Multiple Rank-1 Lattices as Sampling Schemes for Approximation

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joint work with Toni Volkmer

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Outline

Introduction

We consider

- periodic and
- sufficiently smooth
- multivariate

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$$\mathrm{Samp}_{\mathcal{G}}(\mathcal{F}, Y) := \inf_{A\colon \mathbb{C}^M \to Y} \sup_{\|f\|_{\mathcal{F}} \le 1} \left\| f - A(f(\mathbf{x}_j))_{j=1}^M \right\|_Y$$

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and the general sampling numbers

$$g_M(\mathcal{F}, Y) := \inf_{|\mathcal{G}| \le M} \mathrm{Samp}_{\mathcal{G}}(\mathcal{F}, Y).$$

Introduction

Fourier series of sufficiently smooth periodic function

$$f(\mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{Z}^d} c_{\mathbf{k}}(f)\mathrm{e}^{2\pi\mathrm{i}\mathbf{k}\cdot\mathbf{x}}$$

with Fourier coefficients

$$c_{\mathbf{k}}(f) = \int_{\mathbb{T}^d} f(\mathbf{x})\mathrm{e}^{-2\pi\mathrm{i}\mathbf{k}\cdot\mathbf{x}}\mathrm{d}\mathbf{x}$$

can be well approximated by its Fourier partial sum

$$S_{\mathrm{I}}[f](\mathbf{x}) = \sum_{\mathbf{k}\in\mathrm{I}} c_{\mathbf{k}}(f)\mathrm{e}^{2\pi\mathrm{i}\mathbf{k}\cdot\mathbf{x}}.$$

The characteristics of $\mathcal{F}$ predetermine suitable frequency sets I for estimates on the worst case error for approximation. The considered linear algorithms need to uniquely reconstruct at least all trigonometric polynomials that are supported on these frequency sets.

Introduction – Notation

- $p : \mathbb{T}^d \to \mathbb{C}$ multivariate trigonometric polynomial

$$p(\mathbf{x}) = \sum_{\mathbf{k} \in \mathrm{I}} \hat{p}_{\mathbf{k}} \mathrm{e}^{2\pi \mathrm{i} \mathbf{k} \cdot \mathbf{x}}$$

  with $\mathrm{I} \subset \mathbb{Z}^d$, $|\mathrm{I}| < \infty$ and $\hat{\mathbf{p}} = (\hat{p}_{\mathbf{k}})_{\mathbf{k} \in \mathrm{I}} \in \mathbb{C}^{|\mathrm{I}|}$
- discretization in spatial domain $\mathcal{X} \subset \mathbb{T}^d$, $|\mathcal{X}| < \infty$
- evaluation of p at all nodes of $\mathcal{X}$

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- as matrix vector product with $\mathbf{A} = \left(\mathrm{e}^{2\pi \mathrm{i} \mathbf{k} \cdot \mathbf{x}}\right)_{\mathbf{x} \in \mathcal{X}, \mathbf{k} \in \mathrm{I}}$

$$\mathbf{A}\hat{\mathbf{p}} = \mathbf{p}, \quad \hat{\mathbf{p}} = (\hat{p}_{\mathbf{k}})_{\mathbf{k} \in \mathrm{I}} \text{ and } \mathbf{p} = (p(\mathbf{x}))_{\mathbf{x} \in \mathcal{X}}$$

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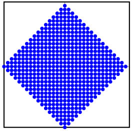

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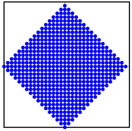

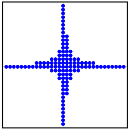

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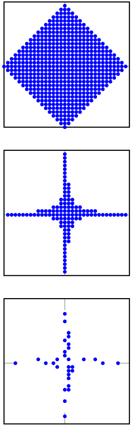

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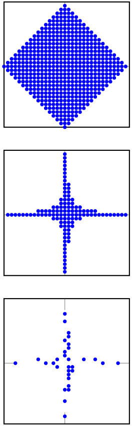

dimension
$d \in \mathbb{N}$
large

Introduction – Sparse Grids & Single Rank-1 Lattices

sparse grid		single rank-1 lattice
✓	construction of sampling sets	✗
✓	oversampling	✗
✓	approximation properties	✓
✗	ease of implementation	✓
✗	flexibility	✓

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Focus on the discretization of trigonometric polynomials with

- unique reconstruction
- efficient reconstruction algorithms
- efficient algorithms for the construction of the sampling sets
- low (i.e. almost optimal) oversampling
- good approximation properties

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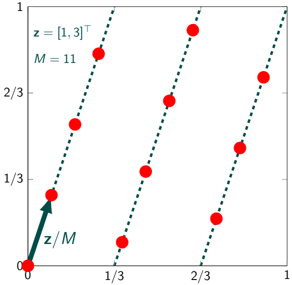
- rank-1 lattice: $\mathbf{z} \in \mathbb{N}^d, M \in \mathbb{N}$

$$\mathbf{x}_j = \frac{j\mathbf{z}}{M} \bmod \mathbf{1},\ j = 0, \ldots, M-1$$



Korobov 59
Maisonneuve 72
Sloan & Kachoyan 84,87,90
Temlyakov 86
Lyness 89
Sloan & Joe 94
Sloan & Reztsov 01
Li & Hickernell 03

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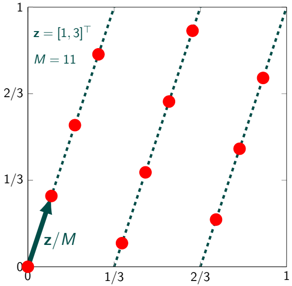

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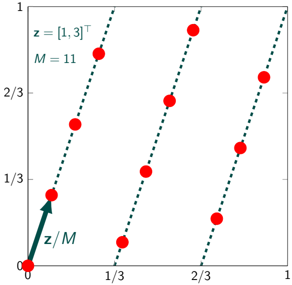

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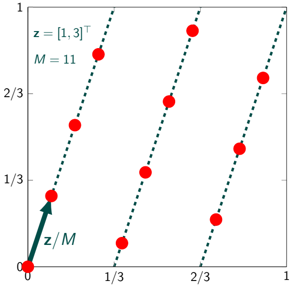

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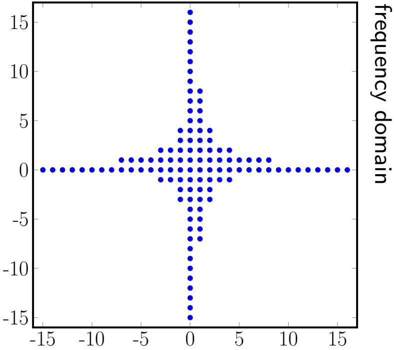

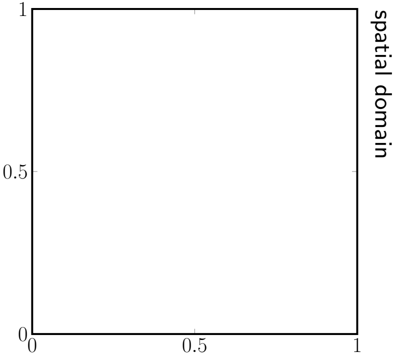

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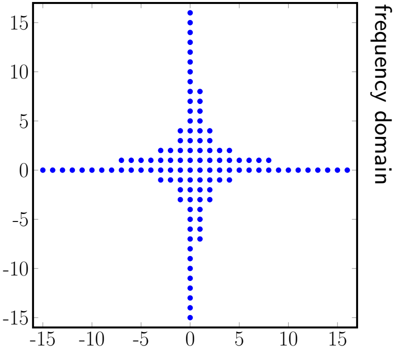

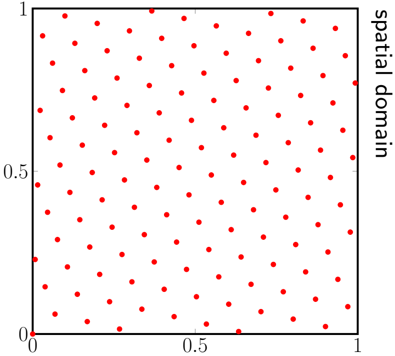

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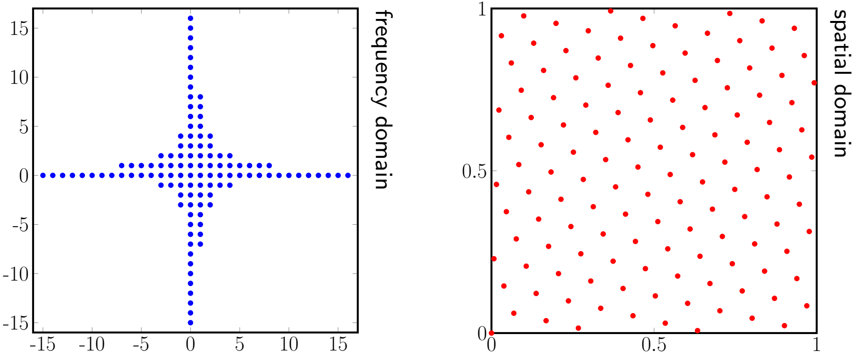

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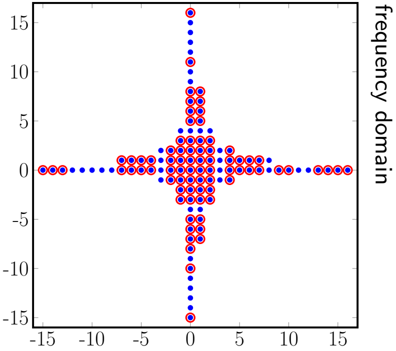

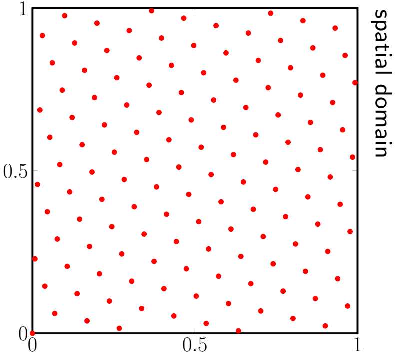

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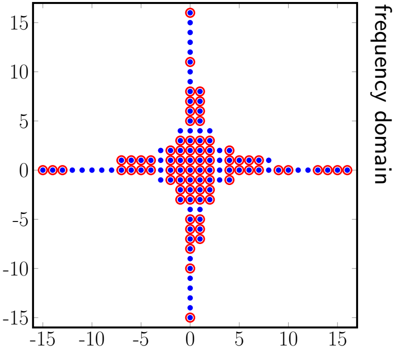

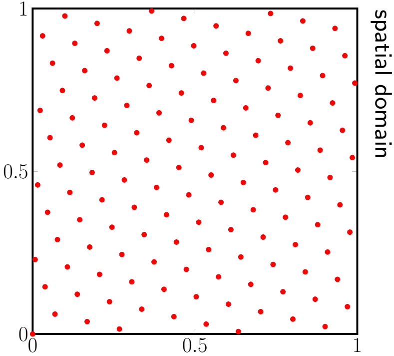

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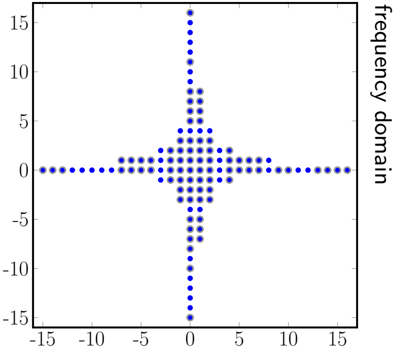

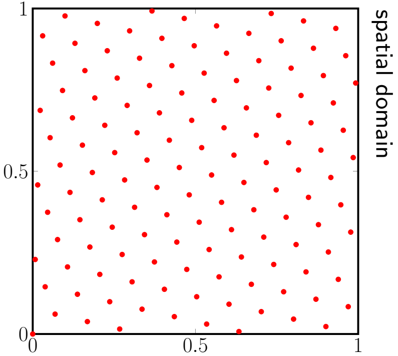

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- collect frequencies of already computed $\hat{p}_{\mathbf{k}}$ in $\mathrm{I}_{\bullet} := \mathrm{I}_{\odot}$

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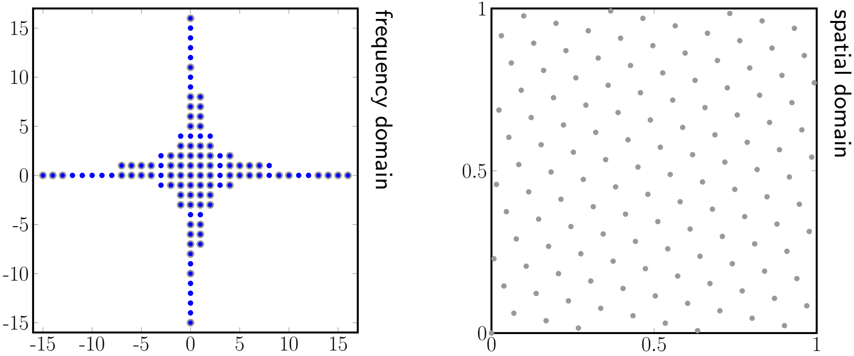

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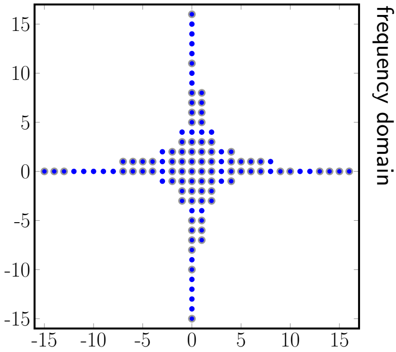

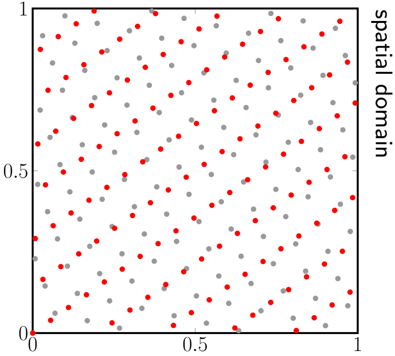

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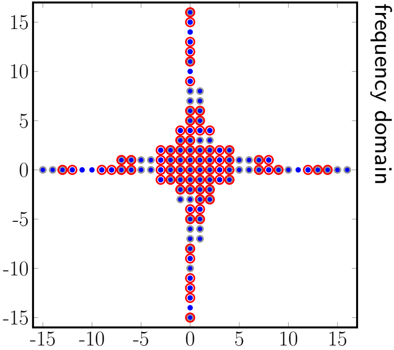

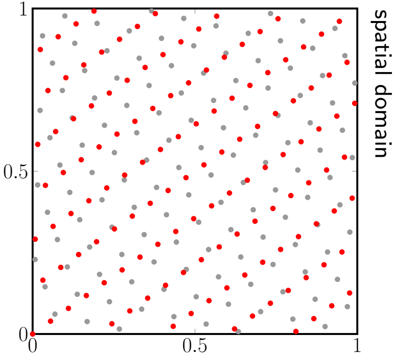

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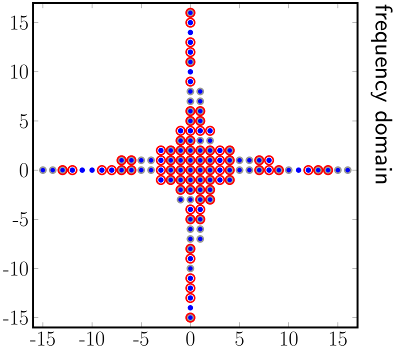

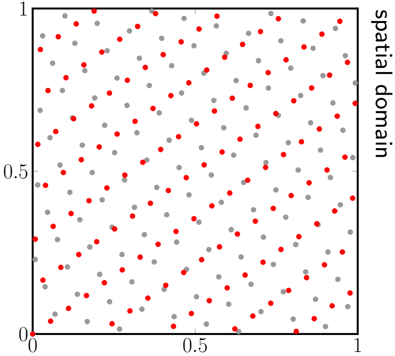

- reconstruct $\hat{p}_{\mathbf{k}}$, $\mathbf{k} \in \mathrm{I}_{\odot}$, using the rank-1 lattice $\Lambda(\mathbf{z}_2, M_2)$

$$\hat{p}_{\mathbf{k}} = \frac{1}{M_2} \sum_{j=0}^{M_2} p\left(\frac{j}{M_2}\mathbf{z}_2\right) \mathrm{e}^{-2\pi \mathrm{i} j \frac{\mathbf{k}\cdot\mathbf{z}_2}{M_2}}, \quad \mathbf{k} \in \mathrm{I}_{\odot}$$

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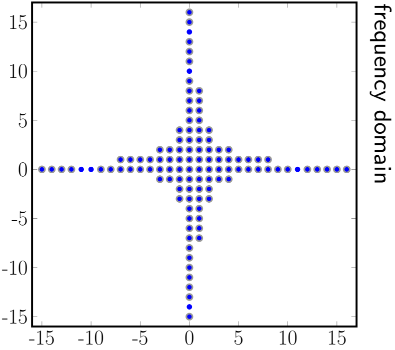

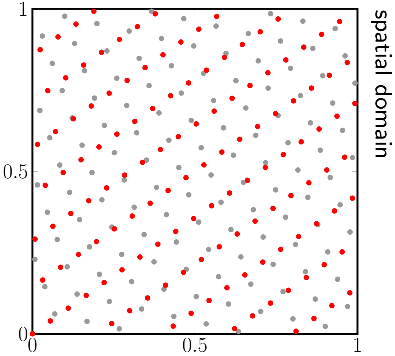

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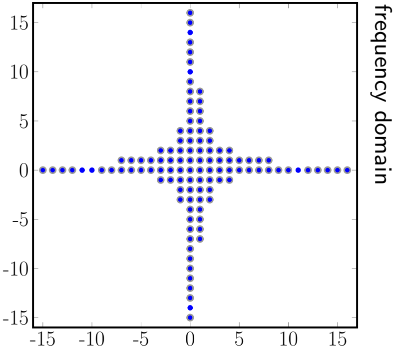

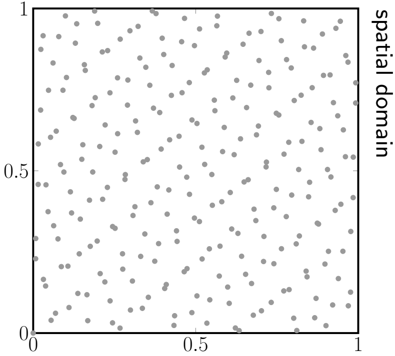

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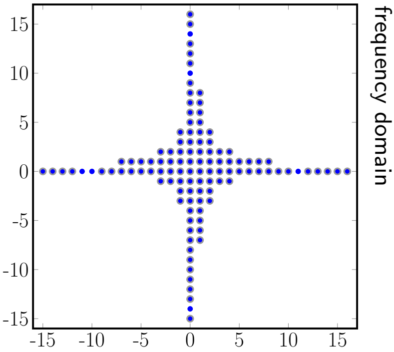

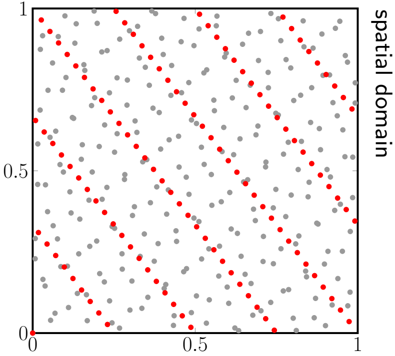

- choose rank-1 lattice $\Lambda(\mathbf{z}_3, M_3)$
  - lattice size $M_3 \sim |\mathrm{I}|$ and
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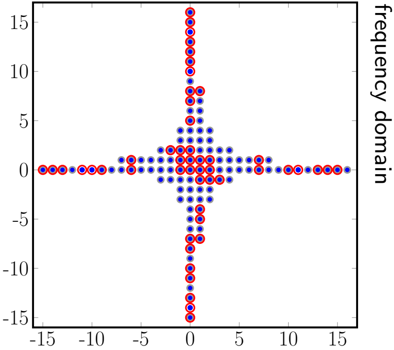

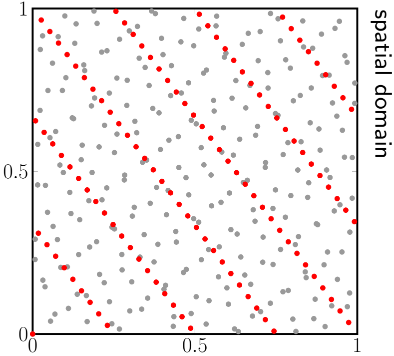

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- determine

$$\mathrm{I}_{\odot} := \{\mathbf{k} \in \mathrm{I} \colon \mathbf{k} \cdot \mathbf{z}_3 \neq \mathbf{h} \cdot \mathbf{z}_3 \pmod{M_3} \text{ for all } \mathbf{h} \in \mathrm{I} \setminus \{\mathbf{k}\}\}$$

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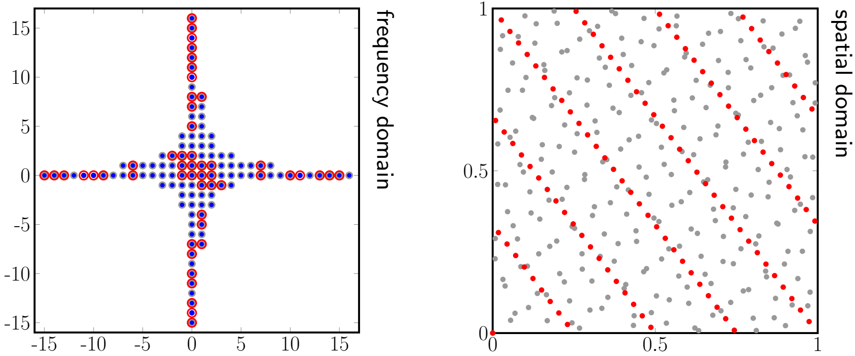

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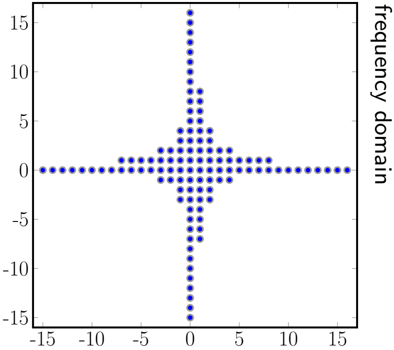

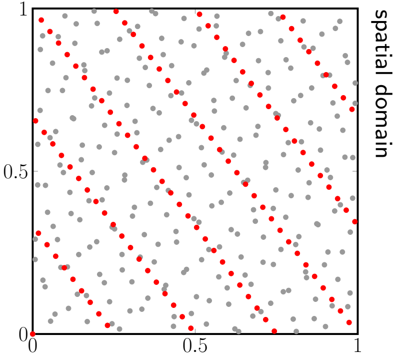

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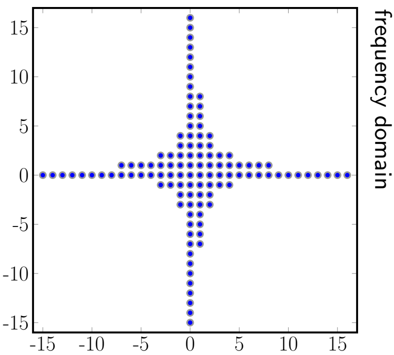
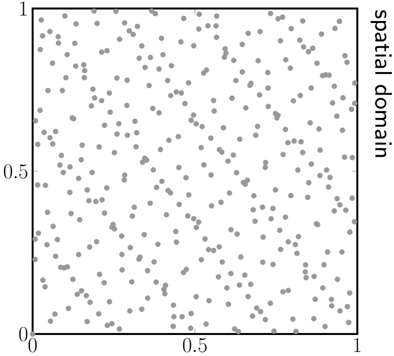
Multiple Rank-1 Lattices – General Idea





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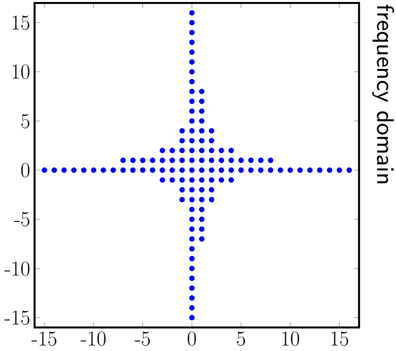

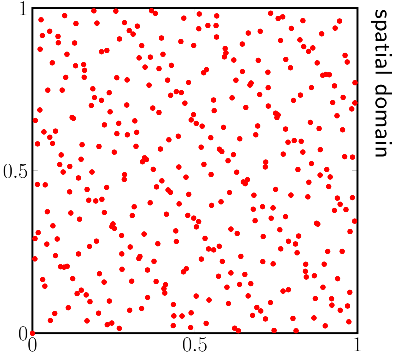

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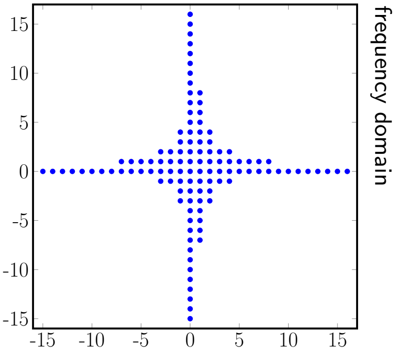

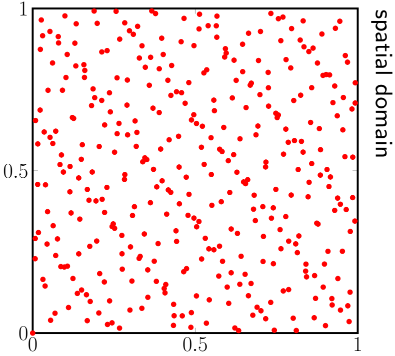

- used $L = 3$ rank-1 lattices, $M \leq M_1 + \ldots + M_L - L + 1$

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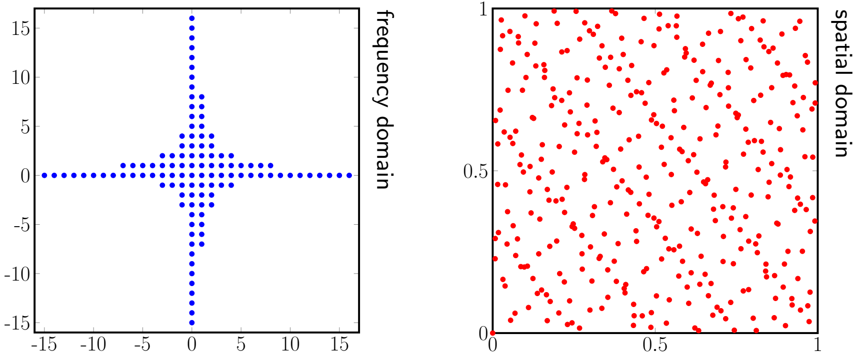

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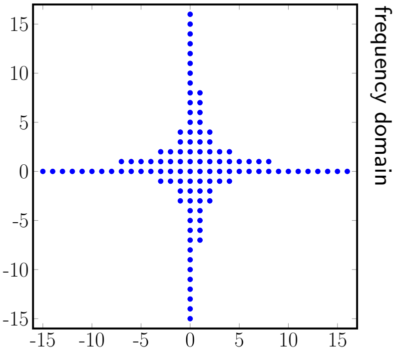
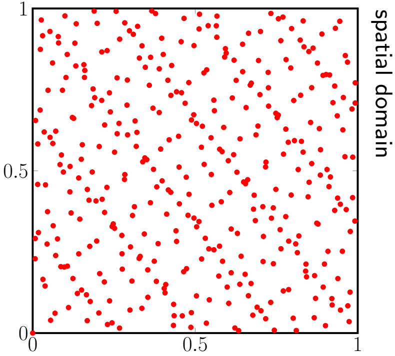
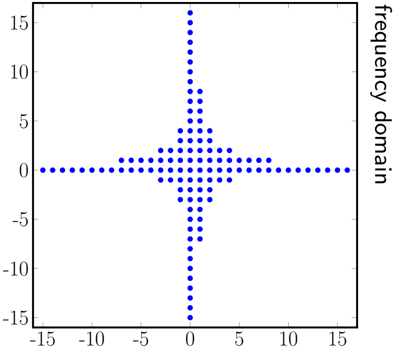
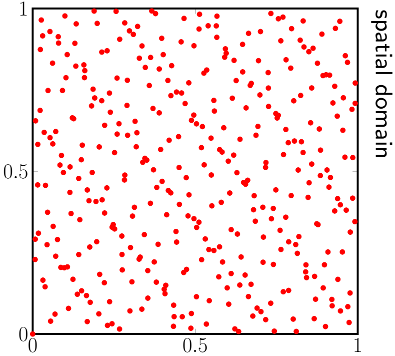




- used $L = 3$ rank-1 lattices, $M \leq M_1 + \ldots + M_L - L + 1$
- L rank-1 lattice FFTs, sorting
- complexity: $\mathcal{O}\left(M \log M + L(d + \log |\mathrm{I}|)|\mathrm{I}|\right)$

Multiple Rank-1 Lattices – General Idea





- number of samples for $M_j \lesssim |\mathrm{I}|$

$$M_1 + \ldots + M_L \leq L \max_{j=1,\ldots,L} M_j \lesssim |\mathrm{I}|\, L$$

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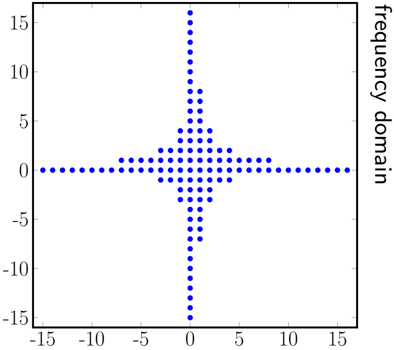

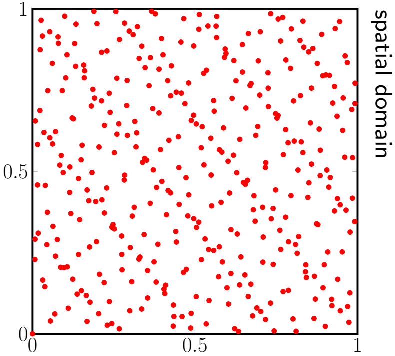

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- complexity of FFT-Algorithm for $M_j \lesssim |\mathrm{I}|$ and $L \lesssim |\mathrm{I}|$

$$M \log M + L(d + \log |\mathrm{I}|)|\mathrm{I}| \lesssim |\mathrm{I}|\,(d + \log |\mathrm{I}|)\, L$$

Multiple Rank-1 Lattices – Estimating L

Theorem (K. 2017)

Let $\mathrm{I} \subset [-N, N]^d \cap \mathbb{Z}^d$, $|\mathrm{I}| < \infty$ *and* $\delta, c \in \mathbb{R}$, $0 < \delta < 1$, $c > 1$, *be given. Choosing primes* $M_j \geq \max(c|\mathrm{I}|, 2N + 1)$ *in the Algorithm above, the number* L *is bounded by*

$$L \leq \left\lceil \left(\frac{c}{c-1}\right)^2 \frac{\log|\mathrm{I}| - \log\delta}{2} \right\rceil$$

with probability at least $1 - \delta$.

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- complexity of construction of spatial discretization $\mathcal{X}$ (w.h.p.)

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- $c|\mathrm{I}| \geq 2N + 1 \Rightarrow M_1 = \ldots = M_L \leq 2c|\mathrm{I}|$ and
  - number of sampling nodes $M < M_1 + \ldots + M_L \lesssim |\mathrm{I}| \log|\mathrm{I}|$
  - complexity of FFT Algorithm $\lesssim |\mathrm{I}| \, (d + \log|\mathrm{I}|) \, \log|\mathrm{I}|$

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with probability at least $1-\delta$.

Advantages:

+ unique reconstruction with low oversampling factors $\frac{M}{|\mathrm{I}|} \lesssim L$
+ fast Fourier transform and fast construction of $\mathcal{X}$

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- \+ fast Fourier transform and fast construction of $\mathcal{X}$
- ? approximation properties

Multiple Rank-1 Lattices – Approximation Setting

periodic Sobolev spaces of generalized mixed smoothness

$$\mathcal{F} = \mathcal{H}^{\alpha,\beta,\boldsymbol{\gamma}}(\mathbb{T}^d) := \left\{ f \in L_1(\mathbb{T}^d) \colon \sqrt{\sum_{\mathbf{k}\in\mathbb{Z}^d} \omega^{\alpha,\beta,\boldsymbol{\gamma}}(\mathbf{k})^2 |c_{\mathbf{k}}(f)|^2} < \infty \right\}$$

$$\omega^{\alpha,\beta,\boldsymbol{\gamma}}(\mathbf{k}) := \max(1, \|\mathbf{k}\|_1)^\alpha \prod_{j=1}^{d} \max(1, \gamma_j^{-1/2} |k_j|^\beta), \qquad \boldsymbol{\gamma} \in (0,1]^d$$

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Apply sampling operator and achieve

$$S_{\mathrm{I}}^{\Lambda}[f](\mathbf{x}) := \sum_{\mathbf{k}\in\mathrm{I}} \hat{f}_{\mathbf{k}}^{\Lambda} \mathrm{e}^{2\pi\mathrm{i}\mathbf{k}\cdot\mathbf{x}}$$

with

$$\mathrm{I} := \left\{ \mathbf{k} \in \mathbb{Z}^d \colon \omega^{\alpha,\beta,\boldsymbol{\gamma}}(\mathbf{k}) \leq N^{\alpha+\beta} \right\}.$$

Multiple Rank-1 Lattices – Approximation Results

smoothness parameters

- isotropic smoothness $\alpha < 0$
- mixed smoothness $\beta > 1/2 - \alpha$

Y	$g_M^{\mathrm{mr1l}}(\mathcal{H}^{\alpha,\beta,\gamma}(\mathbb{T}^d), Y)$	$g_M^{\mathrm{sr1l}}(\mathcal{H}^{\alpha,\beta,\gamma}(\mathbb{T}^d), Y)$	$g_M^{\mathrm{lin}}(\mathcal{H}^{\alpha,\beta,\gamma}(\mathbb{T}^d), Y)$
$L_2(\mathbb{T}^d)$	$\lesssim M^{-(\alpha+\beta)+1/2+\varepsilon}$	$\asymp M^{-\frac{\alpha+\beta}{2}}$	$\asymp M^{-(\alpha+\beta)}$
$L_\infty(\mathbb{T}^d)$	$\lesssim M^{-(\alpha+\beta)+1/2+\varepsilon}$	$\lesssim M^{-\frac{\alpha+\beta-1/2}{2}}$	$\lesssim M^{-(\alpha+\beta)+1/2}$

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smoothness parameters

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$L_2(\mathbb{T}^d)$	$\lesssim M^{-\beta+1/2+\varepsilon}$	$\lesssim M^{-\beta/2+\varepsilon}$	$\lesssim M^{-\beta}(\log M)^{(d-1)(\alpha+1/2)}$
$L_\infty(\mathbb{T}^d)$	$\lesssim M^{-\beta+1/2+\varepsilon}$	$\lesssim M^{-\frac{\beta-1/2}{2}+\varepsilon}$	$\lesssim M^{-\beta+1/2}(\log M)^{(d-1)\alpha}$

Multiple Rank-1 Lattices – Tractability Results

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$$s_{\boldsymbol{\gamma}} := \inf\left\{ s\colon \sum_{j=1}^{\infty} \gamma_j^s < \infty \right\} \qquad \tilde{\beta} = \min\{2\beta, 1/s_{\boldsymbol{\gamma}}\}$$

Y	Λ^{all}	Λ^{std}	$g_M^{\mathrm{mr1l}}(\mathcal{H}^{0,\beta,\boldsymbol{\gamma}}(\mathbb{T}^d), Y)$
$L_\infty(\mathbb{T}^d)$	$\leq c_{\tilde{\beta}} M^{-\frac{\tilde{\beta}-1}{2}}$	$\leq c_{\tilde{\beta},\varepsilon} M^{-\frac{\tilde{\beta}-1}{2}\frac{\tilde{\beta}}{\tilde{\beta}+1}+\varepsilon}$	
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$\Rightarrow$ improvement on rate of convergence for L_∞ approximation and Λ^{std}

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$L_2(\mathbb{T}^d)$	$\leq c_{\tilde{\beta}} M^{-\frac{\tilde{\beta}}{2}}$	$\leq c_{\tilde{\beta},\varepsilon} M^{-\frac{\tilde{\beta}}{2}\frac{\tilde{\beta}}{\tilde{\beta}+1}+\varepsilon}$	[1] $\leq c_{\tilde{\beta},\varepsilon} M^{-\frac{\tilde{\beta}}{4}+\varepsilon}$

$\Rightarrow$ improvement on rate of convergence for L_∞ approximation and Λ^{std}

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[1] result from single rank-1 lattice

Multiple Rank-1 Lattices – Tractability Results

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$$s_{\boldsymbol{\gamma}} := \inf \left\{ s \colon \sum_{j=1}^{\infty} \gamma_j^s < \infty \right\} \qquad \tilde{\beta} = \min\{2\beta, 1/s_{\boldsymbol{\gamma}}\}$$

Y	Λ^{all}	Λ^{std}	$g_M^{\text{mr1l}}(\mathcal{H}^{0,\beta,\boldsymbol{\gamma}}(\mathbb{T}^d), Y)$
$L_\infty(\mathbb{T}^d)$	$\leq c_{\tilde{\beta}} M^{-\frac{\tilde{\beta}-1}{2}}$	$\leq c_{\tilde{\beta},\varepsilon} M^{-\frac{\tilde{\beta}-1}{2} \cancel{\frac{\tilde{\beta}}{\tilde{\beta}+1}} + \varepsilon}$	$\leq c_{\tilde{\beta},\varepsilon} M^{-\frac{\tilde{\beta}-1}{2}+\varepsilon}$
$L_2(\mathbb{T}^d)$	$\leq c_{\tilde{\beta}} M^{-\frac{\tilde{\beta}}{2}}$	$\leq c_{\tilde{\beta},\varepsilon} M^{-\frac{\tilde{\beta}}{2} \frac{\tilde{\beta}}{\tilde{\beta}+1} + \varepsilon}$	[1] $\leq c_{\tilde{\beta},\varepsilon} M^{-\frac{\tilde{\beta}}{4}+\varepsilon}$

$\Rightarrow$ improvement on rate of convergence for L_∞ approximation and Λ^{std}

- still open: MR1L and L_2 approximation

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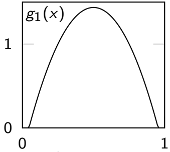
[1] result from single rank-1 lattice

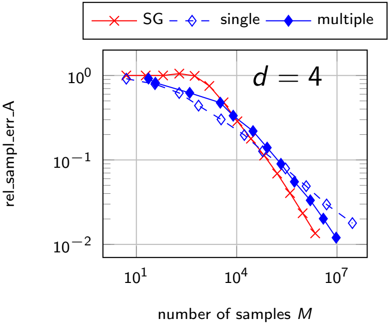
Multiple Rank-1 Lattices – Numerical Example

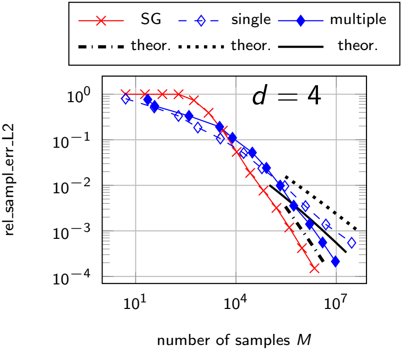
Kink Function

$$g_d(\mathbf{x}) = \prod_{t=1}^{d} \frac{121\sqrt{33}}{100} \max\left\{ \frac{25}{121} - \left(x_t - \frac{1}{2} \right)^2, 0 \right\}$$



$g_d \in \mathcal{H}^{0,3/2-\varepsilon,\mathbf{1}}(\mathbb{T}^d)$, $\|g_d|\mathcal{A}(\mathbb{T}^d)\| \approx (1.84190)^d$, $\|g_d|L_2(\mathbb{T}^d)\| = 1$, $\mathrm{I} = H_n^d$





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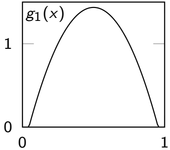

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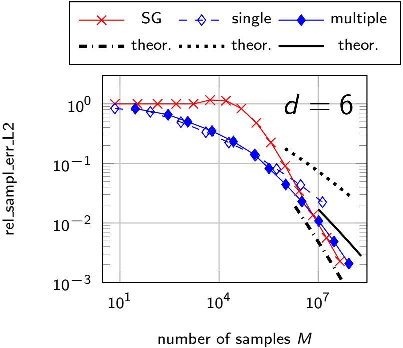

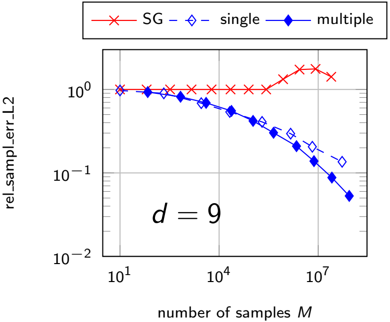

Summary

- multiple rank-1 lattices used as spatial discretizations of trigonometric polynomials with frequencies supported on known index sets I
  - low oversampling factors $M/|\mathrm{I}| \lesssim \log|\mathrm{I}|$
  - fast construction methods $\mathcal{O}(|\mathrm{I}|\,(d+\log|\mathrm{I}|)\,\log|\mathrm{I}|)$
  - FFT algorithm $\mathcal{O}(|\mathrm{I}|\,(d+\log|\mathrm{I}|)\,\log|\mathrm{I}|)$
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Thank you!

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