Dimension-dependence error estimates for sampling recovery on Smolyak grids

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Smolyak grids

- We are interested in d-dependence estimates of the approximation error for sampling recovery on Smolyak grids of 1-periodic functions on the d-torus $\mathbb{T}^d = [0,1]^d$, when d may be very large.
- For $m \in \mathbb{N}$, the periodic Smolyak grids of points in $\mathbb{T}^d$ is

$$G^d(m) := \{\boldsymbol{\xi} = 2^{-\boldsymbol{k}}\boldsymbol{s} : \boldsymbol{k} \in \mathbb{N}^d, \ |\boldsymbol{k}|_1 = m, \ \boldsymbol{s} \in I^d(\boldsymbol{k})\},$$

where

$I^d(\boldsymbol{k}) := \{\boldsymbol{s} \in \mathbb{Z}^d_+ : s_j = 0, 1, ..., 2^{k_j} - 1, \ j \in \{1, ..., d\}\}$

$2^{-\boldsymbol{k}}\boldsymbol{s} := (2^{-k_1}s_1, ..., 2^{-k_d}s_d); \ |\boldsymbol{k}|_1 := \sum_{i=1}^d |k_i|$.

- The grids $G^d(m)$ are very sparse and of size $\le 2^m m^{d-1}$ in comparing with the full standard grids of size 2^{dm}.
- However, both give the same error of the sampling recovery of functions having mixed smoothness.
- Smolyak grids can be considered as a counterpart of hyperbilic cross for sampling recovery.

Lipschitz-Hölder mixed smoothness

- For sampling recovery we take d-variate 1-periodic functions $f \in H_\infty^\alpha(\mathbb{T}^d)$ having Lipschitz-Hölder mixed smoothness $\alpha > 0$.

- The univariate rth difference operator

$$\Delta_h^r(f,x) := \sum_{j=0}^{r} (-1)^{r-j} \binom{r}{j} f(x + jh).$$

- For $u \subset \{1, ..., d\}$, the mixed (r,u)th difference operator $\Delta_{\boldsymbol{h}}^{r,u}$

$$\Delta_{\boldsymbol{h}}^{r,u} := \prod_{i \in u} \Delta_{h_i}^r, \ u \subset \{1, ..., d\}; \quad \Delta_{\boldsymbol{h}}^{r,\varnothing} = I, \ \boldsymbol{h} \in \mathbb{T}^d.$$

Lipschitz-Hölder mixed smoothness

- If $0 < \alpha \leq r$, we introduce the semi-norm

$$|f|_{H^\alpha_\infty(u)} := \sup_{h \in \mathbb{T}^d} \prod_{i \in u} h_i^{-\alpha} \|\Delta_{\boldsymbol{h}}^{r,u}(f)\|_{C(\mathbb{T}^d)}.$$

- The Lipschitz-Hölder space $H^\alpha_\infty(\mathbb{T}^d)$ of mixed smoothness α:

$$\|f\|_{H^\alpha_\infty(\mathbb{T}^d)} := \sup_{u \subset \{1,...,d\}} |f|_{H^\alpha_\infty(u)} < \infty$$

- U^α_∞ is the unit ball in $H^\alpha_\infty(\mathbb{T}^d)$.

Lipschitz-Hölder mixed smoothness

- Zero boundary condition on $[0,1]^d$:
  $\mathring{U}^\alpha_\infty = \left\{ f \in U^\alpha_\infty : \ f(\boldsymbol{x}) = 0 \text{ if } x_j = 0 \text{ for some } j \in \{1, ..., d\} \right\}.$

- A few active variables:
  $U^{\alpha,\nu}_\infty = \left\{ f \in U^\alpha_\infty : \ \text{having at most } \nu \text{ active variables} \right\}.$

- x_j is active for a f if for all $i \neq j$ there are x_i^* s.t.
  $g(t) = f(x_1^*, ..., x_{j-1}^*, t, x_{j+1}^*, ..., x_d^*) \neq \text{const}.$

- $U^{\alpha,\nu}_\infty$ is a model of the objects in a d-variate space depending only a few ν (much smaller than d) unknown variables.

- $U^{\alpha,d}_\infty = U^\alpha_\infty$ (full variables).

- The function classes U^α_∞ and $U^{\alpha,\nu}_\infty$ with large d naturally appear in some high dimrnsional problems, in particular, parametric and stochastic PDEs.

Sampling algorithms on Smolyak grids for $\mathring{U}^\alpha_\infty$

- For $f \in \mathring{U}^\alpha_\infty$, we use the linear sampling algorithm

$$\mathring{S}_m(\mathring{\Phi}_m, f) := \sum_{\boldsymbol{\xi} \in \mathring{G}(m)} f(\boldsymbol{\xi}) \varphi_{\boldsymbol{\xi}},$$

on the Smolyak grids

$$\mathring{G}(m) := \left\{ \boldsymbol{\xi} = 2^{-\boldsymbol{k}} \boldsymbol{s} : \boldsymbol{k} \in \mathbb{N}^d, \ |\boldsymbol{k}|_1 = m, \ \boldsymbol{s} \in \mathring{I}^d(\boldsymbol{k}) \right\},$$

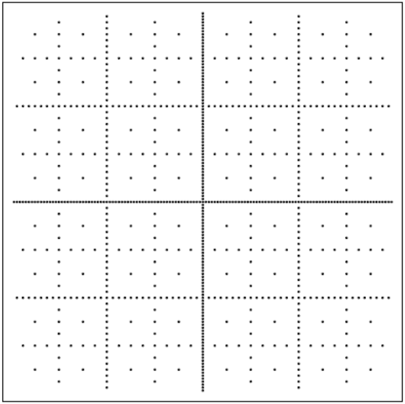
where $\mathring{\Phi}_m = \{\varphi_{\boldsymbol{\xi}}\}_{\boldsymbol{\xi} \in \mathring{G}(m)}$ is a family of functions on $\mathbb{T}^d$,

$$\mathring{I}^d(\boldsymbol{k}) := \left\{ \boldsymbol{s} \in \mathbb{Z}^d_+ : s_j = 1, ..., 2^{k_j} - 1, \ j \in \{1, ..., d\} \right\}.$$

- Optimal sampling on the grids $\mathring{G}(m)$

$$\mathring{s}_m(\mathring{U}^\alpha_\infty)_p := \inf_{\mathring{\Phi}_m} \sup_{f \in \mathring{U}^\alpha_\infty} \|f - \mathring{S}_m(\mathring{\Phi}_m, f)\|_p.$$

Smolyak grids $\mathring{G}(7)$ for $f = 0$ at the boundary of $[0,1]^d$



Sampling algorithms on Smolyak grids for $U^{\alpha,\nu}_\infty$

- For $f \in U^{\alpha,\nu}_\infty$, we use the linear sampling algorithm

$$S^\nu_m(\Phi^\nu_m, f) := \sum_{\boldsymbol{\xi} \in G^\nu(m)} f(\boldsymbol{\xi})\varphi_{\boldsymbol{\xi}},$$

on the Smolyak grids

$$G^\nu(m) := \Big\{\boldsymbol{\xi} = 2^{-\boldsymbol{k}}\boldsymbol{s} : \boldsymbol{k} \in \mathbb{N}^d, \ |\boldsymbol{k}|_1 = m, |\operatorname{supp}(\boldsymbol{k})| \le \nu, \ \boldsymbol{s} \in I^d(\boldsymbol{k})\Big\},$$

where $\Phi^\nu_m = \{\varphi_{\boldsymbol{\xi}}\}_{\boldsymbol{\xi} \in G^\nu(m)}$ is a family of functions on $\mathbb{T}^d$,

$$I^d(\boldsymbol{k}) := \Big\{\boldsymbol{s} \in \mathbb{Z}^d_+ : s_j = 0, 1, ..., 2^{k_j} - 1, \ j = 1, ..., d\Big\},$$

supp$(\boldsymbol{k})$ is the support of $\boldsymbol{k}$ and $|A|$ is the cardinality of A.
- The choice of $G^\nu(m)$ for $U^{\alpha,\nu}_\infty$ is quite natural since the active variables are unknown.
- Optimal sampling on the grids $G^\nu(m)$

$$s^\nu_m(U^{\alpha,\nu}_\infty)_p \ := \ \inf_{\Phi^\nu_m} \ \sup_{f \in U^{\alpha,\nu}_\infty} \|f - S^\nu_m(\Phi^\nu_m, f)\|_p.$$

Numbers of sampling points in Smolyak grids

- The number of points in the grid $\mathring{G}(m)$ is

$$|\mathring{G}(m)| \ \leq \ 2^m \binom{m-1}{d-1}, \ m \geq d.$$

- $\mathring{G}(m)$ is empty if $m < d \Rightarrow$ We should take $\mathring{G}(m)$ with $m \geq d$.

- The number $G^\nu(m)$ of points in the grid $G^\nu(m)$ is

$$|G^\nu(m)| \leq 2^m \binom{d}{\nu} \binom{m-1}{\nu-1}, \ m \geq \nu, \quad |G^d(m)| = 2^m \binom{m-1}{d-1}.$$

- $G^\nu(m)$ is full $\Leftrightarrow m \geq \nu$: $\Rightarrow$ We should take $G^\nu(m)$ with $m \geq \nu$.

Traditional estimates

- Traditional estimates:

$$\mathring{A}'(d)\, 2^{-\alpha m} m^{d-1} \le \mathring{s}_m(\mathring{U}^{\alpha}_{\infty})_p \le \mathring{A}(d)\, 2^{-\alpha m} m^{d-1}, \qquad (1)$$

$$A'(d,\nu)\, 2^{-\alpha m} m^{\nu-1} \le s^{\nu}_m(U^{\alpha,\nu}_{\infty})_p \ \le \ A(d,\nu)\, 2^{-\alpha m} m^{\nu-1}. \qquad (2)$$

- We want to establish upper and lower bounds for $\mathring{s}_m(\mathring{U}^{\alpha}_{\infty})_p$ and $s^{\nu}_m(U^{\alpha,\nu}_{\infty})_p$ explicitly in m and ν, d.
- In (1) and (2) the term $2^{-\alpha m} m^{d-1}$ is *a priori* split from constants $\mathring{A}(d)$, $\mathring{A}'(d)$, $A(d,\nu)$, $A'(d,\nu)$ which are actually a function of dimension parameters d and ν.
- $\Rightarrow$ Any estimate based on (1) and (2) may lead to a rough bound.

More natural forms of estimates

- Estimates of a more natural form (by a combinatorial argument):

$$\mathring{C}'(d)\,2^{-\alpha m}\binom{m}{d-1} \le \mathring{s}_m(\mathring{U}^{\alpha}_{\infty})_p \le \mathring{C}(d)\,2^{-\alpha m}\binom{m}{d-1},$$

$$C'(d,\nu)\,2^{-\alpha m}\binom{m}{\nu-1} \le s^{\nu}_m(U^{\alpha,\nu}_{\infty})_p \ \le \ C(d,\nu)\,2^{-\alpha m}\binom{m}{\nu-1}.$$

- Our purpose is to define "constants" $\mathring{A}(d)$, $\mathring{A}'(d)$, $A(d,\nu)$, $A'(d,\nu)$, $\mathring{C}(d)$, $\mathring{C}'(d)$, $C(d,\nu)$, $C'(d,\nu)$ as explicit functions in variables d,ν.

Previous related results

- There are many papers on high-dimensional approximation with different aspects and applications. We mention only a few of them dirrectly related to our talk.
- The first dimension-dependence estimates of approximation error: Wasilkowski and Wozniakowsi (1995).
- Liberating the dimension for high-dimensional approximation: Wasilkowski and Wozniakowsi (2011).
- High-dimentional sampling recovery on Smolyak grids based on hierachical Lagrangian polynomials: Bungartz and Griebel (2004).
- High-dimensional hyperbolic cross approximation: DD and Ullrich (2013); Kühn, Sickel and Ullrich (2015); Chernov and DD (2016).
- Infinite-dimensional hyperbolic cross approximation and applications to parametric PDEs: DD and Griebel (2016); DD, Griebel, Vu and Rieger (2018).

Periodic Faber basis

- We first consider the sampling recovery on Smolyak grids for functions $f \in H^\alpha_\infty(\mathbb{T}^d)$ with smoothness $0 < \alpha \le 2$. We will construct sampling algorithm by using the periodic Faber basis of hat functions.
- Hat function is the piece-wise linear B-spine with knots at $0, 1, 2$:

$$M_2(x) := \begin{cases} 1 - |x - 1|, & 0 \le x \le 2, \\ 0, & \text{otherwise.} \end{cases}$$

- The support of $M_2(2^{k+1}\cdot)$ is $[0, 2^{-k}] \subset [0, 1]$: $\Rightarrow$ We can extend these functions to 1-periodic functions φ_k on the whole $\mathbb{R}$.
- The univariate periodic Faber basis $\mathcal{F} := \{\varphi_{k,s}\}_{k\in\mathbb{Z}_+,\ s\in Z(k)}$ is a basis for $C(\mathbb{T})$, where

$$\varphi_{0,0}(x) := 1, \quad \varphi_{k,s}(x) := \varphi_k(x - 2^{-k-1}s),$$

$Z(0) := \{0\}$ and $Z(k) := \{0, 1, ..., 2^{k-1} - 1\}$.

Multivariate periodic Faber basis

- Put $Z^d(\boldsymbol{k}) := \prod_{i=1}^d Z(k_i)$ and for $\boldsymbol{k} \in \mathbb{Z}_+^d$, $\boldsymbol{s} \in Z^d(\boldsymbol{k})$,

$$\varphi_{\boldsymbol{k},\boldsymbol{s}}(\boldsymbol{x}) := \prod_{i=1}^d \varphi_{k_i,s_i}(x_i) \quad \text{(tensor product of hat functions)},$$

- The d-variate periodic Faber system

$$\mathcal{F}_d := \left\{\varphi_{\boldsymbol{k},\boldsymbol{s}} : \boldsymbol{s} \in Z^d(\boldsymbol{k}),\ \boldsymbol{k} \in \mathbb{Z}_+^d\right\}.$$

is a basis in $C(\mathbb{T}^d)$.

- For functions f on $\mathbb{T}$, define the univariate functionals

$$\lambda_{k,s}(f) := -\frac{1}{2}\Delta^2_{2^{-k}}(f, 2^{-k+1}s),\ k > 0; \quad \lambda_{0,0}(f) := f(0).$$

- For functions f on $\mathbb{T}^d$, and $\boldsymbol{k} \in \mathbb{Z}_+^d$, $\boldsymbol{s} \in Z^d(\boldsymbol{k})$,

$$\lambda_{\boldsymbol{k},\boldsymbol{s}} := \prod_{i=1}^d \lambda_{k_i,s_i}.$$

Representation by Faber series

- Put $\mathbb{Z}^d_+(u) := \{\boldsymbol{k} \in \mathbb{Z}^d_+ : \operatorname{supp}(\boldsymbol{k}) = u\}$ for $u \subset \{1, ..., d\}$.

Theorem

Let $0 < p \le \infty$ and $0 < \alpha \le 2$. Then a function $f \in H^\alpha_\infty(\mathbb{T}^d)$ can be represented by the series

$$f \; = \sum_{\boldsymbol{k} \in \mathbb{Z}^d_+} q_{\boldsymbol{k}}(f), \quad q_{\boldsymbol{k}}(f) := \sum_{\boldsymbol{s} \in Z^d(\boldsymbol{k})} \lambda_{\boldsymbol{k},\boldsymbol{s}}(f) \varphi_{\boldsymbol{k},\boldsymbol{s}}$$

converging in the norm of $C(\mathbb{T}^d)$. Moreover, for every $\boldsymbol{k} \in \mathbb{Z}^d_+(u)$,

$$\|q_{\boldsymbol{k}}(f)\|_p \; \le \; \left[2^{\alpha+1} (p+1)^{1/p}\right]^{-|u|} 2^{-\alpha|\boldsymbol{k}|_1} \, |f|_{H^\alpha_\infty(u)}.$$

Sampling algorithms on Smolyak grids for $\mathring{U}^\alpha_\infty$

- Since $\mathring{U}^\alpha_\infty = \left\{ f \in U^\alpha_\infty : f = \sum_{\boldsymbol{k}\in\mathbb{N}^d} q_{\boldsymbol{k}}(f) \right\}$, for sampling recovery of $f \in U^\alpha_\infty$, we take the series truncation

$$\mathring{R}_m(f) := \sum_{\boldsymbol{k}\in\mathbb{N}^d :\, |\boldsymbol{k}|_1 \le m} q_{\boldsymbol{k}}(f).$$

- $\mathring{R}_m$ defines a linear sampling algorithm on the Smolyak grid $\mathring{G}(m)$:

$$\mathring{R}_m(f) = \mathring{S}_m(\mathring{\Psi}_m, f) = \sum_{\boldsymbol{\xi}\in\mathring{G}(m)} f(\boldsymbol{\xi})\psi_{\boldsymbol{\xi}}, \quad \mathring{\Psi}_m := \{\psi_{\boldsymbol{\xi}}\}_{\boldsymbol{\xi}\in\mathring{G}(m)},$$

where $\psi_{\boldsymbol{\xi}}(\boldsymbol{x}) = \prod_{i=1}^d \psi_{2^{-k_i}s_i}(x_i)$, $\boldsymbol{\xi} = 2^{-\boldsymbol{k}}\boldsymbol{s}$, $\boldsymbol{k} \in \mathbb{N}^d$, $\boldsymbol{s} \in I^d(\boldsymbol{k})$,

$$\psi_{2^{-k}s} = \begin{cases} 1 - \varphi_{0,0}, & k = 1,\ s = 0 \\ \varphi_{0,0}, & k = 1,\ s = 1 \\ -\frac{1}{2}(\varphi_{k,j} + \varphi_{k,j-1}), & k > 1,\ s = 2j,\ 0 \le j < 2^{k-1} - 1, \\ \varphi_{k,j}, & k > 1, s = 2j+1, 0 < j < 2^{k-1} - 1. \end{cases}$$

Upper bounds of sampling recovery for $\mathring{U}^\alpha_\infty$ $(0 < \alpha \leq 2)$

Theorem

Let $d \geq 2$, $0 < p \leq \infty$, $0 < \alpha \leq 2$. Then we have for every $m \geq d$,

$$\sup_{f \in \mathring{U}^\alpha_\infty} \|f - \mathring{R}_m(f)\|_p \leq 2^{-\alpha} B^{-d}\, 2^{-\alpha m} \binom{m}{d-1} \leq 2^{-\alpha} \frac{B^{-d}}{(d-1)!}\, 2^{-\alpha m}\, m^{d-1}, \tag{3}$$

where $B := 2(2^\alpha - 1)(p+1)^{1/p}$.

Improved upper bounds of sampling recovery for $\mathring{U}^\alpha_\infty$ $(1 < \alpha \le 2)$

Theorem

Let $d \ge 2$, $0 < p \le \infty$, $1 < \alpha \le 2$. *Then we have for every* $m \ge 2(d-1)$,

$$\sup_{f \in \mathring{U}^\alpha_\infty} \|f - \mathring{R}_m(f)\|_p \le (2^\alpha - 2)^{-1} B^{-d} 2^{-\alpha m} \binom{m}{d-1} \le (2^\alpha - 2)^{-1} \frac{B^{-d}}{(d-1)!} 2^{-\alpha m} m^{d-1}, \tag{4}$$

where $B := 2^{\alpha+1}(p+1)^{1/p} > 0$.

Tight bounds for optimal recovery of $\mathring{U}^\alpha_\infty$

Theorem

Let $0 < p \le \infty$ *and* $0 < \alpha \le 2$. *Then for every* $m \ge d$,

$$(2^\alpha - 1)^{-1} 2^{-7d} \le \frac{\mathring{s}_m(\mathring{U}^\alpha_\infty)_p}{2^{-\alpha m}\binom{m}{d-1}} \le (2^\alpha - 2)^{-\alpha+1} B^{-d},$$

where $B := 2(2^\alpha - 1)(p+1)^{1/p}$.

- When $B > 1$? If $\alpha > \log_2\left(1 + \frac{1}{2(p+1)^{1/p}}\right)$ in particular, $\alpha \ge 1$, then $B > 1$. In particular, $B > 1$ if
  - $p = \infty$, $\alpha > \log_2 3 - 1 \approx 0.585$, or
  - $p = 1$, $\alpha > \log_2 5 - 2 \approx 0.322$.
- Some upper bounds of $\|f - \mathring{R}_m(f)\|_p$, $p = 2, \infty$, for functions from the Sobolev space $W^2_2(\mathbb{I}^d)$ of mixed smoothness 2 were obtained by Bungartz&Griebel (2004).

Sampling algorithms on sparse grids for $U_\infty^{\alpha,\nu}$

- Since

$$U_\infty^{\alpha,\nu} = \left\{ f \in U_\infty^{\alpha} : \exists u \subset \{1, ..., d\}, |u| = \nu : f = \sum_{u' \subset u} \sum_{\boldsymbol{k} \in \mathbb{Z}_+^d(u')} q_{\boldsymbol{k}}(f) \right\},$$

for sampling recovery of $f \in U_\infty^{\alpha,\nu}$, we take the series truncation

$$R_m^\nu(f) := \sum_{\boldsymbol{k} \in \mathbb{Z}_+^d : |\operatorname{supp}(\boldsymbol{k})| \le \nu, \ |\boldsymbol{k}|_1 \le m} q_{\boldsymbol{k}}(f).$$

- R_m^ν defines a linear sampling algorithm on the Smolyak grid $G^\nu(m)$:

$$R_m^\nu(f) \ = \ S_m^\nu(\Psi_m^\nu, f) \ = \ \sum_{\boldsymbol{\xi} \in G^\nu(m)} f(\boldsymbol{\xi}) \psi_{\boldsymbol{\xi}}, \quad \Psi_m^\nu := \{\psi_{\boldsymbol{\xi}}\}_{\boldsymbol{\xi} \in G^\nu(m)}.$$

Theorem

Let $0 < p \le \infty$, $0 < \alpha \le 2$ *and* $1 \le \nu \le d$. *Then we have for every* $m \ge \nu$,

$$
\begin{aligned}
s_m^\nu(U_\infty^{\alpha,\nu})_p \ \le \ \sup_{f \in U_\infty^{\alpha,\nu}} \|f - R_m^\nu(f)\|_p \ &\le \ 2^{-\alpha}(1+B^{-1})^\nu \, 2^{-\alpha m} \binom{m}{\nu-1} \\
&\le \ 2^{-\alpha} \frac{(1+B^{-1})^\nu}{(\nu-1)!} \, 2^{-\alpha m} \, m^{\nu-1},
\end{aligned} \tag{5}
$$

where $B := 2(2^\alpha - 1)(p+1)^{1/p}$.

Tight bounds for optimal recovery of $U_\infty^{\alpha,\nu}$

Theorem

Let $0 < p \le \infty$, $0 < \alpha \le 2$ *and* $1 \le \nu \le d$. *Then we have for every* $m \ge \nu$,

$$[1 + 2^7(2^\alpha - 1)]^{-1}\, \underline{B}^\nu \le \frac{s_m^\nu(U_\infty^{\alpha,\nu})_p}{2^{-\alpha m}\binom{m}{\nu-1}} \le 2^{-\alpha}\overline{B}^\nu,$$

where $\underline{B} := 1 + \frac{1}{2^7(2^\alpha-1)} > 1$ *and* $\overline{B} := 1 + \frac{1}{2(2^\alpha-1)(p+1)^{1/p}} > 1$

B-spline quasi-interpolation representations

- We want extend the results on upper bounds of $s_m^\nu(U_\infty^{\alpha,\nu})_p$ to arbitrary smoothness $\alpha > 0$. This can be done by using B-spline quasi-interpolation representations.

- Let M be the B-spline of order $2r$ with knots $0, 1, ..., 2r$, and support $[0, 2r]$. Let $\Lambda = \{\lambda(j)\}_{|j|\le\mu}$ be a given even sequence for some $\mu \ge r - 1$. We define the linear operator Q for $f \in C(\mathbb{R})$ by

$$Q(f, x) := \sum_{s\in\mathbb{Z}} \Lambda(f, s) M(x - s), \tag{6}$$

where

$$\Lambda(f, s) := \sum_{|j|\le\mu} \lambda(j) f(s - j + r). \tag{7}$$

- Q is called a *quasi-interpolation operator in* $C(\mathbb{R})$ if $Q(f) = f$ for every polynomial f of degree at most $\ell - 1$ in each variable.

B-spline quasi-interpolation representations

- If $k \in \mathbb{Z}_+$, we introduce the operator Q_k by

$$Q_k(f,x) := \; Q(f,x;h_k), \; x \in \mathbb{R}, \quad h_k := \; (2r)^{-1}2^{-k},$$

where

$$Q(f;h) := \; \sigma_h \circ Q \circ \sigma_{1/h}(f), \quad \sigma_h(f,x) := \; f(x/h).$$

- We define the integer translated dilation $M_{k,s}$ of M by

$$M_{k,s}(x) := \; M(h_k^{-1}x - s), \; k \in \mathbb{Z}_+, \; s \in \mathbb{Z}.$$

Then we have for $k \in \mathbb{Z}_+$,

$$Q_k(f)(x) \; = \; \sum_{s \in \mathbb{Z}} a_{k,s}(f) M_{k,s}(x), \; \forall x \in \mathbb{R},$$

where the coefficient functional $a_{k,s}$ is defined by

$$a_{k,s}(f) := \; \sum_{|j| \le \mu} \lambda(j) f(h_k(s - j + r)). \tag{8}$$

B-spline quasi-interpolation representations

- Since $M(h_k^{-1}x) = 0$ for $x \notin (0,1)$, we can extend $M(h_k^{-1}\cdot)$ to an 1-periodic function on the whole $\mathbb{R}$. Denote this periodic extension by N_k and define

$$N_{k,s}(x) := \ N_k(x - h_k^{-1}s), \ k \in \mathbb{Z}_+, \ s \in I(k),$$

  where $I(k) := \{0, 1, ..., 2r2^k - 1\}$.

- Then we have for functions f on the torus $\mathbb{T}$,

$$Q_k(f,x) \ = \ \sum_{s \in I(k)} a_{k,s}(f) N_{k,s}(x), \quad \forall x \in \mathbb{T}. \tag{9}$$

Periodic B-spline quasi-interpolation representations

- With $Q_{-1}(f) = 0$ we define the operators $q_{\boldsymbol{k}}$ by

$$q_{\boldsymbol{k}} := \prod_{i=1}^{d} \Big(Q_{k_i} - Q_{k_i - 1} \Big), \ \boldsymbol{k} \in \mathbb{Z}_+^d. \tag{10}$$

- Then we can represent the component functions $q_{\boldsymbol{k}}(f)$ as

$$q_{\boldsymbol{k}}(f) = \sum_{\boldsymbol{s} \in I^d(\boldsymbol{k})} c_{\boldsymbol{k},\boldsymbol{s}}(f) N_{\boldsymbol{k},\boldsymbol{s}}, \tag{11}$$

where $I^d(\boldsymbol{k}) := \prod_{i=1}^d I(k_i)$,

$$N_{\boldsymbol{k},\boldsymbol{s}}(x) := \prod_{i=1}^{d} N_{k_i,s_i}(x_i), \quad c_{\boldsymbol{k},\boldsymbol{s}}(f) = \Big(\prod_{j=1}^{d} c_{k_j,s_j} \Big)(f),$$

and c_{k_j,s_j} are certain coefficient functionals of f as a univariate function in variable x_j.

Periodic B-spline quasi-interpolation representations

Theorem

Let $1 \le p \le \infty$, and $0 < \alpha \le 2r$. Then

(i) every $f \in H_\infty^\alpha(\mathbb{T}^d)$ can be represented by the series

$$f = \sum_{\boldsymbol{k} \in \mathbb{Z}_+^d} q_{\boldsymbol{k}}(f), \quad q_{\boldsymbol{k}}(f) := \sum_{\boldsymbol{s} \in I^d(\boldsymbol{k})} c_{\boldsymbol{k},\boldsymbol{s}}(f) N_{\boldsymbol{k},\boldsymbol{s}}, \tag{12}$$

converging in the norm of $C(\mathbb{T}^d)$, where the coefficient functionals $c_{\boldsymbol{k},\boldsymbol{s}}(f)$ are explicitly constructed as linear combinations of at most m_0 of function values of f for some number $m_0 \in \mathbb{N}$ which is independent of $\boldsymbol{k}, \boldsymbol{s}$ and f.

(ii) Moreover, for every $\boldsymbol{k} \in \mathbb{Z}_+^d(u)$,

$$\|q_{\boldsymbol{k}}(f)\|_p \le a^{|u|}\, b^{d-|u|}\, 2^{-\alpha|\boldsymbol{k}|_1}\, |f|_{H_\infty^\alpha(u)},$$

where a and $b \ge 1$ are explicit functions in α, r, p, Λ.

Upper bounds for sampling recovery ($0 < \alpha \le 2r$)

Theorem

Let $1 \le p \le \infty$, $0 < \alpha \le 2r$ *and* $1 \le \nu \le d$. *Then for* $m \ge \nu$,

$$\begin{aligned} s_m^\nu(U_\infty^{\alpha,\nu})_p &\le \sup_{f \in U_\infty^{\alpha,\nu}} \|f - R_m^\nu(f)\|_p \\ &\le 2^{-\alpha} B^\nu \, 2^{-\alpha m} \binom{m}{\nu - 1} \qquad (13) \\ &\le 2^{-\alpha} \frac{B^\nu}{(\nu-1)!} 2^{-\alpha m} \, m^{\nu-1}, \end{aligned}$$

where $B := \frac{a}{1-2^{-\alpha}} + b > 1$

Improved upper bounds for sampling recovery $(1 < \alpha \le 2r)$

Theorem

Let $1 \le p \le \infty$, $1 < \alpha \le 2r$ *and* $1 \le \nu \le d$. *Then for* $m \ge 2(\nu - 1)$,

$$
\begin{aligned}
s_m^\nu(U_\infty^{\alpha,\nu})_p &\le \sup_{f \in U_\infty^{\alpha,\nu}} \|f - R_m^\nu(f)\|_p \\
&\le (2^\alpha - 2)^{-1} (a+b)^\nu \, 2^{-\alpha m} \binom{m}{\nu - 1} \qquad (14) \\
&\le (2^\alpha - 2)^{-1} \frac{(a+b)^\nu}{(\nu-1)!} \, 2^{-\alpha m} \, m^{\nu - 1}.
\end{aligned}
$$

Remark: For a given number of active variables ν, the upper bounds (in both the theorems) for sampling recovery of $f \in U_\infty^{\alpha,\nu}$ do not depend on the dimension d.

Thank you for your attention!