Bringing pure and applied analysis together via the Wiener-Hopf technique, its generalisations and applications

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From Sommerfeld problems to operator factorisation

1 Introduction

The original Sommerfeld diffraction problem has been solved by series expansion in 1896 NZS04*,Som96 and some 50 years later by the so-called Wiener-Hopf technique in the 1940s, see Jon64,Nob58,Wei69.

Another 40 years later it has been proved to be well posed in the sense of Hadamard Had02 in certain Sobolev space settings MS89,S86

Moreover the resolvent operator has been identified as an explicit formula based upon operator factorisation in the sense of Shinbrot in the context of a wider class of problems DevShi69,Shi64,S85,S86.

*[NZS04] R.J. Nagem, M. Zampolli, and G. Sandri, *Mathematical Theory of Diffraction.* Birkhäuser, Boston 2004.

In eight sections we shall address the following questions:

Part I

- How can we consider the classical Wiener-Hopf procedure as an operator factorisation (OF) and what is the profit of that interpretation?
- What are the characteristics of Wiener-Hopf operators occurring in Sommerfeld half-plane problems and their features in terms of functional analysis?

Part II

- What are the most relevant methods of constructive matrix factorisation in Sommerfeld problems?
- How does OF appear generally in linear boundary value and transmission problems and why is it useful to think about this question?
- What are adequate choices of function(al) spaces and symbol classes in order to analyse the well-posedness of problems and to use deeper results of factorisation theory?

Part III

- A sharp logical concept for equivalence and reduction of linear systems (in terms of OF)? Why is it needed and why does it simplify and strengthen the reasoning?
- Where do we need other kinds of operator relations beyond OF?
- What are very practical examples for the use of the preceding ideas, e.g., in higher dimensional diffraction problems?

So we come to the following work schedule:

Contents

Historical remarks and corresponding references are provided at the end of each section.

The main objective of all this is to demonstrate how real applications naturally guide us to operator factorisation concepts and how useful those are to simplify and to strengthen the reasoning in the applications.

The exposition of these concepts will be accompanied by concrete problems closely connected with the Rawlins problem Raw81,Raw84.

2 From the classical Wiener-Hopf technique to operator factorisation

First we consider the linear (non-singular) convolution equation on the real half-line

$$\mathcal{W}f(x) \quad = \quad af(x) - \int_0^\infty K(x-y)\, f(y)\, dy \, = \, g(x) \quad , \quad x > 0. \tag{1}$$

Herein $a \in \mathbb{C} \setminus \{0\}, K \in L^1(\mathbb{R})$ and $g \in L^p(\mathbb{R}_+), p \in [1, \infty]$ are (arbitrarily) given and $f \in L^p(\mathbb{R}_+)$ is unknown, $\mathbb{R}_+ = (0, \infty)$. It is well known that $\mathcal{W}$ defines a bounded linear operator

$$\mathcal{W} \quad : \quad L^p(\mathbb{R}_+) \to L^p(\mathbb{R}_+) \, , \tag{2}$$

in brief $\mathcal{W} \in \mathcal{L}(L^p(\mathbb{R}_+))$, known as classical *Wiener-Hopf operator* (WHO) GohFel74, Kre58, MikPro86. For simplicity we abbreviate $L^p = L^p(\mathbb{R})$ and choose $a = 1$ and $p = 2$ in this discussion, focusing the scalar case (instead of systems of equations GohKre58).

Now let us look at the steps of the WH technique Mei83a,Nob58 from the viewpoint of operator theory. Certainly one likes to find out whether the equation (1) is uniquely solvable and, in this case, the solution f depends continuously on the given function g, (i.e., whether (1) is well-posed) and also to obtain an explicit analytical formula for the solution, in brief to determine $W^{-1} \in \mathcal{L}(L^2(\mathbb{R}_+))$ if possible.

To this end we need some notation. We denote in (1) the convolution operator on the full real line by A and obtain with the help of the *Fourier transformation* $\mathcal{F}$ from the convolution theorem that

$$\begin{aligned} A &= I + K* = \mathcal{F}^{-1}\,\Phi\cdot\mathcal{F} \quad : \quad L^2 \to L^2\,, \qquad (3) \\ \Phi &= 1 + \mathcal{F}K \quad \subset \quad L^\infty(\mathbb{R}), \\ \mathcal{F}K(\xi) &= \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{ix\xi}\,K(x)\,dx \quad , \quad \xi \in \mathcal{R}\,. \end{aligned}$$

Hence the classical Wiener-Hopf operator can be written as

$$\mathcal{W} \quad = \quad r_+ \, A \, \ell_0 \quad = \quad L^2(\mathbb{R}_+) \,\to\, L^2(\mathbb{R}_+) \tag{4}$$

where r_+ and ℓ_0 denote the restriction and zero extension operator, respectively:

$$r_+ \;:\; L^2(\mathbb{R}) \to L^2(\mathbb{R}_+) \qquad , \qquad \ell_0 \;:\; L^2(\mathbb{R}_+) \to L^2(\mathbb{R}) \,, \tag{5}$$

which are bounded linear operators.

Obviously we have

$$r_+ \, \ell_0 \;=\; I_{L^2(\mathbb{R}_+)} \quad , \quad P \;=\; \ell_0 \, r_+ \;\in\; \mathcal{L}(L^2) \tag{6}$$

where P is a projector acting in L^2 with

$$\begin{aligned} \operatorname{im} P \;&=\; L^2_+ \;=\; \{f \in L^2 \,:\, \operatorname{supp} f \subset \overline{\mathbb{R}_+}\} \\ \ker P \;&=\; L^2_- \;=\; \{f \in L^2 \,:\, \operatorname{supp} f \subset \overline{\mathbb{R}_-}\} \,. \end{aligned} \tag{7}$$

With the help of this basic notation the steps of the Wiener-Hopf technique can be described as follows.

First the equation (1) is extended by zero to an equation holding on the full line:

$$A f_+ \quad = \quad g_+ + g_- \tag{8}$$

where $g_+ = \ell_0\, g \in L^2_+$ is known and $f_+ = \ell_0\, g \in L^2_+$, $g_- \in L^2_-$ are unknown provided (1) is solvable.

In what follows the *Fourier symbol* $\Phi = 1 + \mathcal{F}K$ of A plays a decisive role. Note that this function is also referred to as "Fourier transformed kernel" or simply as "kernel" in some papers such as LawAbr07 which has to be distinguished from the convolution kernel K of the integral operator A and from the kernel (or null space) of a linear operator like A or W (hence we avoid calling Φ a kernel).

Now let us assume that $|\Phi|$ is bounded from below. Since the Fourier symbol Φ of A belongs to the *Wiener algebra* $\mathbb{C} \oplus \mathcal{F}L^1$ (with $\Phi(\infty) = 1$), it allows a factorisation GohKru73,Kre58

$$\Phi = \Phi_- \ \zeta^\kappa \ \Phi_+ \,, \tag{9}$$

$$\kappa = \frac{1}{2\pi} \int_{\mathbb{R}} d \arg \Phi(\xi) d\xi \in \mathbb{Z} \quad , \quad \zeta(\xi) = \frac{\xi - i}{\xi + i} \,, \ \xi \in \mathbb{R} \,,$$

$$\Phi_\pm = \exp\{\mathcal{F} \ell_0 \, r_\pm \mathcal{F}^{-1} \log(\zeta^{-\kappa} \Phi)\} \,.$$

With the help of this factorisation one can continue considering the Fourier transformed equation of (8) (as common in the classical Wiener-Hopf technique Nob58) or, alternatively, making directly use of the operator factorisation that results from (9):

$$A = A_- \; C \; A_+ \,, \tag{10}$$

$$A_\pm = \mathcal{F}^{-1}\,\Phi_\pm \cdot \mathcal{F} \quad , \quad C = \mathcal{F}^{-1}\,\zeta^\kappa \cdot \mathcal{F} \quad : \quad L^2 \to L^2 \,.$$

All the three factors A_-, C, A_+ are isomorphisms in L^2 (i.e., linear homeomorphisms, boundedly invertible operators), and the inverses are convolution operators with Fourier symbols $\Phi_-^{-1}, \zeta^{-\kappa}, \Phi_+^{-1}$ in the Wiener algebra and value 1 at infinity.

The next step in the Wiener-Hopf method is a rearrangement of equation (8) which is different depending on whether $\kappa \geq 0$ or $\kappa \leq 0$ (and coinciding for $\kappa = 0$ where $C = I$), namely

$$CA_+\,f_+ = A_-^{-1}\,g_+ + A_-^{-1}\,g_- \quad , \quad \text{if } \kappa \geq 0, \tag{11}$$

$$A_+\,f_+ = C^{-1}A_-^{-1}\,g_+ + C^{-1}A_-^{-1}\,g_- \;, \; \text{if } \kappa \leq 0\,.$$

Now the classical Wiener-Hopf method Nob58 works with holomorphy properties of the Fourier transformed terms of the equation and an additive decomposition of the first term on the right hand side which results in the same conclusion as applying the projector P on the two equations, namely

$$\begin{aligned} PCA_+ f_+ &= PA_-^{-1} g_+ \quad , \quad \text{if } \kappa \geq 0, \\ PA_+ f_+ &= PC^{-1} A_-^{-1} g_+ \quad , \quad \text{if } \kappa \leq 0 \,. \end{aligned} \tag{12}$$

This is just a consequence of the invariance properties of the factors: A_+ maps $\operatorname{im} P$ onto itself, A_- maps $\ker P$ onto itself. Furthermore, in case of $\kappa \geq 0$, the middle factor C maps $\operatorname{im} P$ into itself, whilst C^{-1} maps $\ker P$ into itself, and in case of $\kappa \leq 0$, it's the other way around.

These properties together are equivalent to the following formulas, so-called *invariance properties* of the factors

$$
\begin{aligned}
PA_+P &= A_+P \,, & PA_+^{-1}P &= A_+^{-1}P \,, \\
PA_-P &= PA_- \,, & PA_-^{-1}P &= PA_-^{-1} \,, \\
PCP &= CP \,, & PC^{-1}P &= PC^{-1} \,, \text{ if } \kappa \geq 0 \,, \\
PCP &= PC \,, & PC^{-1}P &= C^{-1}P \,, \text{ if } \kappa \leq 0 \,.
\end{aligned}
\tag{13}
$$

If $\kappa \geq 0$ the equation (12) can be easily transformed with the help of the *factor properties* (13) into

$$f_+ \;=\; A_+^{-1}\, P\, C^{-1}\, P\, A_-^{-1}\, g_+ \;\in\; L_+^2 \,. \tag{14}$$

In the original setting (1) we obtain

$$f \;=\; r_+\, A_+^{-1}\, P\, C^{-1}\, P\, A_-^{-1}\, \ell_0\, g \;\in\; L^2(\mathbb{R}_+) \,. \tag{15}$$

However, this formula holds only if the equation (1) is solvable (as we started from the assumption that (1) holds). Looking carefully at (11) one can find (for $\kappa > 0$) a necessary solubility condition, namely $PA_-^{-1}g_+ \in \operatorname{im} PCL_+^2$, which can be verified to be sufficient, as well.

Instead of this argumentation one can prove that (15) presents a left inverse of the operator $\mathcal{W}$ in (4) for all $\kappa \geq 0$ by the help of (13). Similarly one verifies that it gives a right inverse of $\mathcal{W}$ in the case of $\kappa \leq 0$.

Denoting these one-sided inverses by $\mathcal{W}^-$ we have in the case $\kappa > 0$ a (non-trivial) projector onto the image of $\mathcal{W}$ given by $\mathcal{W}\,\mathcal{W}^-$ and in the case $\kappa < 0$ a (non-trivial) projector along the kernel of $\mathcal{W}$ given by $\mathcal{W}^-\,\mathcal{W}$.

So we obtain as a common operator theoretical interpretation:

Theorem 2.1 *Let $\mathcal{W}$ be given by (4) where $|\Phi|$ be bounded from below, i.e., $A \in \mathcal{L}(L^2(\mathbb{R})$ is an isomorphism, and let $\mathcal{W}^-$ be the operator from (15). Then we have for*

- $\kappa = 0$: *Equation (1) is uniquely solvable by (15) (where $C = I$) for any $g \in L^2(\mathbb{R}_+)$.*
- $\kappa > 0$: *Equation (1) is solvable for a certain $g \in L^2(\mathbb{R}_+)$ if and only if g satisfies the solubility condition*

$$\mathcal{W}\mathcal{W}^- g \quad = \quad r_+ A_- PCPC^{-1} PA_-^{-1} \ell_0 g \quad = \quad g\,;$$

  In this case (15) presents the unique solution.

- $\kappa < 0$: *Equation (1) is solvable for all $g \in L^2(\mathbb{R}_+)$. The general solution reads*

$$\begin{aligned} f \quad &= \quad \mathcal{W}^- g \ + \ (I - \mathcal{W}^- \mathcal{W})\, h \ , \ \ h \in L^2(\mathbb{R}_+) \\ \mathcal{W}^- \mathcal{W} \quad &= \quad r_+ A_+^{-1} PC^{-1} PCPA_+ \ell_0 \,. \end{aligned}$$

Remark 2.2 *Moreover one can show that $\mathcal{W}$ is a Fredholm operator, i.e., its defect numbers are finite:* $\dim\ker W < \infty$, $\operatorname{codim}\operatorname{im} W < \infty$*. The solubility conditions in the case $\kappa > 0$ can be written as κ orthogonality conditions using a suitable basis of L^2. The kernel of $\mathcal{W}$ can also be explicitly written as a $-\kappa$-dimensional subspace of $L^2(\mathbb{R}_+)$ GohFel74, Kre58, MikPro86. Details are given later in* (91) - (94).

Remark 2.3 *It is well known that the appearance of one-sided invertibility has a deaper background in the Theorem of Coburn, see [BoeSil06, which allows the conclusion that for certain classes of operators (such as classical Toeplitz and Wiener-Hopf operators), the Fredholm property implies one-sided invertibility.*

Hence the question arises if, in a general setting, a factorisation of A with the factor properties (13) is not only sufficient for the one-sided invertibility of $\mathcal{W}$, but moreover if such a factorisation is necessary, as well. For clarity let us first see the case where $\mathcal{W}$ is invertible and $C = I$ coming back to the general case later. To this end we need the following notation.

Let X be a Banach space, $A \in \mathcal{L}(X)$ and $P \in \mathcal{L}(X)$ a projector, and put $Q = I - P$, $PX = \operatorname{im} P = \ker Q$, $QX = \operatorname{im} Q = \ker P$ for convenience. Then

$$W \quad = \quad P\,A|_{PX} \quad : \quad PX \to PX \tag{16}$$

is referred to as *general Wiener-Hopf operator* (WHO).

If A is an isomorphism, an operator pair A_+, $A_- \in \mathcal{L}(X)$ is said to be a *strong (right) WH factorisation* of A *with respect to* (X, P) if

$$A \quad = \quad A_-\,A_+ \tag{17}$$

and the first two lines of the relations (13) are satisfied, i.e., A_+ leaves $\operatorname{im} P$ invariant and A_- leaves $\ker P$ invariant.

As a standard situation in this general setting we shall work only with Banach spaces; other convenient frameworks could be topological vector spaces or Hilbert spaces.

Remark 2.4 *The classical WHO* (4) *is not of the form* (16) *as* $L^2(\mathbb{R}_+)$ *is not a subspace of* $L^2(\mathbb{R})$. *But* $L^2(\mathbb{R}_+)$ *is isomorphic to the subspace* $L^2_+ \subset L^2(\mathbb{R})$, *see* (5) - (7).

With the operator P *of* (6) *we therefore have*

$$\begin{aligned} W &= \ell_0\, r_+\, A\, \ell_0\, r_+|_{L^2_+} = \ell_0\, \mathcal{W}\, r_+ \;:\; L^2_+ \to L^2_+\,, \\ \mathcal{W} &= r_+\, P\, A|_{PX}\, \ell_0 = r_+\, W\, \ell_0 \;:\; L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)\,, \end{aligned}$$

i.e., the classical WHO is equivalent to an operator of the form of a general WHO (16).

Remember that two bounded linear operators S, T acting in Banach spaces are said to be equivalent, if there are isomorphisms E, F such that

$$T = E\; S\; F\,,$$

briefly written as

$$T \sim S\,.$$

Theorem 2.5 (of Devinatz and Shinbrot) *Let X be a Banach space, $P \in \mathcal{L}(X)$ a projector, and $A \in \mathcal{GL}(X)$.*

Then the WHO W is boundedly invertible if and only if A admits a strong WH factorisation $A = A_- A_+$ with respect to (X, P).

In this case the inverse of W is given by

$$W^{-1} \quad = \quad A_+^{-1} \, P \, A_-^{-1}|_{PX} \quad : \quad PX \to PX. \tag{18}$$

The proof is quite elementary, although not at all constructive.

Sufficiency is verified with the help of the factor properties (13).

Necessity is proved as follows: If W is invertible in $\mathcal{L}(PX)$, then $PA + Q$ is invertible in $\mathcal{L}(X)$ because of the well-known relation

$$PAP + Q \quad = \quad (I - PAQ)\,(PA + Q)$$

where $PAP + Q$ is obviously invertible and $(I - PAQ)^{-1} = I + PAQ$. Now put just

$$A \quad = \quad A_- \, A_+ \quad = \quad A_- \, (PA + Q)$$

and verify the factor properties, see S85 for details.

This result was a cornerstone for the study of general Wiener-Hopf operators as given by (16) and moreover for general WHOs in an asymmetric space setting S85,S15 defined as follows.

Let X, Y be Banach spaces, $A \in \mathcal{L}(X, Y)$ an isomorphism, $P_1 \in \mathcal{L}(X)$, $P_2 \in \mathcal{L}(Y)$ two projectors, and put $Q_j = I - P_j$, $P_1 X = \operatorname{im} P_1 = \ker Q_1$ etc. for convenience.

Then

$$W \quad = \quad P_2\, A|_{P_1 X} \quad : \quad P_1 X \rightarrow P_2 Y \tag{19}$$

is referred to as a *general WHO* (*in asymmetric setting*).

The case of one-sided invertible WHOs can be seen as a special case of *generalised invertibility*, i.e., for $T \in \mathcal{L}(X,Y)$ there exists an operator T^- such that

$$T\ T^-\ T \quad = \quad T. \tag{20}$$

This conception allows a unified discussion of many kinds of general WHOs and their realisations. It will be discussed later on.

In what concerns Sommerfeld diffraction problems several modifications of the WHO (1) have to be considered, particularly:

- operators acting between different (Sobolev-like) functional spaces,
- matrix instead of scalar operators,
- more general Fourier symbol classes (rather than the Wiener algebra).

The need of these generalisations will be shown subsequently.

Historical remarks.

Wiener-Hopf equations are named after Norbert Wiener and Eberhard Hopf due to their paper of 1931 (in German) WH31.

Crucial progress in the solution of WH equations was presented in the fundamental paper by Mark Krein in 1958 (in Russian) Kre58 and for systems by Israel Gohberg and Mark Krein in the same year GohKre58. Various function spaces were considered besides the Lebesgue spaces (with similar results) and also various symbol classes besides the Wiener algebra allowed similar results, because the Hilbert transformation is continuous in so-called decomposing algebras (including Hölder continuous functions).

Other symbol classes (like the continuous functions) are not decomposing, the (formal) factors are not in the same class, see the books by I.C. Gohberg and I.A. Fel'dman GohFel74 or Solomon Grigorevich Mikhlin and Siegfried Prössdorf MikPro86 and the most important article on generalised factorisation by I.B. Simonenko from 1968 Sim68.

The Wiener-Hopf method (also called WH technique or WH procedure) developed parallel to the previous advances in applications, mainly by British researchers, see the famous books of Ben Noble from 1958 Nob58, by Douglas Jones from 1964 Jon64, as well as the surveys on the WH method in applications by David Abrahams Abr02 and another one by Jane Lawrie and David Abrahams LawAbr07.

Further remarkable progress was obtained for instance in Canada by Albert Heins and Robert Allan Hurd (loc. cit.), in Russia, see the book of Lev Weinstein Wei69, and later in Turkey by Mithat Idemen Ide79, A. Hamit Serbest et al. Ser96.

General WHOs were introduced by Marvin Shinbrot in 1964 Shi64. Theorem 2.5 is in a paper with Allan Devinatz in 1969 DevShi69 for separable Hilbert spaces with a rather complicated proof, a year after a more general result about one-sided invertibility of ring elements (instead of $W \in \mathcal{L}(X)$) that was already published by Grigorii Chebotarev in 1968 (in Russian) Ceb68, under the name *abstract WHO*.

Some ideas about general WHOs appeared independently under the names projection, compression, or truncation of an operator, see the book of Israel Gohberg and Naum Krupnik GohKru73, for instance.

Also in the context of pseudo-differential operators (acting in Sobolev spaces) we can find similar ideas and related results, see Esk73,Tal73. Some higher dimensional WH equations in applications have been addressed in GolGoh60,MS79, for instance.

The connection between classical and general WHOs was firstly pointed out in DevShi69,MS79, see also Mei86,Mei87,S85.

3 The WH equations in Sommerfeld half-plane problems

A *Sommerfeld half-plane problem* is here referred to as to determine the solution u of a boundary value problem given by the *Helmholtz equation*

$$(\Delta + k^2)\, u(x) \quad = \quad \frac{\partial^2 u(x)}{\partial x_1^2} + \frac{\partial^2 u(x)}{\partial x_2^2} + \frac{\partial^2 u(x)}{\partial x_3^2} + k^2 u(x) \quad = \quad 0$$

in the slit domain $\Omega = \mathbb{R}^3 \setminus \Sigma$ where the screen Σ is a half-plane $\Sigma = \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 \geq 0\,,\, x_2 = 0\}$ on which the field u is known, or its normal derivative or another linear boundary condition is prescribed on the two sides $\Sigma^{\pm}$of the screen (corresponding with $x_1 \geq 0\,,\, x_2 = \pm 0$).

The *wave number* k is assumed to have a positive imaginary part throughout this paper.

With the argument that in certain cases (as for a plane wave incoming perpendicularly to the edge of the screen) the solution will not depend on the third variable, the problem is modified to be a two-dimensional one, briefly written as

$$\begin{aligned} (\Delta + k^2)\, u &= 0 \quad \text{in } \Omega, \\ T_{0,\Sigma^\pm}\, u &= g \quad \text{on } \Sigma^\pm \end{aligned} \tag{21}$$

in case of the *Sommerfeld-Dirichlet problem*, with the trace operator $T_{0,\Sigma^\pm}$.

More precisely, we like to show that the problem is well-posed in a somehow reasonable space setting, and therefore we are looking for a weak solution in the energy space H^1 such that g has to be (arbitrarily) given in the trace space $H^{1/2}(\mathbb{R}_+) = r_+ H^{1/2}$ identifying the two banks $\Sigma^\pm$ of the screen with the half-line $\overline{\mathbb{R}_+}$.

Moreover an explicit solution in closed analytical form is wanted. See MS89,S86 for more details.

We shall work mainly with the spaces of Bessel potentials H^s which coincide with the Sobolev spaces $W^{s,p}$ in the case $s \in \mathbb{Z}$, $p = 2$ and the Sobolev-Slobodeckij spaces for $s \in \mathbb{R}$, $p = 2$, all briefly referred to as *Sobolev spaces* Esk73,HW08,Tri83,Wlo87.

It has been shown S86 that problem (21) is solvable if and only if the difference of the traces of the normal derivative of u on the upper and lower bank of the full line $\{(x_1, x_2) \in \mathbb{R}^2 \;:\; x_2 = 0\}$ (identified with $\mathbb{R}$), respectively, $f \in H^{-1/2}(\mathbb{R})$ satisfies the equation

$$r_+ \, A_\Phi \, f \quad = \quad 2g \tag{22}$$

where $A_\Phi \;=\; \mathcal{F}^{-1}\, \Phi \cdot \mathcal{F}$, $\Phi(\xi) \;=\; (\xi^2 - k^2)^{-1/2}$, $\xi \,\in\, \mathbb{R}$, i.e., A_Φ is a convolution (or pseudodifferential) operator of order -1 and an isomorphism from $H^{-1/2}(\mathbb{R})$ onto $H^{1/2}(\mathbb{R})$.

Furthermore the functional f has to be supported on the positive half-line, which will be indicated by $f \,\in\, H_+^{-1/2}$ (analogously to the definition of L_+^2).

It is important that the formulas (5) - (6) are valid for H^s instead of L^2 if and only if $|s| < 1/2$, see Esk73

$$r_+ \ : \ H^s(\mathbb{R}) \to H^s(\mathbb{R}_+) \quad , \quad \ell_0 \ : \ H^s(\mathbb{R}_+) \to H^s(\mathbb{R}),$$

as a bounded linear operator and

$$r_+ \ell_0 \ = \ I_{H^s(\mathbb{R}_+)} \quad , \quad P \ = \ \ell_0 r_+ \ \in \ \mathcal{L}(H^s).$$

The different analytical nature of the spaces $H^{\pm 1/2}(\mathbb{R}_+)$ and $H_+^{\pm 1/2}$ is crucial. It is well-known Gri85,HW08 that the spaces

$$\widetilde{H}^s(\mathbb{R}_+) \quad = \quad \{f \in H^s(\mathbb{R}_+) \ : \ r_+ f \in H_+^s\} \quad , \quad s \geq -\frac{1}{2},$$

equipped with the norm of H_+^s, satisfy

$$\widetilde{H}^s(\mathbb{R}_+) \quad = \quad H^s(\mathbb{R}_+) \ , \ |s| < \frac{1}{2},$$

$$\widetilde{H}^s(\mathbb{R}_+) \quad \underset{\text{dense}}{\subset} \quad H^s(\mathbb{R}_+) \ , \ s = \pm\frac{1}{2},$$

$$\widetilde{H}^s(\mathbb{R}_+) \quad \subset \quad C(\mathbb{R}_+) \ , \ s > \frac{1}{2}.$$

For these reasons it is not convenient to consider equation (22) as a WH integral equation of the first kind in L^2, but as a WH equation in asymmetric space setting with a WHO defined by

$$W \quad = \quad r_+ \, A_\Phi|_{H_+^{-1/2}} \quad : \quad H_+^{-1/2} \;\to\; H^{1/2}(\mathbb{R}_+) \tag{23}$$

where the convolution $A_\Phi = \mathcal{F}^{-1}\,\Phi\cdot\mathcal{F} : H^{-1/2} \to H^{1/2}$ is invertible by $A_\Phi^{-1} = \mathcal{F}^{-1}\,\Phi^{-1}\cdot\mathcal{F}$.

Remark 3.1 *Note that the operator W does not have the form of a general WHO in asymmetric space setting (19), because $r_+ : H^{1/2} \to H^{1/2}(\mathbb{R}_+)$ is not a projector in $H^{1/2}$, nor is it equivalent to a WHO in $H^{1/2}(\mathbb{R}_+)$ in the same way as in the L^2 setting, because ℓ_0 does not act from $H^{1/2}(\mathbb{R}_+)$ into $H^{1/2}(\mathbb{R})$. However, any continuous extension operator $\ell^c : H^{1/2}(\mathbb{R}_+) \to H^{1/2}(\mathbb{R})$ with $P_2 = \ell^c\, r_+ = P_2^2 \in H^{1/2}(\mathbb{R})$ will serve this purpose and leads to more complicated formulas explained below.*

The square root function that appears in the Fourier symbol of A_Φ (with $\Im m\, k > 0$) is of great importance. We shall denote it by

$$t(\xi) \quad = \quad (\xi^2 - k^2)^{1/2} \quad , \quad \xi \in \mathbb{R}, \tag{24}$$

and take the branch cut from k to $-k$ vertically via ∞ such that

$$t(\xi) \quad \approx \quad |\xi| \quad , \quad \text{as } \xi \to \pm\infty.$$

Its factorisation (with corresponding branches)

$$t(\xi) \;=\; (\xi - k)^{1/2}\; (\xi + k)^{1/2} \;=\; t_-^{1/2}(\xi)\; t_+^{1/2}(\xi), \quad \xi \in \mathbb{R}, \tag{25}$$

is closely conneted with the definition of the one-dimensional Sobolev spaces Esk73 and some important properties.

Firstly we have the mapping properties

$$\begin{aligned} H^s &= A_{t^{-s}}\, L^2, \\ H^s_+ &= A_{t_+^{-s}}\, L^2_+ \quad , \quad H^s_- = A_{t_-^{-s}}\, L^2_-, \\ H^s(\mathbb{R}_+) &= r_+\, A_{t_-^{-s}}\, L^2_+ \quad , \quad H^s(\mathbb{R}_-) = r_-\, A_{t_+^{-s}}\, L^2_-, \end{aligned} \tag{26}$$

for all $s \in \mathbb{R}$ (in a distributional sense for $s < 0$). Note that in these formulas the wave number k takes over the common role of the imaginary unit i because of the assumption of $\Im m\, k > 0$ (the dependence on k is here suppressed for brevity).

Moreover they serve to construct projectors and extension operators (mentioned before) that are most convenient in the present context CDS14,CS00.

To make the connection between the WHO in Sobolev spaces (23) and the general WHO (19) it is necessary to find convenient projectors and extension operators. This step is a little technical, however leads to a rigorous understanding of classical as general WHOs.

For any $s \in \mathbb{R}$ we find projectors in H^s abbreviating now (for cosmetic reasons) $P_+ = \ell_0 r_+ \in \mathcal{L}(L^2)$ and $P_- = I - P_+ = \ell_0 r_-$ by putting

$$
\begin{aligned}
P_+^{(s)} &= A_{t_+^{-s}}\, P_+\, A_{t_+^s} \quad \text{onto} \quad H_+^s, \\
P_-^{(s)} &= A_{t_-^{-s}}\, P_-\, A_{t_-^s} \quad \text{onto} \quad H_-^s, \\
\Pi_+^{(s)} &= A_{t_-^{-s}}\, P_+\, A_{t_-^s} \quad \text{along} \quad H_-^s, \\
\Pi_-^{(s)} &= A_{t_+^{-s}}\, P_-\, A_{t_+^s} \quad \text{along} \quad H_+^s,
\end{aligned}
\tag{27}
$$

which are orthogonal for $k = i$.

Hence $P_+^{(s)} + \Pi_-^{(s)} = I_{H^s}$ and $P_-^{(s)} + \Pi_+^{(s)} = I_{H^s}$ for any $s \in \mathbb{R}$.

Suitable extension operators are (because of (26)) given by

$$\ell_+^{(s)} = A_{t_-^{-s}} P_+ A_{t_-^s} \ell^{(s)} \quad : \quad H^s(\mathbb{R}_+) \to H^s(\mathbb{R}),$$
$$\ell_-^{(s)} = A_{t_+^{-s}} P_- A_{t_+^s} \ell^{(s)} \quad : \quad H^s(\mathbb{R}_-) \to H^s(\mathbb{R}),$$

where $\ell^{(s)}$ stands for an arbitrary extension into H^s, see Esk73.

Obviously we have the analogue of (6)

$$r_+ \ell_+^{(s)} = I_{H^s(\mathbb{R}_+)} \quad , \quad \Pi_+^{(s)} = \ell_+^{(s)} r_+ \in \mathcal{L}(H^s).$$

Now we are in the position for a strict understanding of (23) in the sense of an asymmetric general WHO and its inversion in the spirit of Theorem 2.5. First we see that W is equivalent to the operator

$$\ell_+^{(1/2)} W \;=\; \Pi_+^{(1/2)} A_\Phi|_{\mathrm{im}\, P_+^{(-1/2)}} \quad : \quad H_+^{-1/2} \;\to\; \ell_+^{(1/2)} H^{1/2}(\mathbb{R}_+), \tag{28}$$

which has the form of (19). Second, the idea of a strong factorisation (10) (with $C = I$) extends to the asymmetric case

$$\begin{aligned} A_\Phi \quad &= \quad A_{t_-^{-1/2}} \quad A_{t_+^{-1/2}} \\ &: H^{1/2} \longleftarrow L^2 \longleftarrow H^{-1/2}, \end{aligned} \tag{29}$$

a factorisation into isomorphisms between the indicated spaces with factor properties analogous to the symmetric space case (17)

$$\begin{aligned} PA_+P_1 \quad &= \quad A_+P_1 \quad , \quad P_1A_+^{-1}P \quad = \quad A_+^{-1}P\,, \\ P_2A_-P \quad &= \quad P_2A_- \quad , \quad PA_-^{-1}P_2 \quad = \quad PA_-^{-1}\,, \end{aligned} \tag{30}$$

with the previous interpretation of operators, namely $P_1 = P_+^{(-1/2)}$, $P = \ell_0 r_+$, $P_2 = \Pi_+^{(1/2)}$.

It follows that the equation (22) is solved by the asymmetric analogue of (18):

$$f \quad = \quad W^{-1}\, 2g \quad = \quad A_+^{-1}\, P\, A_-^{-1}|_{P_2 Y}\, 2g. \tag{31}$$

However, the idea of Theorem 2.5 works for general WHOs in asymmetric space setting, as well, see S85 or S17:

Theorem 3.2 *Let* X, Y *be Banach spaces,* $A \in \mathcal{L}(X, Y)$ *an isomorphism and* $P_1 \in \mathcal{L}(X)$, $P_2 \in \mathcal{L}(Y)$ *two projectors.*

Then the (general) WHO

$$W \quad = \quad P_2 \, A|_{P_1 X} \quad : \quad P_1 X \to P_2 Y \tag{32}$$

is boundedly invertible if and only if A *admits a strong WH factorisation with respect to* (X, Y, P_1, P_2)*:*

$$\begin{aligned} A \quad &= \qquad A_- \qquad A_+ \\ &: \quad Y \longleftarrow Z \longleftarrow X, \end{aligned} \tag{33}$$

where Z *is a Banach space,* $P \in \mathcal{L}(Z)$ *a projector, and the relations* (30) *are satisfied.*

In this case the inverse of W *is given by*

$$W^{-1} \quad = \quad A_+^{-1} \, P \, A_-^{-1}|_{P_2 Y} \quad : \quad P_2 Y \to P_1 X. \tag{34}$$

The previous concept is applicable to a large class of diffraction problems that lead to WH equations and systems of WH equations, as well.

As an example we consider the *Rawlins problem* MS89,Raw84,S86 where, instead of the Dirichlet condition on both banks of the screen (see (21)), a Dirichlet condition on the upper bank and a Neumann condition on the lower bank is prescribed, in brief

$$\begin{aligned} (\Delta + k^2)\, u &= 0 \quad \text{in } \Omega, \qquad (35)\\ T_{0,\Sigma^+}\, u &= u(.,x_2)|_{x_2=+0} = g_0 \in H^{1/2}(\mathbb{R}_+)\\ T_{1,\Sigma^-}\, u &= \partial u/\partial x_2\, u(.,x_2)|_{x_2=-0} = g_1 \in H^{-1/2}(\mathbb{R}_+). \end{aligned}$$

The resulting 2×2 system of WH equations can be written (after some elementary substitutions) as

$$r_+ A_\Phi f \quad = \quad g \tag{36}$$

where A acts in a topological product (Banach) space (like a 2×2 matrix operator):

$$\begin{aligned} A &= \mathcal{F}^{-1} \Phi \cdot \mathcal{F} \quad : \quad H^{1/2} \times H^{-1/2} \;\to\; H^{1/2} \times H^{-1/2}, \qquad (37) \\ \Phi &= \begin{pmatrix} -1 & t^{-1} \\ t & 1 \end{pmatrix}, \\ f &\in H_+^{1/2} \times H_+^{-1/2}, \\ g &= (g_0\,,\, g_1) \;\in\; H^{1/2}(\mathbb{R}_+) \times H^{-1/2}(\mathbb{R}_+). \end{aligned}$$

It turns out that a strong factorisation in the form (33) can be explicitly obtained with an intermediate space MS89

$$Z \quad = \quad H^{1/4} \times H^{-1/4}. \tag{38}$$

This is an interpretation of the explicit factorisation of the matrix Φ, a spectacular result of Tony Rawlins in 1975 Raw75,Raw81, which will be discussed in a wider frame subsequently and leads us to the rigorous solution of a series of boundary-transmission problems and operator theoretical insights, presented below.

Hence the problem (35) is well-posed in this space setting and the explicit solution (resolvent operator) can be obtained by the help of the inversion formula of Theorem 3.2. Moreover the asymptotical behavior of the solution near the edge of the screen is related to the order of the Sobolev spaces in (38):

$$\operatorname{grad} u \quad \sim \quad |x|^{-3/4} \text{ as } x \to 0.$$

Further boundary-transmission problems can be studied by analogy when the two boundary conditions in (35) are replaced by an arbitrary linear combination of the two Dirichlet data $T_{0,\Sigma^\pm}$ in the first place and an arbitrary linear combination of the two Neumann data $T_{1,\Sigma^\pm}$ in the second place MS89,S89:

$$(\Delta + k^2)\, u \quad = \quad 0 \quad \text{in } \; \Omega,$$

$$a_0 u_0^+ + b_0 u_0^- \;=\; g_0 \quad , \quad a_1 u_1^+ + b_1 u_1^- \;=\; g_1 \quad \text{on} \quad \Sigma.$$

They lead to a modification in the derived WH equation, namely replacing the Fourier symbol Φ in (37) by

$$\sigma_\lambda \quad = \quad \begin{pmatrix} 1 & t^{-1} \\ t & \lambda \end{pmatrix} \tag{39}$$

where the parameter $\lambda \in \mathbb{C} \setminus \{0, 1\}$ is an algebraic combination of the coefficients a_0, b_0, a_1, b_1 (under slight conditions on the coefficients that make the boundary operators to be of "normal type", see S89, formulas (1.5), (1.6)). We shall present the explicit factorisation in the next section and a complete discussion of all $\lambda \in \mathbb{C}$ in Section 5, after introducing some auxiliary material.

As a remarkable result the exponent $1/4$ in the intermediate space (39) has to be replaced by $\delta/2$ where $\delta \in (0,1]$ is simply computed from λ and determines the order of the singularity of $\operatorname{grad} u$ at the origin:

$$\delta \;=\; \Re e \left(\frac{i}{\pi} \log \frac{\sqrt{\lambda}+1}{\sqrt{\lambda}-1} \right) \quad , \quad \operatorname{grad} u \;\sim\; |x|^{\delta/2-1} \text{ as } x \to 0.$$

It turns out that, under the assumption $\lambda \notin [1, +\infty)$, all these problems are well-posed in the given setting and explicitly solvable by Theorem 3.2 with the help of a factorisation of the matrix (39), see MS89,S89 for details.

The same holds true for transmission problems where a pair of functions $u = (u^+, u^-)$ in the upper and lower half-plane $\Omega^\pm = \{x \in \mathbb{R}^2 : \pm x_2 > 0\}$ satisfies

$$\begin{aligned} (\Delta + k^2)\, u^\pm &= 0 \quad \text{in } \Omega^\pm, \\ a_0 u_0^+ + b_0 u_0^- &= g_0 \quad , \quad a_1 u_1^+ + b_1 u_1^- = g_1 \quad \text{on} \quad \Sigma \\ a_0' u_0^+ + b_0' u_0^- &= g_0' \quad , \quad a_1' u_1^+ + b_1' u_1^- = g_1' \quad \text{on} \quad \Sigma' \end{aligned} \tag{40}$$

where $u_0^\pm$, $u_1^\pm$ are the traces of u and its normal derivative, respectively, on $x_2 = \pm 0$ and the coefficients are any complex numbers.

The Fourier symbol of the derived WH system is in this case given by

$$\Phi = \frac{1}{a_0' b_1' + b_0' a_1'} \begin{pmatrix} a_0 b_1' + b_0 a_1' & -(a_0 b_0' + b_0 a_0')\, t^{-1} \\ -(a_1 b_1' + b_1 a_1')\, t & a_1 b_0' + b_1 a_0' \end{pmatrix}. \tag{41}$$

Sommerfeld half-plane problems with first and second kind boundary conditions can be seen as a special case of (40) where the third line is replaced by the homogeneous jump conditions

$$u_0^+ - u_0^- = 0 \quad , \quad u_1^+ - u_1^- = 0 \quad \text{on} \quad \Sigma'. \tag{42}$$

Another phenomenon appears when the two boundary conditions (or transmission conditions) have the same order, for instance the two Dirichlet data $T_{0,\Sigma^\pm} u = g^\pm \in H^{1/2}(\mathbb{R}_+)$ are given (DD) or two Neumann data $T_{1,\Sigma^\pm} u = g^\pm \in H^{-1/2}(\mathbb{R}_+)$ (NN) or two impedance data are given (II)

$$\begin{aligned} T_{1,\Sigma^+} u \; + \; i\, p^+ T_{0,\Sigma^\pm} u &= g^+ \in H^{-1/2}(\mathbb{R}_+) \\ T_{1,\Sigma^-} u \; + \; i\, p^- T_{0,\Sigma^\pm} u &= g^- \in H^{-1/2}(\mathbb{R}_+) \end{aligned} \tag{43}$$

with different impedance numbers $p^\pm$, $\Im m\, p^\pm > 0$, and possibly different data in all cases.

Obviously it is necessary that the difference of the given data $g^+ - g^-$ (considered as function(al) on $\mathbb{R}_+$) has to be extensible by 0 to $\mathbb{R}_-$ within the data space $H^{1/2}$ or $H^{-1/2}$, respectively. I.e., the following compatibility conditions are necessary for the solubility of the corresponding problem MS86,S89:

$$\begin{aligned} g^+ - g^- &\in \widetilde{H}^{1/2}(\mathbb{R}_+) \quad \text{in case } DD, \\ g^+ - g^- &\in \widetilde{H}^{-1/2}(\mathbb{R}_+) \quad \text{in case } NN \text{ and } II. \end{aligned} \tag{44}$$

As a matter of fact these conditions arise automatically from an operator theoretical approach named minimal normalisation of WHOs of the form (1) in scales of Sobolev spaces MoS97,MoST96,MoST98.

Historical remarks.

Sommerfeld half-plane problems have their origin in the famous article ”Mathematische Theorie der Diffraction” by Arnold Sommerfeld 1896 (in German) Som96 which was based on his habilitation thesis. In 2004 there appeared a carefully annotated translation in English as a book by R.J. Nagem, M. Zampolli, and G. Sandri NZS04.

Several books and survey papers demonstrate the attention on Sommerfeld problems. For an overview about classical WH methods the reader may consult Abr02,Jon64,LawAbr07,Mei86,Mei87,Nob58 which report particularly on the pioneering work in diffraction theory by Edward Copson Cop46, Albert Heins Hei50,Hei82, Allan Hurd Hur87, Douglas Jones Jon52, Ernst Lüneburg LH84, Anthony Rawlins Raw75,Raw81,Raw84, Thomas Senior Sen52, and W.E. Williams Wil84, to mention a few.

Formulations of Sommerfeld half-plane problems as boundary value and transmission (or interface) problems in Sobolev spaces started in the 1970s with Giorgio Talenti Tal73, continued in the 1980s by Erhard Meister and his co-authors, touching the theory of partial and pseudo-differential equations, for instance by Eli Shamir Sha63,Sha67, Gregory Eskin Esk73, Joseph Wloka Wlo87, George Hsiao and Wolfgang Wendland HW08, to mention some closely related work.

We point out particularly the first solutions of the Sommerfeld-Dirichlet and the Sommerfeld-Neumann problems by Edward Copson in 1946 Cop46 with the classical WH technique and the factorisation of the matrix coming up from the mixed Dirichlet-Neumann problem by Antony Rawlins in 1981 Raw75,Raw81,Raw84.

Erhard Meister and the author started to present resolvent operators in Sobolev spaces for much wider classes of Sommerfeld boundary value and transmission problems in 1985 HMS89,MS89,S86,S89,S04. They also stressed the importance of the intermediate space in operator factorisations.

The bridge between WHOs and ΨDOs acting in Sobolev spaces was build by Gregory Eskin in the early 1970s Esk73. He also introduced the most convenient notation (23) (avoiding the more complicated writing (28) presumably for cosmetic reasons) which allowed the consideration of the WHO in a scale of Sobolev spaces, its lifting (cf. Proposition 7.1) etc.

The equivalence with operators of the form of a general WHO (19) was pointed out by the author in 1983 S83,S85 and systematically used in subsequent publications.

BVPs for systems of PDEs such as the Lamé equations (in crack problems) can be found in Abr02,DW95,MS89e for instance. They are, however, beyond the scope of this article.

4 Constructive factorisation of non-rational matrix functions

This topic can be considered as a separate area of research with components in algebra, analysis, matrix and operator theory, which is strongly related to applications.

According to the width of the area we confine ourselves to a rough overview after focusing one distinguished class of matrix functions (39) as a prototype of those related to the theme of this survey article, which also gave a tremendous impact to factorisation theory MS89,Raw75,Raw81,S04.

Let us recall the WHO with symbol (39) written as

$$W \quad = \quad r_+ A|_{P_1 X} \quad : \quad H_+^{1/2} \times H_+^{-1/2} \to H^{1/2}(\mathbb{R}_+) \times H^{-1/2}(\mathbb{R}_+) \tag{45}$$

where $X = Y = H^{1/2}(\mathbb{R}) \times H^{-1/2}(\mathbb{R})$, $A = \mathcal{F}^{-1}\sigma_\lambda \cdot \mathcal{F}$ with

$$\sigma_\lambda \quad = \quad \begin{pmatrix} 1 & t^{-1} \\ t & \lambda \end{pmatrix}, \tag{46}$$

and $t(\xi) = (\xi^2 - k^2)^{1/2}$, $\xi \in \mathbb{R}$, $\lambda \in \mathbb{C} \setminus \{0, 1\}$.

The connection with a general WHO is given by (32) where, strictly speaking, $P_1 = P_+^{(1/2)} \otimes P_+^{(-1/2)} \sim \text{diag}\,(P_+^{(1/2)}, P_+^{(-1/2)})$ and $P_2 = \Pi_+^{(1/2)} \otimes \Pi_+^{(-1/2)}$.

A certain factorisation (with the common holomorphy properties and "low" increase at infinity) is derived with the help of Khrapkov's formulas and Daniele's trick S89:

$$\sigma_{\lambda+} = (1-\lambda^{-1})^{-1/4} \begin{pmatrix} c_+ & -s_+\sqrt{\lambda}/t \\ -c_+\xi/\sqrt{\lambda} - s_+t/\sqrt{\lambda} & s_+\xi/t + c_+ \end{pmatrix},$$

$$\sigma_{\lambda-} = (1-\lambda^{-1})^{-1/4} \begin{pmatrix} c_- - s_-\xi/t & -s_-\sqrt{\lambda}/t \\ -s_-t/\sqrt{\lambda} + c_-\xi/\sqrt{\lambda} & c_- \end{pmatrix}, \tag{47}$$

where

$$c_\pm(\xi) = \cosh[C\log\gamma_\pm(\xi)]\,,$$

$$s_\pm(\xi) = \sinh[C\log\gamma_\pm(\xi)]\,,$$

$$\gamma_\pm(\xi) = \frac{\sqrt{k\pm\xi} + i\sqrt{k\mp\xi}}{\sqrt{2k}}\ ,\ \ \xi\in\mathbb{R}\,,$$

$$C = \frac{i}{\pi}\log\frac{\sqrt{\lambda}+1}{\sqrt{\lambda}-1}\quad .$$

Because of the asymptotic behavior of $\sigma_{\lambda\pm}$ at infinity, the corresponding factorisation of $A = \mathcal{F}^{-1}\sigma_\lambda \cdot \mathcal{F}$ represents a (so-called strong or canonical) Wiener-Hopf factorisation through a vector Sobolev space, provided $\lambda \notin [1, +\infty)$:

$$A_\lambda \quad = \quad A_{\lambda-} \quad A_{\lambda+} \quad = \quad \mathcal{F}^{-1}\,\sigma_{\lambda-} \cdot \mathcal{F} \quad \mathcal{F}^{-1}\sigma_{\lambda+} \cdot \mathcal{F},$$

$$H^{1/2} \times H^{-1/2} \leftarrow \; Z \; \leftarrow H^{1/2} \times H^{-1/2} \tag{48}$$

$$Z \quad = \quad H^{\vartheta}(\mathbb{R}) \quad , \quad \vartheta = (\vartheta_1, \vartheta_2) = (\frac{1}{2}(1-\delta), \frac{1}{2}(\delta - 1))$$

where $\delta = \Re e\, C = \frac{-1}{\pi} \arg \frac{\sqrt{\lambda}+1}{\sqrt{\lambda}-1} \in]0, 1]$.

The crucial point is that $|\vartheta_j| < 1/2$, as we shall see (cf. S15, (6.4)), briefly speaking: the Sobolev space orders where r_+ is a left inverse of ℓ_0.

The asymptotic behavior of the gradient of the solution follows by analogy of the conclusions from (38), see S89 for details.

$$\operatorname{grad} u \quad \sim \quad |x|^{\delta/2-1} \text{ as } x \to 0.$$

The factorisation procedure consists of two steps:

First the matrix belongs to the (commutative) so-called Daniele-Krapkov class which admit a meromorphic factorisation within the same class in which the two factors have an algebraic behavior at infinity, however is not related to a generalised factorisation in the sense of Simonenko BoeSil06,GohKru73,Sim68 and therefore does not directly imply a formula for the inverse of the WHO.

The Daniele trick corrects this deficiency by twofold factorisation of a rational matrix function, related to the behavior of the previous at infinity, and a rearrangement of the first factorisation by adding factors in the middle of the first factorisation which compensate the increase Dan84,MS89,Raw84,S86.

It turns out that the final factorisation can be considered as a factorisation into unbounded factors generating a bounded inverse and as a factorisation into bounded factors in the sense of (33), as well, provided $|\vartheta_j| < 1/2$ MS89,S89.

To understand this technique in detail and to answer the question "What happens in case of $\lambda \in [0, +\infty)$?" completely, it is necessary to study the so-called lifting of WHOs in Sobolev spaces to WHOs in L^2 spaces presented in Section 6. Hence we come back to this question later.

Now let us have a wider look at the field of constructive matrix factorisation (related to Sommerfeld problems). The progress in this area is based upon many different ideas. We refer to the books ClaGoh81,LitSpi87 and the survey papers Abr02,BoeSpi13,ES02,Jon84 LawAbr07,RogMis16,SpiTas89,Tal73 and point out some basic ideas through relevant keywords that demonstrate the variety of facets in constructive matrix factorisation:

- factorisation of rational matrix functions by means of linear algebra ClaGoh81,
- commutative matrix factorisation Hei50,Jon84,
- the Daniele-Khrapkov class Ceb68,Dan84,Khr71,
- the Wiener-Hopf-Hilbert method Hur76,Hur87,
- understanding of a matrix factorisation as generalised factorisation in L^p spaces PS90,Sim68,
- triangular matrix functions LitSpi87,PriRog18,
- asymmetric space settings S85,
- generalised inverses for a unified approach S85,

- lifting Dud79,Esk73,S85,S89,
- separation of analytic and algebraic properties MS89,
- classification of matrix functions with respect to the number of rationally independent entries ES02,PS90,
- reduction by rational transformation ES02,
- meromorphic factorisation CLS92,CamMal08,Ide79,MS89e,
- connection with corona problems and Riemann surfaces BKST98,CS
- operator factorisation through an intermediate space CS95,S17.

In further classes occurring of canonical diffraction problems (wave guides, wedges etc.), one finds other species containing exponential terms, e.g., with an exhausting an still increasing amount of literature, see BKST98,BKS02, for instance.

Features such as approximate factorisation and stability questions as well as numerical aspects exceed the scope of this exposition. In the present context we refer to Abr00,CLS92,Kis15,MisRog16.

Historical remarks.

This wide field of research developed mainly in the last three decades. We highlight only a few milestones which are most important in the present context. A first basic insight was that matrix functions with rational symbols and those with non-rational symbols are quite different in nature.

Rational matrix functions have great importance in system theory BGK84 and admit a rather complete approach ClaGoh81.

Non-rational Fourier symbols enter the scene with the square root function appearing in the representation formulas, see Proposition 5.4. Some corresponding difficulties in matrix factorisation have been recognized already in the 1950s, see the book of Ben Noble Nob58.

The importance of the so-called Daniele-Khrapkov class in diffraction theory was recognized by A.A. Khrapkov in 1971. The pioneering results of Anthony Rawlins Raw81,Raw84,RawWil81 and Vito Daniele in 1984 Dan84 released a tremendous investigation of constructive factorisation of matrix functions from this class and its applications, see for instance CSB95,ES02,LawAbr07,MeiPen92,MS89 PS90,RogMis16,S04.

The most important lines of research in view of the applications in diffraction theory are connected with the names of Abrahams, Daniele, Heins, Hurd, Jones, Lüneburg, Meister, Mishuris, Noble, Rawlins, Rogosin, Senior, Talenti, and Williams.

Related theoretical work (existence questions, functional analytic aspects, etc.) can be found in the work of Böttcher, Câmara, Ehrhardt, Feldman, Gohberg, Krupnik, Litvinchuk, Prössdorf, dos Santos, Silbermann, Simonenko, Spitkovsky, and the author.

5 Operator factorisation in boundary value problems

Now we demonstrate the connection between the solution of a boundary value problem (BVP) or a transmission problem (TRP) and operator factorisation starting with a quite basic contemplation S13, which leads us to an explicit representation of resolvent operators in the case of so-called canonical diffraction problems. Typically an elliptic linear boundary value problem is written in the form

$$\begin{array}{rcllr} Au & = & f & \texttt{in} \ \ \Omega & \text{(pde in nice domain)} \\ Bu & = & g & \texttt{on} \ \ \Gamma = \partial\Omega & \text{(boundary condition).} \end{array} \qquad (49)$$

More precisely the problem is: Determine the (general*) solution of the system (49) (in a certain form**) where the following are given: Ω is a Lipschitz domain in $\mathbb{R}^n$ (e.g.), $A \in \mathcal{L}(\mathcal{X}, Y_1), B \in \mathcal{L}(\mathcal{X}, Y_2)$ are bounded linear operators in Banach spaces of function(al)s living on Ω or $\Gamma = \partial\Omega$, respectively. The data (f, g) are arbitrarily given in a (known product) space $Y = Y_1 \times Y_2$.

The situation becomes a bit more transparent if we consider the *operator associated with the boundary value problem*

$$L = \begin{pmatrix} A \\ B \end{pmatrix} : \mathcal{X} \to Y = Y_1 \times Y_2 \tag{50}$$

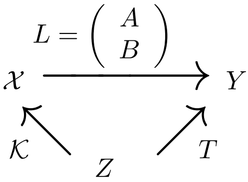
where the data space Y and the solution space $\mathcal{X}$ are usually assumed to be known (eventually modified later for practical reasons and in contrast to free boundary problems or certain inverse problems).

It is clear that a linear boundary value problem in the abstract setting (49) is well-posed if and only if the operator L is an isomorphism.

Thus the main problem is: Find (in a certain form) the inverse (resolvent) of the associated operator L (or a generalised inverse etc.).

Associated operators of BVPs were systematically used, e.g., in the work of Bou71, CST04, CST06, ENS11, ENS14, Esk73, MoST96, Wlo87. Linear transmission problems can be considered in the same way, cf. S86,S17.

The classical idea to present the possible solution $u \in \mathcal{X}$ by surface and/or volume potentials can be seen as an operator factorisation:



If $\mathcal{K}$ is an isomorphism, then L is *equivalently reduced* to $T = \mathcal{K}L$ in the sense that the two operators are *(algebraically and topologically) equivalent*, i.e., that T is representable as

$$T \quad = \quad E \quad L \quad F \tag{51}$$

where E, F are linear homeomorphisms. Here $E = \mathcal{K}$ is a potential operator and $F = I$, $Z = Y$.

Equation (51) defines an equivalence relation between classes of bounded linear operators in the genuine mathematical sense (reflexive, symmetric and transitive) and practically it includes the idea of a substitution in the solution and in the data space. For the existence of a relation (51) we write

$$T \quad \sim \quad L. \tag{52}$$

Remark 5.1 *Note that for this relation it is not sufficient to take one-to-one substitutions* E , F *BT91 but, if they are also bounded linear operators in Banach spaces, the relation* (52) *follows from the inverse mapping theorem. Sometimes it happens that a potential operator* $\mathcal{K}$ *is not surjective CST06.*

However, in potential theory the reasoning is often based on a proof that a BVP is uniquely solvable if and only if a system of boundary integral equations is uniquely solvable, see DW95,HW08.

This would be a consequence ("transfer property" CS98) of (52)*, however there are other important relations which also imply the equivalence of the unique solubility and much more, cf. Section 8.*

Let us take a look at *canonical diffraction problems*.

There is no strict definition for this kind of boundary value and transmission problems, however experts of the area arrived at a consensus about what it means in conferences Mei97,Ser96 dedicated to the one hundredth anniversary of Sommerfeld's famous paper Som96 as reported in the introduction of CST04:

The word canonical stands for

- particular geometrical situations like axi-parallel, rectangular or circular configurations,
- constant coefficients in the corresponding partial differential equations and in the boundary conditions, and
- the importance of the problem to describe basic phenomena in linear time harmonic wave propagation governed by the Helmholtz equation.

As a prototype we focus again the Sommerfeld half-plane problems (40) with jump conditions (42) on Σ'.

Let $\mathcal{H}^1(\Omega)$ denote the weak solutions HW08 of the Helmholtz equation in the two-dimensional slit domain $\Omega = \mathbb{R}^2 \setminus \Sigma$ where $\Sigma = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0\,,\, x_2 = 0\}$ is the half-line identified with $\mathbb{R}_+$ as in Section 3.

Sommerfeld half-plane problems with two *transmission conditions of first and second kind*, respectively, are briefly written as S89

$$\begin{aligned} u &\in \mathcal{H}^1(\Omega), \\ B_1\, u &= a_0 u_0^+ + b_0 u_0^- = g_0 \in H^{1/2}(\mathbb{R}_+), \\ B_2\, u &= a_1 u_1^+ + b_1 u_1^- = g_1 \in H^{-1/2}(\mathbb{R}_+). \end{aligned} \tag{53}$$

Remark 5.2 *Obviously we have a special case of the BVP (49) where the right hand side of the PDE $f = 0$, hence the associated operator is, instead of (50), the following one:*

$$L^0 \;=\; B|_{\ker A} \;:\; \mathcal{H}^1(\Omega) \to Y_2 = H^{1/2}(\mathbb{R}_+) \times H^{-1/2}(\mathbb{R}_+)\,. \quad (54)$$

The relation between L and L^0 will be discussed later in Section 8.

Remark 5.3 *Note that we avoided the notation $u \in H^1(\Omega)$ here and before in the context of the slit domain $\Omega = \mathbb{R}^2 \setminus \Sigma$ (in Section 3), because otherwise we had $H^1(\Omega) = H^1(\mathbb{R}^2)|_\Omega$ according to the definition of Sobolev spaces which does not allow different traces of u on $\Sigma^\pm$. Hence $\mathcal{H}^1(\Omega)$ strictly speaking denotes functions from $L^2(\Omega)$ for which any restriction to a proper sub-cone $\Omega' \subset \Omega$ is a $H^1(\Omega')$ solution of the Helmholtz equation in Ω'.*

The following representation formula for $u \in \mathcal{H}^1(\Omega)$ is well-known MS89,S86.

Proposition 5.4 *A function $u \in L^2(\mathbb{R}^2)$ belongs to $\mathcal{H}^1(\Omega)$ if and only if*

$$u^{\pm} = u|_{\Omega^\pm} \in H^1(\Omega^\pm) \text{ and hence } u_0^{\pm} = u|_{x_2=\pm 0} \in H^{1/2}(\mathbb{R}),$$

$$u_0^+ - u_0^- = 0 \quad \text{and} \quad A_t\,(u_0^+ + u_0^-) = 0 \quad \text{on } \mathbb{R}_-, \tag{55}$$

$$u(x) = \mathcal{F}^{-1}_{\xi\mapsto x_1}\, e^{-x_2 t(\xi)}\widehat{u_0^+}(\xi)\, 1_+(x_2) + \mathcal{F}^{-1}_{\xi\mapsto x_1}\, e^{x_2 t(\xi)}\widehat{u_0^-}(\xi)\, 1_-(x_2)\,,$$

$$x = (x_1, x_2) \in \Omega^\pm$$

where $u_0^\pm$ are taken in the sense of the trace theorem, $\widehat{u_0^\pm} = \mathcal{F}\, u_0^\pm$, and $1_\pm$ stand for the characteristic functions of $\mathbb{R}_\pm$.

We denote the *potential operator* in (55) as $\mathcal{K}$ and write briefly

$$\mathcal{K} \quad : \quad H^{1/2} \times H^{1/2} \quad \to \quad \mathcal{H}^1(\Omega^+) \times \mathcal{H}^1(\Omega^-), \tag{56}$$

$$u \quad = \quad \mathcal{K}(u_0^+\,,\, u_0^-).$$

It is easily shown that $\mathcal{K}$ is an isomorphism.

To solve the problem (53) we now determine Φ from (41) - (42) (plugging (55) into the transmission conditions (53)) as

$$\begin{aligned} \Phi &= \frac{1}{2}\begin{pmatrix} a_0 - b_0 & -(a_0+b_0)\,t^{-1} \\ -(a_1+b_1)\,t & a_1 - b_1 \end{pmatrix} \qquad (57) \\ &= \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -\frac{a_1+b_1}{a_0+b_0} \end{pmatrix} \begin{pmatrix} 1 & t^{-1} \\ t & \lambda \end{pmatrix} \begin{pmatrix} a_0+b_0 & 0 \\ 0 & -(a_0+b_0) \end{pmatrix} \\ &= \qquad E \qquad\qquad \sigma_\lambda \qquad\qquad F \end{aligned}$$

with $\lambda = (a_1 - b_1)/(a_1 + b_1)$ by elementary transformation with invertible matrices $E, F \in \mathbb{C}^{2\times 2}$ provided $(a_0 \pm b_0) \neq 0$ and $(a_1 + b_1) \neq 0$.

We define $A_\Phi = \mathcal{F}^{-1}\,\Phi\cdot\mathcal{F} : X \to X$ where $X = H^{1/2}\times H^{-1/2}$, $P_1 = P_+^{(1/2)} \otimes P_+^{(-1/2)}$, $P_2 = \Pi_+^{(1/2)} \otimes \Pi_+^{(-1/2)}$ (sometimes written as diagonal matrix operators) and carry out the factorisation of σ_λ as given in (47) $\sigma_\lambda = \sigma_{\lambda-}\,\sigma_{\lambda+}$, $A_{\sigma_\lambda} = A_{\sigma_{\lambda-}}\,A_{\sigma_{\lambda+}}$.

Further put

$$W = P_2\,A_\Phi|_{P_1X} \quad , \quad W^{-1} = F^{-1}\;A^{-1}_{\sigma_{\lambda+}}\,P\,A^{-1}_{\sigma_{\lambda-}}|_{P_2X}\;E^{-1} \tag{58}$$

in accordance with Theorem 3.2 where $P = \ell_0\,r_+$.

The following result is taken from S89.

Theorem 5.5 *The BVP (53) is well posed, if and only if $\lambda \notin [1, +\infty)$.*

In this case, the operator L^0 associated to the (semi-homogeneous) BVP is equivalent to an invertible WHO W and the resolvent operator reads

$$(L^0)^{-1} \quad = \quad \mathcal{K} W^{-1}. \tag{59}$$

A complete discussion in dependence of the parameter λ will be possible later in Section 7 after equivalent reduction of W to a WHO W_0 acting in L^2 spaces, the so-called lifting process.

In brief the proof is based upon an equivalence relation

$$L^0 \quad \sim \quad W \quad = \quad L^0 \mathcal{K} \quad \sim \quad W_0 \tag{60}$$

and a complete analysis of $W_0 \in \mathcal{L}(L^2(\mathbb{R}_+)^2)$ as a generalisation of the WHO in Section 2 which is only possible after some additional preparation in Section 6.

Further concepts of operator factorisation in boundary value problems shall be discussed in Section 7.

Historical remarks.

In the present sense, operators L associated with elliptic BVPs have been used most consequently in the book of Joseph Wloka Wlo87. In different context it appeared already in the area of pseudo-differential operators, see Bou71 for instance.

The question of whether a BVP is well-posed goes back to Jacques Hadamard (1865-1963), see Had02,MazSha98. Besides the unique solubility he asked for a continuous dependence of the solution from the data (here from f and g). Therefore a complete formulation of the linear BVP (49) needs at least topological vector spaces (for the sets of data and solutions). The restriction of the consideration to Banach or Hilbert spaces comes from practical reasons and physical motivation (the energy space). See HW08,Wlo87 for more details.

The formulation of an "equivalent reduction" of a BVP to a boundary integral equation, e.g., has many different forms in the literature, see for instance HW08, Section 5.6.5, or DW95. In recent years it reached a higher level of precision by use of operator relations CST04,MS89,S13,S17 and became subject of independent research BGK84,BT91,HR13.

6 On the choice of function spaces and symbol classes

To tackle a (linear) problem in mathematical physics without mentioning the underlying function (or functional) spaces involves the difficulty that, e.g., the notion of well-posedness remains senseless, because "unique solubility" needs a prescribed set of solutions and "continuous dependence of the solution on the data" needs a topological structure in both, a solution and a data space.

Therefore a good strategy consists in starting with a certain space setting (of topological vector spaces such as Banach or Hilbert spaces) from the very beginning and adapting it later eventually for some reason. This process is sometimes called normalisation.

It makes also sense to admit first a variety of spaces such as L^p or $W^{p,s}$ spaces, $1 \leq p < \infty$, $s \in \mathbb{R}$, and to specify the parameters p, s later, for instance to make the problem well-posed or to discuss the smoothness of solutions.

The motivation for the choice of Bessel potential spaces H^s, which coincide with the Sobolev-Slobodetskii spaces $W^{p,s}$ in the case of $p = 2$, is related to physical relevance (the energy space reasoning, consideration of diffracted waves, fields without sources) SanTei89 and mathematical convenience (Hilbert space structure, boundedness of the Hilbert transformation for additive decomposition and factorisation of bounded functions into bounded factors, e.g.), see Esk73,HW08,Wlo87 and GohKru73,MikPro86, respectively.

This leads us to study BVPs such as the Sommerfeld half-plane problems in scales of solution spaces $\mathcal{H}^s(\Omega)$, $s \geq 1$, data spaces such as $H^{s-1/2}(\mathbb{R}_+)$ and $H^{s-3/2}(\mathbb{R}_+)$.

Weighted Sobolev spaces are also convenient Tri83 in view of asymptotics and smoothness of solutions to a BVP.

In what concerns the classes of WHOs it is quite important and convenient to study not only Fourier symbols which are rational functions or which belong to a decomposing algebra but which are *related to a decomposing algebra* such as the Wiener algebra or the Banach algebra of *Hölder continuous functions* on the one-point compactification of the real line $\dot{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$:

$$C^{\mu}(\dot{\mathbb{R}}) = \Big\{ \phi \in C^{\mu}(\mathbb{R}) : \phi(\infty) = \lim_{\xi \to \infty} \phi(\xi) \text{ exists and } \phi(\xi) - \phi(\infty) = \mathcal{O}\left(|\xi|^{-\mu} \right) \text{ as } \xi \to \infty \Big\}, \quad \mu \in (0,1). \tag{61}$$

It is well known GohKru73,MikPro86 that $C^\mu(\dot{\mathbb{R}})$ forms a unital algebra, which is decomposing, i.e., the *Hilbert projectors* (also writable with the help of the Hilbert transformation which is here omitted)

$$\mathcal{F}^{-1}\ P_\pm\ \mathcal{F} \quad = \quad \mathcal{F}^{-1}\ \ell_0\, r_\pm\ \mathcal{F} \quad = \quad C_0^\mu(\dot{\mathbb{R}}) \to C_0^\mu(\dot{\mathbb{R}}) \tag{62}$$

are continuous where $C_0^\mu(\dot{\mathbb{R}}) = \{\phi \in C^\mu(\dot{\mathbb{R}}) : \phi(\infty) = 0\}$ and we have an additive decomposition into complemented subspaces

$$C_0^\mu(\dot{\mathbb{R}}) \quad = \quad C_+^\mu(\dot{\mathbb{R}}) \oplus C_-^\mu(\dot{\mathbb{R}}). \tag{63}$$

Roughly speaking these are the subspaces of $C_0^\mu(\dot{\mathbb{R}})$ functions which are holomorphically extensible into the upper and into the lower complex half-plane, respectively, with zero limit at infinity.

The regular functions of the algebra (61)

$$\mathcal{G}\,C^\mu(\dot{\mathbb{R}}) \quad = \quad \{\phi \in C^\mu(\dot{\mathbb{R}})\ ,\ \phi(\xi) \neq 0 \ \text{in} \ \dot{\mathbb{R}}\} \tag{64}$$

form a group which is inverse-closed, i.e., $\phi^{-1} \in \mathcal{G}\,C^\mu(\dot{\mathbb{R}})$ if $\phi \in \mathcal{G}\,C^\mu(\dot{\mathbb{R}})$. These functions allow a factorisation of the form (9) (after splitting $\phi(\infty)$) within $\mathcal{G}\,C^\mu(\dot{\mathbb{R}})$ with all the consequences (10) - (15).

If, instead of the situation before, there are two different limits $\phi(\pm\infty)$ at infinity, we have also a unital algebra denoted by

$$C^\mu(\ddot{\mathbb{R}}) = \Big\{\phi \in C^\mu(\mathbb{R}) : \phi(\pm\infty) = \lim_{\xi\to\pm\infty} \phi(\xi) \text{ exist and } \phi(\xi) - \phi(\pm\infty) = \mathcal{O}\left(|\xi|^{-\mu}\right) \text{ as } \xi \to \pm\infty\Big\}, \quad \mu \in (0,1) \tag{65}$$

which is inverse-closed, as well.

Unfortunately $C^\mu(\ddot{\mathbb{R}})$ is not decomposing, a factorisation like (10) with factors in $C^\mu(\ddot{\mathbb{R}})$ does not exist whenever $\phi(-\infty) \neq \phi(+\infty)$.

However, this algebra is crucial in the treatise of canonical diffraction problems since many Fourier symbols Φ in the related WH equations are generated by polynomials and square root functions like $t^{\pm 1}, t_+^{\pm 1/2}, t_-^{\pm 1/2}$, see the examples in Section 3.

More precisely, all Fourier symbols that appear in Sommerfeld half plane problems (two-dimensional, only one wave number involved) belong to a certain algebra, the functions generated by polynomials and one square root function t which can be represented by

$$\Phi \quad = \quad a_1\, Q_1 \, + \, a_2\, Q_2 \tag{66}$$

where a_j are scalar functions and Q_j are rational 2×2 matrix functions. We say: The *number of rationally independent entries* $\operatorname{nr} \Phi = 2$ PS90.

These facts finally led to the intensive study of the algebra of matrix functions $C^\mu(\ddot{\mathbb{R}})^{n\times n}$, $n \in \mathbb{N}$, particularly in case of $n = 1$ and $n = 2$ CDS06,CS05,CST04,MoST98.

Let us have a look at the following factorisation theorem for scalar symbols which at the end is no more than a moderate modification of the formulas in Section 2 Dud79,MoST98.

Theorem 6.1 *Let* $\Phi \in \mathcal{G}\, C^{\mu}(\ddot{\mathbb{R}})$*. Then* Φ *admits a factorisation*

$$\begin{aligned}
\Phi &= \Phi_- \ \zeta^{\kappa} \ \Phi_+ \quad \textit{where} && (67)\\
\kappa &= \max\{z \in \mathbb{Z} : z \leq \Re(\omega) + \frac{1}{2}\},\\
\omega &= \frac{1}{2\pi i}\int_{\mathbb{R}} d\log\Phi(\xi)d\xi \in \mathbb{C},\\
\Psi &= \zeta^{-\omega}\,\Phi^{-1}(+\infty)\,\Phi \quad , \quad \zeta(\xi) = \frac{\xi - i}{\xi + i} \ , \ \xi \in \mathbb{R},\\
\Psi_{\pm} &= \exp\{\mathcal{F}\,\ell_0\,r_{\pm}\mathcal{F}^{-1}\ \log\Psi\} \ \in \ C^{\mu}_{\pm}(\dot{\mathbb{R}}),\\
\Phi_- &= \zeta_-^{\omega-\kappa}\Psi_- \quad , \quad \Phi_+ = \Phi(+\infty)\,\zeta_+^{\kappa-\omega}\Psi_+
\end{aligned}$$

where $\zeta_{\pm}(\xi) = \xi \pm i\,,\ \xi \in \mathbb{R}$*.*

Remark 6.2 *The complex winding number* $\omega = \omega(\Phi)$ *of* Φ *can be written as*

$$\omega \;=\; \kappa + \eta + i\tau \quad \textit{where} \quad \kappa \in \mathbb{Z}\,,\, \eta \in (-1/2, +1/2]\,,\, \tau \in \mathbb{R}$$

The function Ψ *belongs to* $C^\mu(\dot{\mathbb{R}})$, $\Psi(\mp\infty) = 1$, *and* $\omega(\Psi) = 0$.

Note that $\zeta(\xi)$ *may be replaced by* $\zeta_k(\xi) = (\xi - k)/(\xi + k) = t_-(\xi)/t_+(\xi)$ *which may have advantage in diffraction problems (cf. (76)).*

Again we formally define the operator factorisation corresponding with (10)

$$A \;=\; A_- \;C\; A_+\,, \tag{68}$$

$$A_\pm \;=\; \mathcal{F}^{-1}\,\Phi_\pm \cdot \mathcal{F} \quad , \quad C \;=\; \mathcal{F}^{-1}\,\zeta^\kappa \cdot \mathcal{F} \quad : \quad L^2 \to L^2.$$

These formulas admit two different interpretations.

Firstly (67) represents a generalised factorisation in L^2 in the sense of Sim68, provided

$$\Re(\omega) + \frac{1}{2} \quad \notin \quad \mathbb{Z}, \tag{69}$$

see CS95,GohKru73,MikPro86,MoST98. I.e., $t_+^{-1}\,\Phi_+^{\pm 1} \;\in\; L^2(\mathbb{R}_+)$, $t_-^{-1}\,\Phi_-^{\pm 1} \;\in\; L^2(\mathbb{R}_-)$. $A_\pm$ in (68) must be regarded as unbounded operators in $L^2(\mathbb{R})$, densely defined and such that

$$W^- \quad = \quad A_+^{-1}\,P\,C^{-1}\,P\,A_-^{-1}|_{L_+^2} \tag{70}$$

admits a bounded extension by continuity to $\mathcal{L}(L^2(\mathbb{R}))$.

Depending on whether κ is positive or negative, this formula defines a right or a left inverse of $W = P\,A|_{PX}$ for $X = L^2$, $P = \ell_0\,r_+$, and an inverse of W for $\kappa = 0$, all this provided $\eta \neq 1/2$.

The second interpretation is a factorisation through an intermediate space S15

$$\begin{aligned} A &= \quad A_- \quad C \quad A_+ \\ &: \quad X \longleftarrow Z \longleftarrow Z \longleftarrow X, \end{aligned} \tag{71}$$

where $X = L^2$ and $Z = H^\eta$ with the advantage to avoid unbounded operators. The consequences are the same as before but formulas are composed of bounded instead of unbounded operators.

In the case of $\eta = 1/2$ the operator W turns out to be not normally solvable. More precisely its image is not closed.

A normalisation can be achieved systematically by choosing a smaller image space imposing a compatibility condition (in the image space) and a corresponding norm such that the normalised operator is one-sided invertible and (70) becomes a one-sided inverse in the new space setting. See MoST96,MoST98 for details.

The generalisation of Theorem 6.1 to the systems case where Φ is a squared matrix function

$$\Phi \quad \in \quad \mathcal{G}C^{\mu}(\ddot{\mathbb{R}})^{m\times m} \tag{72}$$

combines ideas of the factorisation of matrix functions with elements in decomposing algebras such as $\Phi \in C^{\mu}(\dot{\mathbb{R}})^{m\times m}$ GohKre58, with the factorisation technique of Theorem 6.1, see CDS06. The case $m = 2$ is needed for a complete analysis of (60).

In the next section we shall see the abstract analogue of this factorisation (71) as a sufficient and necessary condition for the generalised invertibility of a general WHO in symmetric and in asymmetric space setting, respectively.

Historical remarks.

For a glance at the history of relevant function spaces we refer to the book by Hans Triebel Tri83. The use of Sobolev-like spaces in BVPs started in the 1930s with the famous work of Sergei Sobolev, L.N. Slobodeckij, Beppo Levi, and Nachman Aronzjan and became a standard tool in the solubility theory for partial differential and pseudo-differential equations in the 1950s, see the work of Mikhail Agranovic, Nicolas Bourbaki, and Lars Hörmander.

For general questions about smoothness and asymptotic behavior of solutions see Esk73,HW08,Wlo87.

The edge and radiation conditions SanTei89,Som49 are not relevant in the present context, where a solution u denotes a diffracted wave (without sources and sinks) in a suitable (energy) space.

In what concerns the most relevant classes of Fourier symbols we refer to BoeSil06,GohFel74,MikPro86. In the applications the classes of matrix functions with entries from $C^\mu(\dot{\mathbb{R}})$ and $C^\mu(\ddot{\mathbb{R}})$ play a decisive role, see GohKre58,Kre58 for the origin and Dud79,MoST98, respectively, for some relevant applications.

The idea of an intermediate space in WH factorisation (related to Sommerfeld problems) arose in S86 and gained further interest in Cas94,CS95,MoST98,S15.

7 Equivalence and reduction via operator factorisation

To reduce a problem or a system of equations $Tf = g$ can often be seen as to find a "simpler operator" S somehow related to the operator T such that conclusions for S imply conclusions for T. As before we focus bounded linear operators in Banach spaces.

The easiest case: Two operators are equivalent, i.e.,

$$T \quad = \quad E \ S \ F \tag{73}$$

where E and F are isomorphisms.

In this case, many properties of S are transferred to properties of T, e.g., invertibility, the Fredholm property, explicit representation of (generalised) inverses, etc. CS98 (and vice versa). Conclusions of that kind appeared already in Section 5.

So we speak about operator relations and their transfer properties.

Now we outline four further examples of this rather strong equivalence relation which are most important in WHO theory and its applications:

- lifting from Sobolev to Lebesgue spaces Dud79,Esk73,S89,
- shifting of WHOs in a scale of Sobolev spaces MoST98,
- equivalence of a WHO with the truncation of a cross factor S83,S85,
- symmetrisation of general WHOs BoeSpe16.

First we describe the so-called *lifting of WHOs* from Sobolev to Lebesgue spaces in the scalar case. In view of (23) consider an operator of the form

$$W \quad = \quad r_+ \; A_\Phi|_{H^r_+} \quad : \quad H^r_+ \;\to\; H^s(\mathbb{R}_+) \tag{74}$$

where $A_\Phi : X = H^r \to Y = H^s$ is a bounded convolution operator, i.e., $t^{s-r}\Phi \in L^\infty$. Then we have as a consequence of (26) - the mapping properties of Bessel potential operators:

Proposition 7.1 *Under these assumptions W is equivalent to a WHO acting in L^2 spaces:*

$$\begin{aligned} W \quad \sim \quad W_0 \quad &= \quad r_+ \; A_{\Phi_0}|_{L^2_+} \quad : \quad L^2_+ \;\to\; L^2(\mathbb{R}_+) \qquad (75) \\ \Phi_0 \quad &= \quad t^r_- \, \Phi \, t^{-s}_+ \, . \end{aligned}$$

The same idea works perfectly for matrix operators.

As an example consider (45) again. There we get

$$\begin{aligned} W &= r_+ A_{\sigma_\lambda}|_{H_+^{1/2}\times H_+^{-1/2}} \; : \; H_+^{1/2}\times H_+^{-1/2} \to H^{1/2}(\mathbb{R}_+)\times H^{-1/2}(\mathbb{R}_+) \\ \sim W_0 &= r_+ A_{\sigma_{\lambda,0}}|_{L_+^2\times L_+^2} \; : \; L_+^2\times L_+^2 \to L^2(\mathbb{R}_+)\times L^2(\mathbb{R}_+) \end{aligned}$$

with

$$\begin{aligned} \sigma_{\lambda,0} &= \begin{pmatrix} t_-^{1/2} & 0 \\ 0 & t_-^{-1/2} \end{pmatrix} \begin{pmatrix} 1 & t^{-1} \\ t & \lambda \end{pmatrix} \begin{pmatrix} t_+^{-1/2} & 0 \\ 0 & t_+^{1/2} \end{pmatrix} \\ &= \begin{pmatrix} \zeta_k^{1/2} & 1 \\ 1 & \lambda\,\zeta_k^{-1/2} \end{pmatrix}. \end{aligned} \tag{76}$$

This allows a direct application of the theory of systems of WH equations in L^2 spaces GohKre58,MikPro86 and ends up with a complete discussion of properties of W in dependence of the parameter λ, see S89, as follows:

1. $\lambda = 0$: $\sigma_{\lambda,0}$ is triangular. W_0 decomposes into two single WH equations that are invertible;
2. $\lambda = 1$: $\sigma_{\lambda,0}$ degenerates and $\operatorname{im} W_0$ is not closed;
3. $\lambda \in (0,1)$: $\sigma_{\lambda,0}$ admits a bounded strong factorisation, W_0 is invertible;
4. $\lambda \in (1,\infty)$: $\sigma_{\lambda,0}$ admits a function theoretic factorisation but not a generalised factorisation in $L^2(\mathbb{R})^2$, and W is not Fredholm in $L^2(\mathbb{R}_+)^2$;
5. $\lambda \notin [0,\infty)$: $\sigma_{\lambda,0}$ admits an unbounded (generalised) factorisation with vanishing partial indices, W_0 is invertible.

The preceding idea can also be used to discuss regularity properties of (75), say, i.e., to look at smoother solutions of the equation $W f = g$ (for smoother data g). A convenient way is to consider the restricted operator

$$W_s \quad = \quad \operatorname{Rst} r_+ \ A_{\Phi_0}|_{H^s_+} \quad : \quad H^s_+ \ \to \ H^s(\mathbb{R}_+) \quad , \quad \text{for } \ s > 0. \tag{77}$$

This makes sense because H^s is an invariant subspace of the convolution operator A_{Φ_0}.

The consideration in larger spaces may be interesting, as well. Hence we also define

$$W_s \quad = \quad \operatorname{Ext} r_+ \ A_{\Phi_0}|_{H^s_+} \quad : \quad H^s_+ \ \to \ H^s(\mathbb{R}_+) \quad , \quad \text{for } \ s < 0 \tag{78}$$

by continuous extension since $H^s \subset L^2$, $H^s_+ \subset L^2_+$, and $H^s(\mathbb{R}_+) \subset L^2(\mathbb{R}_+)$ are densely embedded for negative s.

These operators (77) and (78) are referred to as *shifted operators* MoST98,PenSpe93.

Now the point is that the shifted operator can also be lifted to the L^2 level and that lifted shifted WHO has a symbol $\Phi_{s,0} = t_-^s \, \Phi_0 \, t_+^{-s} = \zeta^s \, \Phi_0$ in the scalar case. Hence, if Φ_0 admits a factorisation like (67) then all lifted shifted Fourier symbols admit a factorisation of this kind, as well.

This technique can be used, e.g., in the cases where the operator W_0 of (76) is not invertible. Namely, W can be normalised by a shift in the scale of Sobolev spaces, i.e., it becomes invertible as an operator restricted like

$$W^{(\varepsilon)} \quad : \quad H_+^{1/2+\varepsilon} \times H_+^{-1/2+\varepsilon} \;\to\; H^{1/2+\varepsilon}(\mathbb{R}_+) \times H^{-1/2+\varepsilon}(\mathbb{R}_+)$$

with a suitable $\varepsilon \in (0, 1/2)$.

For more consequences and details see MoST98 and CDS06 for the matrix case.

For the following let us recall some properties of generalised inverses T^- of $T \in \mathcal{L}(X, Y)$ as defined already in (20). It is well-known that the following assertions are equivalent:

- T is generalised invertible, i.e., $T\,T^-\,T = T$ for some $T^- \in \mathcal{L}(Y, X)$,
- T has a reflexive generalised inverse, i.e., $T^-\,T\,T^- = T^-$ holds additionally,
- $\ker T$ and $\operatorname{im} T$ are complemented subspaces of X and Y, respectively,
- there exist projectors $P_0 \in \mathcal{L}(X)$ and $P_1 \in \mathcal{L}(Y)$ onto $\ker T$ and onto $\operatorname{im} T$, respectively,
- there exist projectors $P_0 \in \mathcal{L}(X)$ and $P_1 \in \mathcal{L}(Y)$ such that, for arbitrary $g \in Y$, the equation $T\,f = g$ is solvable if and only if $P_1\,g = g$ and, in this case, the general solution of the equation is given by $f = T^-\,g + P_0\,h$, $h \in X$.

For the proof choose $T^-\,T\,T^-$ as a reflexive generalised inverse, and $P_0 = I - T^-\,T$, $P_1 = T\,T^-$ in the corresponding places.

A third example for an equivalent reduction of a WHO by an operator equivalence (73) was touched already in (10).

The following result is just an interpretation of those formulas, extended to the asymmetric case.

Proposition 7.2 *Let* $W = P_2\, A|_{P_1 X} : P_1 X \to P_2 Y$ *be a general WHO with the assumptions of* (19). *Further let*

$$\begin{array}{cccccccc} A & = & A_- & & C & & A_+ & \\ & : & X \longleftarrow & Z & \longleftarrow & Z & \longleftarrow & X, \end{array} \tag{79}$$

be a factorisation into three isomorphisms with an intermediate Banach space Z, *and a projector* $P \in \mathcal{L}(Z)$ *such that*

$$P A_+ P_1 \;=\; A_+ P_1 \quad , \quad P_2 A_- P \;=\; P_2 A_- \,. \tag{80}$$

Then W *satisfies*

$$W \;=\; P_2\, A_-|_{PZ}\, P\, C|_{PZ}\, P\, A_-|_{P_1 X} \;\sim\; P\, C|_{PZ}\,. \tag{81}$$

I.e., W *can be symmetrised (is equivalent to a WHO in symmetric space setting).*

Various examples show that this idea of equivalent reduction yields direct results, provided the operator C has certain properties.

We call an isomorphism $C \in \mathcal{L}(Z)$ a *cross factor* with respect to a projector $P \in \mathcal{L}(Z)$ if $C^{-1}PCP$ is idempotent, or equivalently if

$$C^{-1}PCP = PC^{-1}PCP. \tag{82}$$

It follows that C splits the space Z twice into four subspaces with

$$\begin{array}{ccccc} Z & = & \overbrace{X_1 \dot{+} X_0}^{PZ} & \dot{+} & \overbrace{X_2 \dot{+} X_3}^{QZ} \\ & & \downarrow & C \nearrow\!\!\!\!\searrow & \downarrow \\ & = & \underbrace{Y_1 \dot{+} Y_2}_{PZ} & \dot{+} & \underbrace{Y_0 \dot{+} Y_3}_{QZ} \end{array} \tag{83}$$

where $Q = I_Z - P$ and where C maps each X_j onto Y_j , j = 0, 1, 2, 3 , i.e., the complemented subspaces $X_0, X_1, ..., Y_3$ are images of corresponding projectors $p_0, p_1, ..., q_3$, namely $X_0 = p_0 Z = C^{-1}QCPZ$, $X_1 = p_1 Z = C^{-1}PCPZ, ..., Y_3 = q_3 Z = CQC^{-1}QZ$.

A WHO W which is a truncation of a cross factor, $W = PC|_{PX}$, allows very strong conclusions for the solution of the corresponding equation $W f = g$:

Proposition 7.3 *Let X be a Banach space, $P \in \mathcal{L}(X)$, $P^2 = P$, and $C \in \mathcal{GL}(X)$ a cross factor with respect to P. Then we have:*

- $W = PC|_{PX}$ *is generalised invertible,*
- $W^- = PC^{-1}|_{PX}$ *is a reflexive generalised inverse of* W,
- $\ker W = PC^{-1}QCPX = C^{-1}QCPX$,
- $\operatorname{im} W = PC^{-1}PCPX = C^{-1}PCPX$,
- *for any $g \in PX$ the equation $W f = g$ is solvable in PX if and only if $C^{-1}PCg = g$ and, in this case, the general solution of the equation is given by*

$$f \quad = \quad W^- g + C^{-1}QCh \quad \textit{for arbitrary} \quad h \in PX\,. \tag{84}$$

The proof of this result is elementary and can be seen as a consequence of the diagram (83).

Moreover the equivalence of a general WHO (16) in symmetric space setting with $A \in \mathcal{GL}(X)$ to a truncation of a cross factor (in the same space) is characteristic for the generalised invertibility of W:

Theorem 7.4 *The WHO (16) is generalised invertible if and only if A admits a cross factorisation (with respect to X and P), i.e.,*

$$A = A_- \; C \; A_+ \tag{85}$$

where $A_\pm$ are strong WH factors (satisfying the first two lines of (13)) and C is a cross factor (with respect to P). In this case we have

$$\begin{aligned} W &= P \; A_- \; P \; C \; P \; A_+|_{PX} \\ &= P A_-|_{PX} \; P C|_{PX} \; P A_+|_{PX} \quad \sim \quad P C|_{PX} \, . \end{aligned} \tag{86}$$

A reflexive generalised inverse of W is given by

$$W^- = P \; A_+^{-1} \; P \; C^{-1} \; P \; A_-^{-1}|_{PX} \, . \tag{87}$$

Attention: Herein we assume that the space setting is symmetric.

The proof of sufficiency is obvious, a verification of W^- to be a reflexive generalised inverse based on the factor properties of $A_\pm$ and C.

The proof of necessity is very tricky, either by use of a space decomposition generated by a reflexive generalised inverse S83 or by a direct formula of a cross factorisation in terms of a generalised inverse, see S85, Chapter 6, and my lecture next Monday.

However, a corresponding result for general WHOs in asymmetric space setting $W = P_2\, A|_{P_1 X}$ (see (19)) has been published only recently BoeSpe16,S15, under an additional condition that guarantees the symmetrization of W:

Theorem 7.5 *Consider the general WHO (19) (in asymmetric space setting where A is an isomorphism). Then the following assertions are equivalent:*

1. *A admits a WH factorisation through an intermediate space (FIS)*

$$A = A_- \quad C \quad A_+ \quad : \quad Y \longleftarrow Z \longleftarrow Z \longleftarrow X, \tag{88}$$

   where $A_\pm$ are strong WH factors, see (30), and C is a cross factor;

2. *W is generalised invertible and $P_1 \sim P_2$, i.e.,*

$$\ker P_1 \cong \ker P_2 \quad \text{and} \quad \operatorname{im} P_1 \cong \operatorname{im} P_2 \,. \tag{89}$$

In this case a reflexive generalised inverse of W is given by

$$W^- = P_1 \; A_+^{-1} \; P \; C^{-1} \; P \; A_-^{-1}|_{P_2 Y} \,. \tag{90}$$

This last result contains the idea to symmetrise a general WHO and (hence) the space setting itself, see BoeSpe16 which is important in applications, since it simplifies the reasoning and enables the use of other results, e.g., about operators in Lebesgue spaces (by lifting, as mentioned before).

However, the conditions of the last theorem are not necessary for a WHO to be generalised invertible, see the cross factorisation theorem in asymmetric settings in S85 and further going research in S18.

More important for applications is the stability of factorisations against a change of the underlying function spaces for normalisation and regularity results, see MoST98 for instance.

As an example that fits best to the present context consider the WHO (4) again where $C = \mathcal{F}^{-1}\zeta^{\kappa} \cdot \mathcal{F}$, $\kappa \in \mathbb{Z}$. It is not hard to prove the following facts.

- Formula (10) represents a cross factorisation provided Φ is invertible in the Wiener algebra ;
- If $\kappa < 0$ the subspaces of (83) are

$$\begin{aligned} X_0 &= \ker PC|_{PX} = \operatorname{span}\{\phi_j\,,\, j = 1, ..., -\kappa\} \\ \phi_j(x) &= \mathcal{F}^{-1}_{\xi\mapsto x}\frac{(\xi - i)^{j-1}}{(\xi + i)^{\kappa}}\,,\, x \in \mathbb{R}\,, \end{aligned} \tag{91}$$

  and X_2 is not present, etc.
- If $\kappa > 0$ the subspaces of (83) are

$$\begin{aligned} X_2 &= \ker QC|_{QX} = \operatorname{span}\{\psi_j\,,\, j = 1, ..., \kappa\} \\ \psi_j(x) &= \mathcal{F}^{-1}_{\xi\mapsto x}\frac{(\xi + i)^{j-1}}{(\xi - i)^{\kappa}}\,,\, x \in \mathbb{R}\,, \end{aligned} \tag{92}$$

  and X_0 is not present, etc.

In the first case $\kappa < 0$, the equation $W f = g$ is solvable for all $g \in L^2(\mathbb{R}_+)$ by

$$f = W^- g + \sum_{j=1}^{-\kappa} a_j A_+^{-1} \phi_j \text{ with } a_j \in \mathbb{C}. \tag{93}$$

In the second case $\kappa > 0$, the equation $W f = g$ is solvable if and only if κ conditions are satisfied

$$\int_{\mathbb{R}_+} g P A_-^{-1} P C^{-1} \psi_j = 0 , \quad j = 1, ..., \kappa. \tag{94}$$

The solution reads $f = W^- g$ and is unique in this case.

These results are standard conclusions for Fredholm operators and their generalised inverses.

Finally we like to remark that, in a symmetric space setting, every general WHO $W = PA|_{PX}$ which is generalised invertible can be presented as a truncation of a (suitable) cross factor S18.

Historical remarks.

The idea of equivalent reduction in the sense of (73) is a genuine element of WH factorisation. The idea of the lifting process and its consequences were settled by Gregory Eskin and Roland Duduchava in the 1970s, see Dud79,Esk73.

For the history of generalised inverses see NasRal76. The generalised inversion of general WHOs was firstly studied by the author in S83,S85 where also the term "cross factor" was introduced (in symmetric and in asymmetric Banach space settings). Intermediate spaces in the context of generalised factorisation of matrix functions in the sense of I.B. Simonenko Sim68 were firstly studied by Luís Castro Cas94,CS95. The complete proof of Theorem 7.5 was presented only in 2016 BoeSpe16,S15.

The question whether a general WHO (19) in asymmetric setting can be equivalently reduced to a WHO (16) in symmetric setting was recently analysed in BoeSpe16. For further results about equivalent reduction of general WHOs to more convenient forms see S18.

8 From operator factorisation to operator relations

The operator factorisations considered before can all be seen as equivalence relations in the sense of (73). However, there are other operator relations which play a crucial role. We point out two of them, again coming up from concrete applications.

It is well known that the diffraction problem where the trace of the diffracted field is given on a plane screen $\Sigma \in \mathbb{R}^3$ (equally on the two sides of the screen) leads to an (boundary integral) equation

$$r_\Sigma \, A_{t^{-1}} \, f \quad = \quad g \tag{95}$$

where Σ is seen as a (Lipschitz) domain in $\mathbb{R}^2$, $A_{t^{-1}} = \mathcal{F}^{-1} \, t^{-1} \cdot \mathcal{F}$, $\mathcal{F}$ denotes here the two-dimensional Fourier transformation, $t(\xi_1, \xi_2) = (\xi_1^2 + \xi_2^2 - k^2)^{1/2}$, $g \in H^{1/2}(\Sigma)$ is given and $f \in H_\Sigma^{-1/2}$ is unknown, in analogy to the Sommerfeld-Dirichlet problem: f is an element of $H^{-1/2}(\mathbb{R}^2)$ supported on $\overline{\Sigma}$.

Assuming that the *complementary screen* $\Sigma' = \mathbb{R}^2 \setminus \overline{\Sigma}$ is also a Lipschitz domain, we consider the Sommerfeld-Neumann problem for Σ' and arrive by analogy at an equation

$$r_{\Sigma'} A_t f_* \quad = \quad g_* \tag{96}$$

where $g_* \in H^{-1/2}(\Sigma')$ is given and $f_* \in H^{1/2}_{\Sigma'}$ is unknown. Now, Lipschitz domains allow continuous extension operators Wlo87, say

$$\ell_{\Sigma}^{(1/2)} \;:\; H^{1/2}(\Sigma) \to H^{1/2}(\mathbb{R}^2) \quad , \quad \ell_{\Sigma'}^{(-1/2)} \;:\; H^{-1/2}(\Sigma') \to H^{-1/2}(\mathbb{R}^2)$$

such that the projector $P_2 = \ell_{\Sigma}^{(1/2)} r_{\Sigma}$ acts in $H^{1/2}(\mathbb{R}^2)$ along $H^{1/2}_{\Sigma'}$ and $Q_1 = \ell_{\Sigma'}^{(-1/2)} r_{\Sigma'}$ acts in $H^{-1/2}(\mathbb{R}^2)$ along $H^{-1/2}_{\Sigma}$.

In analogy to the half-line case in Section 3 we conclude that the operators in the foregoing equations (95) and (96) are equivalent to operators in the form of general WHOs

$$r_\Sigma \, A_{t^{-1}}|_{H_\Sigma^{-1/2}} \sim W = P_2 \, A_{t^{-1}}|_{P_1 X} \,, \tag{97}$$
$$r_{\Sigma'} \, A_t|_{H_{\Sigma'}^{1/2}} \sim W_* = Q_1 \, A_t|_{Q_2 Y} \,,$$

where $X = H^{-1/2}$, $P_1 + Q_1 = I|_{H^{-1/2}}$ and $Y = H^{1/2}$, $P_2 + Q_2 = I|_{H^{1/2}}$, cf. (19). Using the space decompositions $X = P_1 X \oplus Q_1 X \cong P_1 X \times Q_1 X$ (identifying the direct sum with the topological product space considered as a Banach space written in matrix form) and $Y = P_2 Y \oplus Q_2 Y \cong P_2 Y \times Q_2 Y$, as well, we have

$$A_{t^{-1}} \quad \sim \quad \begin{pmatrix} W & \cdot \\ \cdot & \cdot \end{pmatrix} \quad = \quad \begin{pmatrix} \cdot & \cdot \\ \cdot & W_* \end{pmatrix}^{-1} \quad \sim \quad A_t^{-1} \tag{98}$$

where the dots stand for obvious entries $P_2 \, A_{t^{-1}}|_{Q_1 X}$ etc., i.e., W and W_* are *matricially coupled* (MC) operators BGK84,BT91 and *associated WHOs* in the sense of S83,S85.

The MC relation between two bounded linear operators T, S in Banach spaces is often written in the form

$$\begin{pmatrix} T & \cdot \\ \cdot & \cdot \end{pmatrix} = \begin{pmatrix} \cdot & \cdot \\ \cdot & S \end{pmatrix}^{-1} . \tag{99}$$

It is well known that (99) is closely related to another operator relation which is called *equivalence after extension* (EAE) relation defined by

$$\begin{pmatrix} T & 0 \\ 0 & I_{Z_1} \end{pmatrix} = E \begin{pmatrix} S & 0 \\ 0 & I_{Z_2} \end{pmatrix} F \tag{100}$$

where Z_1, Z_2 are Banach spaces and E, F are isomorphisms acting between the corresponding spaces. In this case one writes

$$T \overset{*}{\sim} S . \tag{101}$$

The two relations (99) and (100) look quite different, however it turns out that they coincide:

Theorem 8.1 (of Bart and Tsekanovskii) *Two bounded linear operators acting in Banach spaces are matricially coupled if and only if they are equivalent after extension.*

There are some important facts to be pointed out:

- EAE is an equivalence relation in the genuine mathematical sense (reflexive, symmetric, and transitive) and such is MC;
- $T \sim S$ implies $T \overset{*}{\sim} S$, but not conversely (see the example below);
- The two relations (99) and (100) can be computed from each other BT91,S17a.

The following conclusion is most important in applications CS98.

Theorem 8.2 *If* (100) *is satisfied, and if* S^- *is a reflexive generalised inverse of* S*, then a reflexive generalised inverse of* T *is given by the formula*

$$T^- = R_{11}\left(F^{-1}\begin{pmatrix} S^- & 0 \\ 0 & I_{Z_2}\end{pmatrix}E^{-1}\right) \tag{102}$$

where R_{11} *denotes the restricted operator of the first matrix entry to the first component spaces.*

An example for the ease of a MC relation appears in diffraction theory in the context of diffraction of time-harmonic waves from plane screens in $\mathbb{R}^3$. It turns out that the operators associated to certain problems for complementary plane screens are matricially coupled, which was denoted as "abstract Babinet principle" in S13a.

Therefore the construction of the resolvent operator for one of the problems implies a representation of the resolvent operator for the other one. Since the resolvent operators can be computed for certain convex screens, the preceeding idea yields the explicit solution of diffraction problems for a much wider class of plane screens, the so-called "polygonal-conical screens" CDS14.

Another application of the EAE relation is the following. A process commonly denoted by "equivalent reduction" is the step from a boundary value problem (BVP) to a semi-homogeneous BVP. This is not reflected by equivalence between the two associated operators but by equivalence after extension.

Consider the abstract BVP (5.1)-(5.2) and its associated operator (50). Moreover consider the semi-homogeneous (abstract) boundary value problem briefly written as

$$L^0 u = \begin{pmatrix} A \\ B \end{pmatrix} u = \begin{pmatrix} 0 \\ g \end{pmatrix} \in \{0\} \times Y_2 \cong Y_2 \tag{103}$$

with associated operator

$$B|_{\ker A} : X_0 = \ker A \to Y_2. \tag{104}$$

By analogy we may define $A|_{\ker B} : \ker B \to Y_1$.

Theorem 8.3 *Let* $L = \begin{pmatrix} A \\ B \end{pmatrix} \in \mathcal{L}(\mathcal{X}, Y_1 \times Y_2)$ *be a bounded linear operator in Banach spaces.*

Then

$$AR = I \ \text{ for some } \ R \in \mathcal{L}(Y_1, \mathcal{X}) \qquad \text{implies} \qquad L \overset{*}{\sim} B|_{\ker A},$$
$$BR = I \ \text{ for some } \ R \in \mathcal{L}(Y_2, \mathcal{X}) \qquad \text{implies} \qquad L \overset{*}{\sim} A|_{\ker B}.$$

The proof (for the first case) is based upon the operator identity

$$L = ETF = \begin{pmatrix} 0 & A|_{X_1} \\ I|_{Y_2} & B|_{X_1} \end{pmatrix} \begin{pmatrix} B|_{X_0} & 0 \\ 0 & I|_{X_1} \end{pmatrix} \begin{pmatrix} P \\ Q \end{pmatrix} \qquad (105)$$

which holds if A is right invertible: $AR = I|_{Y_1}$, say, with $X_0 = \ker A$, $X_1 = \operatorname{im} R$, $Q = RA, P = I - RA$. Formula (105) is an EAE relation between L and $B|_{X_0}$, more precisely an equivalence after one-sided extension (EAOE) relation, since the operator L on the left side is not extended.

For details and further consequences (transfer properties, normalization etc.) see S13.

As a consequence, the resolvent operator $(L^0)^{-1}$ of the semi- homogeneous BVP (103) implies a representation of the resolvent operator L^{-1} of the full problem by the help of formula (102).

For further operator relations, their interplay, and further applications see BT94,Cas98,CS98,HMR15,HR13,Tim14.

Historical remarks.

The present concepts of equivalence and reduction are much stronger than those usually used in potential theory where, for instance elliptic BVPs are said to be equivalent to certain boundary integral equations if, roughly speaking, there are mappings which generate solutions from each other, see HW08 and MS89, for instance.

Various other equivalence relations appear in the literature, e.g., defined by the fact that two operators are simultaneously Fredholm or not BDS95.

Those properties can be regarded as consequences of the MC and EAE relations and were denoted as ”transfer properties” in S13,S18.

The MC and EAE relations were introduced by Harm Bart, Israel Gohberg, and Rien Kaashoek in the early 1980s BGK84 where the conclusion from MC to EAE was of great importance in system theory, see also S17a.

The proof of the inverse conclusion from EAE to MC was obtained by Harm Bart and V.E. Tsekanovskii BT91. It gave a great impact to the study of operator relations in general, see HMR15,HR13 S13,Tim14.

Its usefulness in the study of various classes of operators was demonstrated already by Luís Castro and the author in 1998 Cas98 CS98,CS00 and recently also for applications in diffraction theory, together with Roland Duduchava CDS14,S13a,S17.

9 Sommerfeld problems in higher dimensions

In this final section we describe briefly an idea of how to study the Sommerfeld problems of Section 3 in higher dimensions, i.e., where Ω is three-dimensional or even n-dimensional which mathematically does not make a great difference to some extend.

The FIS concept (see Theorem 7.5) improves to be very efficient, because the intermediate space turns out to be an anisotropic Sobolev space and the reasoning of an operator factorisation with bounded instead of unbounded operators applies as before in two dimensions.

Instead of a half-line (as in Section 3) consider the half-plane $\Sigma = \mathbb{R}^2_{1+} = \{x \in \mathbb{R}^2 : x_1 > 0\}$, the (orthogonal) projectors $P_+ = \ell_0 r_+ : L^2(\mathbb{R}^2) \to L^2(\mathbb{R}^2)$ onto $L^2_\Sigma = L^2_{\mathbb{R}^2_{1+}}$, $P_- = I - P_+$ and the Bessel potential operators DS92,DS93,Esk73 of order $s \in \mathbb{R}$:

$$\Lambda^s_+ = A_{\lambda^s_+} = \mathcal{F}^{-1}\,\lambda^s_+ \cdot \mathcal{F} \quad , \quad \lambda^s_+(\xi) = \left(\xi_1 + i\sqrt{\xi_2^2+1}\right)^s, \xi \in \mathbb{R}^2$$

$$\Lambda^s_- = A_{\lambda^s_-} = \mathcal{F}^{-1}\,\lambda^s_- \cdot \mathcal{F} \quad , \quad \lambda^s_-(\xi) = \left(\xi_1 - i\sqrt{\xi_2^2+1}\right)^s, \xi \in \mathbb{R}^2$$

where $\mathcal{F}$ now denotes the two-dimensional Fourier transformation. For any $s \in \mathbb{R}$ we find the orthogonal projectors CS00

$$\begin{aligned} P^{(s)}_+ &= \Lambda^{-s}_+ P_+ \Lambda^s_+ \quad \text{onto} \quad H^s_\Sigma \\ P^{(s)}_- &= \Lambda^{-s}_- P_- \Lambda^s_- \quad \text{onto} \quad H^s_{\Sigma'} \\ \Pi^{(s)}_+ &= \Lambda^{-s}_- P_+ \Lambda^s_- \quad \text{along} \quad H^s_{\Sigma'} \\ \Pi^{(s)}_- &= \Lambda^{-s}_+ P_- \Lambda^s_+ \quad \text{along} \quad H^s_\Sigma. \end{aligned}$$

Hence $P^{(s)}_+ + \Pi^{(s)}_- = I_{H^s}$ and $P^{(s)}_- + \Pi^{(s)}_+ = I_{H^s}$.

As mentioned in Remark 6.2 one can work with $\sqrt{\xi_2^2 - k^2}$ instead of $\sqrt{\xi_2^2 + 1}$. In that case, the corresponding projectors are not anymore orthogonal in the usual sense.

Now, let us go to the higher-dimensional case ($m = n - 1 \geq 2$) where Σ is a half-space, cf. CDS03,Esk73,GolGoh60,Sha67,S85.

According to the traditional notation Esk73 x_n denotes now the "normal derivative direction" (corresponding to x_2 in Section 3) and $x_m = x_{n-1}$ denotes the "WH direction" (corresponding to x_1 in Section 3) whilst the remaining variables $x_1, ..., x_{m-1}$ play the role of parameters.

Putting

$$X = Y = H^{1/2}(\mathbb{R}^m) \times H^{-1/2}(\mathbb{R}^m) \ , \ \Sigma = \mathbb{R}^m_+ = \mathbb{R}^{m-1} \times]0, \infty[$$

and $t(\xi) = (\xi_1^2 + ... + \xi_m^2 - k^2)^{1/2}$, $\xi = (\xi', \xi_m) \in \mathbb{R}^m$, we can consider the same factorisation given by (47) replacing k by $(k^2 - \xi'^2)^{1/2}$, i.e., the previous factorisation as to be parameter-dependent of $\xi' \in \mathbb{R}^{m-1}$.

It turns out that the factorisation (47) can be seen as a canonical FIS of A with an intermediate space which is an anisotropic vector Sobolev space

$$\begin{aligned} Z &= H^{\vartheta}(\mathbb{R}^m) \times H^{-\vartheta}(\mathbb{R}^m)\,, \qquad (106) \\ H^{\vartheta}(\mathbb{R}^m) &= \mathcal{F}(w_\vartheta L^2(\mathbb{R}^m)), \\ w_\vartheta(\xi) &= (1+|\xi'|^2)^{\vartheta_1/2})\,(1+\xi_m^2)^{\vartheta_2/2}), \\ \vartheta &= (\vartheta_1, \vartheta_2) = (\frac{1}{2}(\delta-1), \frac{1}{2}(1-\delta))\,, \end{aligned}$$

see S17 for more details.

Adding in the factorisation $\sigma_\lambda = \sigma_{\lambda-}\,\sigma_{\lambda+}$ (cf. (47)) a middle factor of the form $\operatorname{diag}(\zeta^{\kappa_1}, \zeta^{\kappa_2})$, where $\zeta(\xi) = (\xi_m - i\,|\xi'|)/(\xi_m + i\,|\xi'|)$, we find further examples with non-canonical FIS which are not Fredholm according to a non-trivial kernel (or co-kernel) that is translation invariant with respect to $x' = (x_1, ..., x_{m-1}) \in \mathbb{R}^{m-1}$, hence infinite dimensional. However, generalised inverses can be constructed as before according to the FIS interpretation.

One can find plenty of further boundary value and transmission problems in higher dimensions ($n \geq 3$) where the associated operators are not anymore Fredholm but generalised invertible, just adding one or more variables, see CST06,LMoS97,MoS97,S17, for instance.

Many problems in mathematical physics, for instance in elasticity theory, make sense only in three dimensions. They typically lead us to Wiener-Hopf equations, anisotropic function spaces and normalisation problems, see DW95,HW08,MS89e, e.g.

Conclusion

The operator theoretical formulation allows

- a clear and compact description of results about the solution of linear BVPs,
- a simultaneous treatise of large classes of problems,
- and the solution of new classes of BVPs, as well.

For instance, the described ideas are applicable to more complicated geometrical situations, particularly the conception of equivalent reduction via operator relations, see further work on diffraction by wedges or polygonal-conical screens etc. such as CDS03,CDS14 and CST06.

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