorem 4.2 (F. Dai and W. Ye, 2018)

If $1 \leqslant p \leqslant \frac{2\lambda_\kappa+2}{\lambda_\kappa+2}$, $\|R^*Rf\|_{\kappa,p'} = \|f *_\kappa \widehat{d\sigma}\|_{\kappa,p'} \leqslant C\|f\|_{\kappa,p}$.

Note that

$$\|R\|^2_{L^p(\mathbb{R}^d,h_\kappa^2)\to L^2(\mathbb{S}^{d-1},h_\kappa^2)} = \|R^*\|^2_{L^2(\mathbb{S}^{d-1},h_\kappa^2)\to L^{p'}(\mathbb{R}^d,h_\kappa^2)} = \|R^*R\|_{L^p(\mathbb{R}^d,h_\kappa^2)\to L^{p'}(\mathbb{R}^d,h_\kappa^2)}.$$

Corollary 4.3 (Global Restriction Theorem)

If $1 \leqslant p \leqslant p_\kappa$, *$R$ extends to a bounded operator from* $L^p(\mathbb{R}^d, h_\kappa^2)$ *to* $L^2(\mathbb{S}^{d-1}, h_\kappa^2)$, *and R^* extends to a bounded operator from* $L^2(\mathbb{S}^{d-1}, h_\kappa^2)$ *to* $L^{p'}(\mathbb{R}^d, h_\kappa^2)$.

Global restriction theorem

Theorem 4.2 (F. Dai and W. Ye, 2018)

If $1 \leqslant p \leqslant \frac{2\lambda_\kappa+2}{\lambda_\kappa+2}$, $\|R^*Rf\|_{\kappa,p'} = \|f *_\kappa \widehat{d\sigma}\|_{\kappa,p'} \leqslant C\|f\|_{\kappa,p}$.

Note that

$$\begin{aligned}\|R\|^2_{L^p(\mathbb{R}^d,h_\kappa^2)\to L^2(\mathbb{S}^{d-1},h_\kappa^2)} &= \|R^*\|^2_{L^2(\mathbb{S}^{d-1},h_\kappa^2)\to L^{p'}(\mathbb{R}^d,h_\kappa^2)} \\ &= \|R^*R\|_{L^p(\mathbb{R}^d,h_\kappa^2)\to L^{p'}(\mathbb{R}^d,h_\kappa^2)}.\end{aligned}$$

Corollary 4.3 (Global Restriction Theorem)

If $1 \leqslant p \leqslant p_\kappa$, *$R$ extends to a bounded operator from* $L^p(\mathbb{R}^d, h_\kappa^2)$ *to* $L^2(\mathbb{S}^{d-1}, h_\kappa^2)$, *and R^* extends to a bounded operator from* $L^2(\mathbb{S}^{d-1}, h_\kappa^2)$ *to* $L^{p'}(\mathbb{R}^d, h_\kappa^2)$.

Local restriction theorem

Theorem 4.4 (F. Dai and W. Ye, 2018)

Let $c_0 \in (0,1)$ be a parameter depending only on d and κ, and $B = B(\omega, \theta)$ be a ball centered at $\omega \in \mathbb{R}^d$ and having radius $\theta \geqslant c_0 > 0$. If $f \in L^p(\mathbb{R}^d; h_\kappa^2)$ is supported in B, $d \geqslant 2$, $1 \leqslant p \leqslant p_\kappa := \frac{2+2\lambda_\kappa}{\lambda_\kappa+2}$ and $\frac{1}{p} + \frac{1}{p'} = 1$, then

$$\left(\int_B |f *_\kappa \widehat{d\sigma}(x)|^{p'} h_\kappa^2(x)\, dx \right)^{\frac{1}{p'}} \leqslant C \left[\frac{\theta^{2\lambda_\kappa+1}}{meas_\kappa(B)} \right]^{\frac{2}{p}-1} \|f\|_{\kappa,p}.$$

Note that $1 \leqslant p \leqslant p_\kappa$ implies $\frac{2}{p} - 1 \geqslant \frac{1}{\lambda_\kappa+1} > 0$. Since

$$\frac{\theta^{2\lambda_\kappa+1}}{\mathrm{meas}_\kappa B(\omega,\theta)} \sim \prod_{j=1}^{d} \left(1 + \frac{|\omega_j|}{\theta}\right)^{-2\kappa_j},$$

the local restriction theorem is stronger than the global restriction theorem when θ is small.

Local restriction theorem

Theorem 4.4 (F. Dai and W. Ye, 2018)

Let $c_0 \in (0,1)$ be a parameter depending only on d and κ, and $B = B(\omega, \theta)$ be a ball centered at $\omega \in \mathbb{R}^d$ and having radius $\theta \geqslant c_0 > 0$. If $f \in L^p(\mathbb{R}^d; h_\kappa^2)$ is supported in B, $d \geqslant 2$, $1 \leqslant p \leqslant p_\kappa := \frac{2+2\lambda_\kappa}{\lambda_\kappa+2}$ and $\frac{1}{p} + \frac{1}{p'} = 1$, then

$$\left(\int_B |f *_\kappa \widehat{d\sigma}(x)|^{p'} h_\kappa^2(x)\, dx \right)^{\frac{1}{p'}} \leqslant C \left[\frac{\theta^{2\lambda_\kappa+1}}{meas_\kappa(B)} \right]^{\frac{2}{p}-1} \|f\|_{\kappa,p}.$$

Note that $1 \leqslant p \leqslant p_\kappa$ implies $\frac{2}{p} - 1 \geqslant \frac{1}{\lambda_\kappa+1} > 0$. Since

$$\frac{\theta^{2\lambda_\kappa+1}}{\text{meas}_\kappa B(\omega,\theta)} \sim \prod_{j=1}^{d} \left(1 + \frac{|\omega_j|}{\theta}\right)^{-2\kappa_j},$$

the local restriction theorem is stronger than the global restriction theorem when θ is small.

Local restriction theorem

Corollary 4.5 (Local Restriction Theorem)

Let $c_0 \in (0,1)$ be a parameter depending only on d and κ, and $B = B(\omega, \theta)$ be a ball centered at $\omega \in \mathbb{R}^d$ and having radius $\theta \geqslant c_0 > 0$.

(i) If $f \in L^p(\mathbb{R}^d; h_\kappa^2)$ is supported in B and $1 \leqslant p \leqslant p_\kappa$, then

$$\|\widehat{f}\|_{L^2(\mathbb{S}^{d-1};h_\kappa^2)} \leqslant C \left[\frac{\theta^{2\lambda_\kappa+1}}{meas_\kappa(B)}\right]^{\frac{1}{p}-\frac{1}{2}} \|f\|_{L^p(\mathbb{R}^d;h_\kappa^2)}.$$

(ii) If $2 + \frac{2}{\lambda_\kappa} \leqslant q \leqslant \infty$, and $f \in L^2(\mathbb{S}^{d-1}; h_\kappa^2)$, then

$$\|\widehat{f d\sigma}\|_{L^q(B;h_\kappa^2)} \leqslant C \left[\frac{\theta^{2\lambda_\kappa+1}}{meas_\kappa(B)}\right]^{\frac{1}{2}-\frac{1}{q}} \|f\|_{L^2(\mathbb{S}^{d-1};h_\kappa^2)}.$$

A_p-weight

Let $w(x)$ be a non-negative, locally integrable function on $\mathbb{R}^d$. We say w is an A_p weight for some $1 < p < \infty$, if

$$\sup_{B \subseteq \mathbb{R}^d} \left[\frac{\int_B w(x) d\mu_\kappa(x)}{\text{meas}_\kappa(B)} \right] \left[\frac{\int_B w(x)^{\frac{1}{1-p}} d\mu_\kappa(x)}{\text{meas}_\kappa(B)} \right]^{p-1} \leqslant C,$$

where $d\mu_\kappa = h_\kappa^2(x)dx$ and the supremum is taken over all balls $B \subseteq \mathbb{R}^d$.

We say w is an A_1 weight, if

$$\sup_{B \subseteq \mathbb{R}^d} \left[\frac{1}{\text{meas}_\kappa(B)} \int_B w(x) d\mu_\kappa(x) \right] \leqslant w(x), \quad \text{a.e. } x \in B.$$

w is an A_p weight implies that w is an A_q weight if $1 \leqslant p < q$.

A_p-weight

Let $w(x)$ be a non-negative, locally integrable function on $\mathbb{R}^d$. We say w is an A_p weight for some $1 < p < \infty$, if

$$\sup_{B\subseteq\mathbb{R}^d}\left[\frac{\int_B w(x)d\mu_\kappa(x)}{\text{meas}_\kappa(B)}\right]\left[\frac{\int_B w(x)^{\frac{1}{1-p}}d\mu_\kappa(x)}{\text{meas}_\kappa(B)}\right]^{p-1}\leqslant C,$$

where $d\mu_\kappa = h_\kappa^2(x)dx$ and the supremum is taken over all balls $B \subseteq \mathbb{R}^d$.

We say w is an A_1 weight, if

$$\sup_{B\subseteq\mathbb{R}^d}\left[\frac{1}{\text{meas}_\kappa(B)}\int_B w(x)d\mu_\kappa(x)\right]\leqslant w(x), \quad \text{a.e. } x\in B.$$

w is an A_p weight implies that w is an A_q weight if $1 \leqslant p < q$.

A_p-weight

Let $w(x)$ be a non-negative, locally integrable function on $\mathbb{R}^d$. We say w is an A_p weight for some $1 < p < \infty$, if

$$\sup_{B\subseteq\mathbb{R}^d}\left[\frac{\int_B w(x)d\mu_\kappa(x)}{\mathrm{meas}_\kappa(B)}\right]\left[\frac{\int_B w(x)^{\frac{1}{1-p}}d\mu_\kappa(x)}{\mathrm{meas}_\kappa(B)}\right]^{p-1}\leqslant C,$$

where $d\mu_\kappa = h_\kappa^2(x)dx$ and the supremum is taken over all balls $B \subseteq \mathbb{R}^d$.

We say w is an A_1 weight, if

$$\sup_{B\subseteq\mathbb{R}^d}\left[\frac{1}{\mathrm{meas}_\kappa(B)}\int_B w(x)d\mu_\kappa(x)\right]\leqslant w(x), \quad \text{a.e. } x\in B.$$

w is an A_p weight implies that w is an A_q weight if $1 \leqslant p < q$.

Weighted Littlewood-Paley inequality

Let $\Psi(\xi)$ be a radial Schwartz function supported in $\{\xi \in \mathbb{R}^d : \frac{1}{16} \leqslant \|\xi\| \leqslant 16\}$. Let $\Psi_j(x) = 2^{j(2\lambda_\kappa+1)}\Psi(2^j x)$, $j \in \mathbb{Z}$. Define the square function

$$L(f)(x) := \Big(\sum_{j\in\mathbb{Z}} |f *_\kappa \Psi_j(x)|^2\Big)^{\frac{1}{2}}.$$

Theorem 5.1 (F. Dai and W. Ye, 2018)

Suppose $L(f)$ is the square function and w is an A_p weight for some $1 < p < \infty$. If $w(\sigma x) = w(x)$ for all $\sigma \in \mathbb{Z}_2^d$, then

$$\|L(f)\|_{L^p(w)} \leqslant C\|f\|_{L^p(w)}.$$

If, in addition, $\sum_{j\in\mathbb{Z}} |\widehat{\Psi}(2^{-j}\xi)|^2 = 1$ for all $\xi \in \mathbb{R}^d \setminus \{0\}$, then

$$\|L(f)\|_{L^p(w)} \sim \|f\|_{L^p(w)}.$$

Weighted Littlewood-Paley inequality

Let $\Psi(\xi)$ be a radial Schwartz function supported in $\{\xi \in \mathbb{R}^d : \frac{1}{16} \leqslant \|\xi\| \leqslant 16\}$. Let $\Psi_j(x) = 2^{j(2\lambda_\kappa+1)}\Psi(2^j x)$, $j \in \mathbb{Z}$. Define the square function

$$L(f)(x) := \Big(\sum_{j\in\mathbb{Z}} |f *_\kappa \Psi_j(x)|^2\Big)^{\frac{1}{2}}.$$

Theorem 5.1 (F. Dai and W. Ye, 2018)

Suppose $L(f)$ is the square function and w is an A_p weight for some $1 < p < \infty$. If $w(\sigma x) = w(x)$ for all $\sigma \in \mathbb{Z}_2^d$, then

$$\|L(f)\|_{L^p(w)} \leqslant C\|f\|_{L^p(w)}.$$

If, in addition, $\sum_{j\in\mathbb{Z}} |\widehat{\Psi}(2^{-j}\xi)|^2 = 1$ for all $\xi \in \mathbb{R}^d \setminus \{0\}$, then

$$\|L(f)\|_{L^p(w)} \sim \|f\|_{L^p(w)}.$$

Almost everywhere convergence

Theorem 6.1

If $p \geqslant 2$ *and* $\delta > \max\{0, \delta_\kappa(p)\}$, *then for all* $f \in L^p(\mathbb{R}^d; h_\kappa^2)$,

$$\lim_{R\to\infty} B_R^\delta(h_\kappa^2; f)(x) = f(x), \quad a.e. \ x \in \mathbb{R}^d.$$

Theorem 6.2 (A. Carbery, Rubio de Francia, L. Vega)

If $d \geqslant 2$, $\delta > 0$ *and* $2 \leqslant p < \frac{2d}{d-1-2\delta}$, *then for all* $f \in L^p(\mathbb{R}^d)$,

$$\lim_{R\to\infty} B_R^\delta(f)(x) = f(x), \quad a.e. \ x \in \mathbb{R}^d.$$

The conditions in the Theorem 6.2 are equivalent to

$$p \geqslant 2 \ \text{ and } \ \delta > \max\{0, \ \delta_0(p)\}.$$

Almost everywhere convergence

Theorem 6.1

If $p \geqslant 2$ *and* $\delta > \max\{0, \delta_\kappa(p)\}$, *then for all* $f \in L^p(\mathbb{R}^d; h_\kappa^2)$,

$$\lim_{R\to\infty} B_R^\delta(h_\kappa^2; f)(x) = f(x), \quad a.e.\ x \in \mathbb{R}^d.$$

Theorem 6.2 (A. Carbery, Rubio de Francia, L. Vega)

If $d \geqslant 2$, $\delta > 0$ *and* $2 \leqslant p < \frac{2d}{d-1-2\delta}$, *then for all* $f \in L^p(\mathbb{R}^d)$,

$$\lim_{R\to\infty} B_R^\delta(f)(x) = f(x), \quad a.e.\ x \in \mathbb{R}^d.$$

The conditions in the Theorem 6.2 are equivalent to

$$p \geqslant 2 \ \text{ and } \ \delta > \max\{0,\ \delta_0(p)\}.$$

Thanks

Thank you!