Factorisation of matrix functions:
New techniques and applications
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Wiener-Hopf factorisation through an intermediate space and applications to diffraction theory

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General Wiener-Hopf operators [DevShi69, S83, S85]

Overall assumptions:

$$X, Y \text{ are Banach spaces,}$$
$$P_1 \in \mathcal{L}(X)\,,\ P_2 \in \mathcal{L}(Y) \text{ are projectors,}$$
$$A \in \mathcal{L}(X, Y) \text{ is a bounded linear operator.}$$

Then the operator

$$W \quad = \quad P_2 A|_{P_1 X} \quad = \quad P_1 X \to P_2 Y \tag{1}$$

is referred to as a general Wiener-Hopf operator (WHO) in asymmetric space setting in contrast to the case where $X = Y$ and $P_1 = P_2$:

$$W \quad = \quad T_P(A) \quad = \quad PA|_{PX} \quad = \quad PX \to PX\,. \tag{2}$$

We shall also call W truncation of the underlying operator A.

Generalized inverses

Given $T \in \mathcal{L}(X,Y)$ an operator $T^- \in \mathcal{L}(Y,X)$ is said to be a generalized inverse of T if

$$T\ T^-\ T \quad = \quad T\,.$$

It is called a reflexive generalized inverse if $T^-\,T\,T^- = T^-$ holds additionally. In both cases T is said to be generalized invertible (since the existence of a generalized inverse implies the existence of a reflexive generalized inverse, replacing T^- by T^-TT^-).

It is well-known that T is generalized invertible if and only if its kernel and image are complemented subspaces of X and Y, respectively (which includes the cases of T to be Fredholm, one-sided invertible and others).

The knowledge of a generalized inverse yields solubility conditions and an explicit representation of the general solution of the operator equation $Tf = g$.

Cross factorization [S83, S85]

Let A be boundedly invertible. An operator triple A_-, C, A_+ with

$$\begin{aligned} A \quad &= \quad A_- \quad C \quad A_+ \\ &: \quad Y \leftarrow Y \leftarrow X \leftarrow X\,. \end{aligned} \tag{3}$$

is referred to as a cross factorization of A (with respect to X, Y, P_1, P_2), in brief CFn, if the factors $A_\pm$ are strong WH factors, i.e.,

$$\begin{aligned} A_+ \in \mathcal{GL}(X) \quad &, \quad A_- \in \mathcal{GL}(Y)\,, \\ A_+P_1X = P_1X \quad &, \quad A_-Q_2Y = Q_2Y\,, \end{aligned} \tag{4}$$

and the cross factor $C \in \mathcal{L}(X, Y)$ is boundedly invertible such that

$$C^{-1}P_2CP_1 \ \textit{is a projector (idempotent) in } \mathcal{L}(X). \tag{5}$$

The name cross factor (briefly CF) comes from the fact that it appears in operator factorizations and has following (equivalent) property: it splits the spaces X, Y both into four subspaces such that

$$
\begin{array}{ccccccc}
X & = & \overbrace{X_1 \dot{+} X_0}^{P_1 X} & \dot{+} & \overbrace{X_2 \dot{+} X_3}^{Q_1 X} & & \\
 & & \downarrow & C \swarrow\!\!\!\!\searrow & \downarrow & & (6) \\
Y & = & \underbrace{Y_1 \dot{+} Y_2}_{P_2 X} & \dot{+} & \underbrace{Y_0 \dot{+} Y_3}_{Q_2 X} . & &
\end{array}
$$

This means that C maps each X_j onto Y_j , j = 0, 1, 2, 3 , i.e., the complemented subspaces $X_0, X_1, ..., Y_3$ are images of corresponding projectors $p_0, p_1, ..., q_3$, namely

$$
\begin{array}{llll}
X_1 = p_1 X = C^{-1} P_2 C P_1 X & , & X_0 = p_0 X = C^{-1} Q_2 C P_1 X , \\
X_2 = p_2 X = C^{-1} P_2 C Q_1 X & , & X_3 = p_3 X = C^{-1} Q_2 C Q_1 X , \\
Y_1 = q_1 Y = C P_1 C^{-1} P_2 Y & , & Y_2 = q_2 Y = C Q_1 C^{-1} P_2 Y , \\
Y_2 = q_1 Y = C P_1 C^{-1} Q_2 Y & , & Y_3 = q_3 Y = C Q_1 C^{-1} Q_2 Y .
\end{array}
$$

The cross factorization theorem [S83, S85]

Let A be boundedly invertible. Then W is generalized invertible if and only if a cross factorization of A exists and, in this case, a formula for a reflexive generalized inverse of W is given by

$$W^- = A_+^{-1} P_1 C^{-1} P_2 A_-^{-1}|_{P_2 Y} \quad : \quad P_2 Y \to P_1 X\,. \tag{7}$$

Sufficiency is proved by inspection. Necessity is more complicated.

It yields necessary and sufficient conditions for the solution of the equation $Wf = g$ and an explicit formula for the general solution, as well.

A most important fact is the equivalence of W and $P_2 C|_{P_1 X}$, in brief $W \sim P_2 C|_{P_1 X}$, namely

$$W \quad = \quad P_2 A_-|_{P_2 Y} \; P_2 C|_{P_1 X} \; P_1 A_+|_{P_1 X} \quad = \quad E \; P_2 C|_{P_1 X} \; F \tag{8}$$

where E, F are isomorphisms (boundedly invertible linear operators).

WH factorization through an intermediate space

Now we study another type of factorization, which is quite different from the previous and more interesting for many applications.

$$\begin{aligned} A &= \quad A_- \quad C \quad A_+ \\ &: \quad Y \leftarrow Z \leftarrow Z \leftarrow X \,. \end{aligned} \tag{9}$$

is referred to as a WH factorization through an intermediate space Z (with respect to X, Y, P_1, P_2) (in brief FIS), if the factors $A_\pm$ and C are linear and boundedly invertible in the above setting with an additional Banach space Z called intermediate space [CS95] and if there is a projector $P \in \mathcal{L}(Z)$ such that

$$A_+ P_1 X = PZ \quad , \quad A_- QZ = Q_2 Y \tag{10}$$

with $Q = I_Z - P$ and such that $C \in \mathcal{GL}(Z)$ and

$$C^{-1}PCP \text{ is a projector (idempotent).} \tag{11}$$

I.e., C splits the space Z twice into four subspaces:

$$
\begin{array}{ccccccc}
Z & = & \overbrace{X_1 \dot{+} X_0}^{PZ} & \dot{+} & \overbrace{X_2 \dot{+} X_3}^{QZ} \\
 & & \downarrow & C \nearrow\!\!\!\!\swarrow\!\!\searrow & \downarrow \\
 & = & \underbrace{Y_1 \dot{+} Y_2}_{PZ} & \dot{+} & \underbrace{Y_0 \dot{+} Y_3}_{QZ}
\end{array}
$$

where C maps each X_j onto Y_j.

Again $A_\pm$ are called strong WH factors and C is said to be a cross factor, now acting from a space Z into the same space Z.

By analogy to the cross factorization theorem, the following conclusion is straightforward, as well: A FIS of A implies a reflexive generalized inverse of W by putting

$$
W^- = A_+^{-1} P C^{-1} P A_-^{-1}|_{P_2 Y} \quad : \quad P_2 Y \to P_1 X \,. \tag{12}
$$

The inverse conclusion is not true in general, as we shall see later.

Unbounded FIS [S15]

Let $A \in \mathcal{L}(X, Y)$ be boundedly invertible.

A factorization $A = A_- C A_+$ is said to be an unbounded WH factorization through an intermediate (Banach) space Z (unbounded FIS), if the factors $A_\pm^{\pm 1}$ are densely defined injective linear operators in the above-mentioned spaces, P and $C \in \mathcal{L}(Z)$ have the same properties as before, and the operator

$$T = A_+^{-1} P C^{-1} P A_-^{-1} \quad : \quad Y \to X \tag{13}$$

is also densely defined and admits a bounded extension to the full space, in brief $T \in \mathcal{L}(Y, X)$.
I.e., the factorization holds with a cross factor $C \in \mathcal{GL}(Z)$ and with $A_\pm^{\pm 1}$ being densely defined injective linear operators with the above factor properties and with continuous extension in the sense of a FIS.

By analogy, an unbounded cross factorization may be defined.

If $C = I$, the previous factorizations are called canonical.

Full range factorization [S15]

Let $T \in \mathcal{L}(X,Y)$ and

$$T \quad = \quad L \; R \tag{14}$$

where $R \in \mathcal{L}(X,Z), L \in \mathcal{L}(Z,Y)$, X,Y,Z are Banach spaces, R is right invertible and L is left invertible. Then (13) is said to be a full range factorization (FRF) of T.

This notion is well-known from matrix theory as full rank factorization ($\dim Z =$ rank T). Evidently a FRF implies that

$$T^- \quad = \quad R^- \; L^-$$

is a reflexive generalized inverse of T provided $RR^- = I_Z = L^- L$. The intermediate space Z is isomorphic to the image (or range) of T and to any complement of the kernel of T, as well.

Remark 1 In contrast to (14) a factorisation $T = R\,L$ does not produce immediate results.

Remark 2

Let T, S be bounded linear operators in Banach spaces and

$$T \quad = \quad L \; S \; R$$

where $L^{-}L = I$, $RR^{-} = I$ in the corresponding Banach spaces.

If S is generalised invertible, say $SS^{-}S = S$, then T is generalised invertible, as well, and a generalised inverse of T is given by the reverse order law

$$T^{-} \quad = \quad R^{-} \; S^{-} \; L^{-} \, .$$

This relation between T and S is not reflexive: The inverse conclusion is not valid in general.

Basic results of [S15]

Let $A \in \mathcal{L}(X,Y)$ be boundedly invertible, $W = P_2 A|_{P_1 X}$ as before.

Theorem 1. *W is invertible if and only if A admits a canonical FIS:*

$$\begin{aligned} A &= A_- \quad A_+ \\ &: \quad Y \leftarrow Z \leftarrow X\,. \end{aligned} \tag{15}$$

Theorem 2. *The following assertions are equivalent:*
(i) *W is generalized invertible and $P_1 \sim P_2$ holds,*
(ii) *A admits a FIS.*

Herein the condition $P_1 \sim P_2$ is not redundant.

Theorem 3. *Every canonical unbounded FIS can be regarded as a canonical bounded FIS, by a change of the intermediate space.*

Corollary

The following statements are equivalent:

(j) *A admits a CFn (with respect to X, Y, P_1, P_2),*

(jj) *W admits a full range factorization,*

and moreover, if $P_1 \sim P_2$ holds,

(jjj) *A admits a FIS (with respect to X, Y, P_1, P_2).*

In all cases W is generalized invertible, a generalized inverse of W is given in terms af the factorization and a factorization of one type can be computed from any other (via W^-).

Remark

In 2015 the question appeared if $P_1 \sim P_2$ is also necessary for A to admit a FIS which was finally proved in [BoeS16] (filling a gap in the proof of Theorem 2 and correcting also a misprint in Corollary 2.9 of [S15]).

Example 1: Generalized or Φ-factorization

Let $\Gamma \in \mathbb{C}$ be a closed contour which divides $\mathbb{C} \cup \infty$ into two domains D_+ and D_- such that $0 \in D_+, \infty \in D_-$ and $\partial D_+ = \partial D_- = \Gamma$. $L^p_\pm \subset L^p(\Gamma)$ $(p > 0)$ are the spaces of functions which are boundary values of functions holomorphic in $D_\pm$ in the sense of Privalov (see [LitSpi87] for details). For simplicity we consider the unit circle $\Gamma = \Pi_0 = \{z \in \mathbb{C} : |z| = 1\}$.

A (right, generalized) factorization of $G \in L^\infty(\Gamma)^{n\times n}$ in $L^p, 1 < p < \infty$, relative to Γ is a representation in the form

$$G(z) \quad = \quad G_-(z)\ \Lambda(z)\ G_+(z) \quad , \quad z \in \Gamma \tag{16}$$

where $G_- \in L^p_-, G_+ \in L^q_+, G_-^{-1} \in L^q_-, G_+^{-1} \in L^p_+, q = p/(p-1)$ and the matrix function Λ has the form

$$\Lambda(z) \quad = \quad \operatorname{diag}(z^{\kappa_1}, ..., z^{\kappa_n}) \quad , \quad z \in \Gamma \tag{17}$$

where $z^{\kappa_1} \geq ... \geq z^{\kappa_n}$ are integers. Λ can be seen as a cross facor in the following context:

Consider the Toeplitz operator

$$T \quad = \quad P\,G\cdot|_{(L^p_+)^n} \tag{18}$$

in the space of vector functions $X = (L^p_+)^n, p \in (1,\infty)$, where $A = G\cdot$ denotes the multiplication operator and P the Riesz projection. T can be seen as an example for a general Wiener-Hopf operator W in symmetric setting $X = Y$, $P_1 = P_2 = P$.

Let us assume that $G \in \mathcal{G}L^\infty(\Gamma)^{n\times n}$ admits a (right, generalized) factorization in L^p [Sim68] for some $p \in (1,\infty)$. Then T is normally solvable (and moreover Fredholm) if and only if

$$K \;=\; G_+^{-1}\cdot\Lambda\,Q\,G_-^{-1} \quad \texttt{and} \quad K_1 \;=\; G_-\cdot P\,G_-^{-1} \tag{19}$$

are bounded in $(L^p)^n$. Then (16) is said to be a Φ-factorization of G [LitSpi87].

The Φ-factorization can be interpreted in the sense of an unbounded FIS (still with $X = Y$) and the formulas for a generalized inverse in terms of the factorization are applicable. We have $Z = \operatorname{im} A_+ = \operatorname{im} A_-^{-1}$ with the induced norm and $P, C \in \mathcal{L}(Z)$. This was pointed out already in [CasS95] where the nature of those spaces was studied.

Ex 2: WHOs in diffraction from plane screens

The following species appears particularly in problems of diffraction from plane screens ($n = 2$ or $n = 3$) such as the Sommerfeld diffraction problem [MS89], its various generalizations, see [CDS14] for instance, and other elliptic boundary value problems [Esk81, HW08, Wlo87], in the scalar case (pure Dirichlet, Neumann, or impedance conditions):

$$W_{\Phi,\Sigma} \quad = \quad r_\Sigma \mathcal{F}^{-1}\Phi \cdot \mathcal{F}|_{H^r_\Sigma} \quad : \quad H^r_\Sigma \to H^s(\Sigma) \tag{20}$$

where $\Sigma \subset \mathbb{R}^{n-1}$ is an open set (subset of a hyperplane in $\mathbb{R}^n$), $\mathcal{F}$ the $n-1$-dimensional Fourier transformation, $A = \mathcal{F}^{-1}\Phi \cdot \mathcal{F}$ a translation invariant operator, elliptic of order $r - s$, i.e., $\lambda^{s-r}\Phi \in \mathcal{G}L^\infty$ where $\lambda(\xi) = (|\xi|^2 + 1)^{1/2}, \xi \in \mathbb{R}^{n-1}, r, s \in \mathbb{R}$ for $n = 2$ (and $r, s \in \mathbb{R}^2$ for $n = 3$), and $H^r_\Sigma, H^s(\Sigma)$ Sobolev spaces of distributions supported on $\overline{\Sigma}$ or restricted to Σ, respectively, see [Esk81] for instance.

The operator (20) is not directly of the form of a general WHO (1) but equivalent, provided $\operatorname{int}\operatorname{clos}\Sigma = \Sigma$ holds and Σ has the strong extension property [HW08], i.e., there exists (for any $s \in \mathbb{R}$) an extension operator $E^s_\Sigma \in \mathcal{L}(H^s(\Sigma), H^s(\mathbb{R}^{n-1}))$ which is left invertible by restriction: $r_\Sigma E^s_\Sigma = I_{H^s(\Sigma)}$. Then we have

$$W_{\Phi,\Sigma} \quad = \quad r_\Sigma W \quad \sim \quad W \quad = \quad P_2 A|_{P_1 X} \tag{21}$$

where $X = H^r, Y = H^s, P_1$ is a projector in H^r onto H^r_Σ and P_2 is a projector in H^s along $\Sigma' = \mathbb{R}^{n-1} \setminus \overline{\Sigma}$, for instance $P_2 = E^s_\Sigma r_\Sigma$.

Concrete examples are known from diffraction theory where also matrix operators with entries like (19) are relevant [CDS14,S14], considered later in Example 6.

Ex 3: Classical WHOs on a half-line

In a simple subclass of (20) we find $n-1=1$ and $r=s=0$, i.e., $X=Y=L^2(\mathbb{R}), \Sigma=\mathbb{R}_+=]0,\infty[$ and $\Phi\in\mathcal{G}C^\nu(\dot{\mathbb{R}})$ where $\dot{\mathbb{R}}$ denotes the one-point compactification of $\mathbb{R}$. A Wiener-Hopf factorisation (in the decomposing algebra $C^\nu(\dot{\mathbb{R}})$) is given by the well-known formulas

$$\begin{aligned} A &= A_- \, C \, A_+ = \mathcal{F}^{-1}\Phi_-\cdot\mathcal{F} \quad \mathcal{F}^{-1}\ \zeta^\kappa\cdot\mathcal{F} \quad \mathcal{F}^{-1}\Phi_+\cdot\mathcal{F} \\ \kappa &= \operatorname{ind}\Phi = \frac{1}{2\pi}\int_{\mathbb{R}} d\arg\Phi \qquad (22) \\ \Phi_\pm &= \exp\{P^\pm\log(\zeta^{-\kappa}\Phi)\} \end{aligned}$$

where $\kappa\in\mathbb{Z}$ is the winding number of Φ , $\zeta(\xi)=\frac{\xi-i}{\xi+i}$, $\xi\in\mathbb{R}$, $P^\pm=\frac{1}{2}(I\pm H)$ and H is the Hilbert transformation.

The factorisation of A in (22) (and of Φ, as well) can be considered as a special case of a CFn (in symmetric space setting $X=Y$) or a FIS (through $Z=X=Y$). The corresponding WHO is always one-sided invertible which in this class is equivalent to be Fredholm, generalised invertible or normally solvable, as the Fourier symbol does not vanish on $\dot{\mathbb{R}}$, cf. Coburn's Theorem for Toeplitz operators [BoeSil06].

Ex 4: Just a jump at infinity

In applications to diffraction theory we hardly find the preceding class of non-rational Fourier symbols but more likely $\Phi \in \mathcal{G}C^{\nu}(\ddot{\mathbb{R}})$ with a "jump at infinity", i.e., being Hölder continuous functions with respect to the two-point compactification of $\mathbb{R}$ which is important for asymptotic results. This is because Φ belongs to an algebra $\mathcal{A}$ generated by rational functions and a square-root function $\zeta_k^{1/2}(\xi) = \sqrt{\frac{\xi - k}{\xi + k}}$ (originating from the Helmholtz symbol and the lifting process where the wave number k is assumed to have a positive imaginary part). $\mathcal{A}$ is not a decomposing algebra nor an $\mathcal{R}$-algebra [GohKru79,MP86] and therefore Φ allows only a generalised factorisation provided an additional condition at infinity is satisfied, namely

$$\frac{1}{2\pi} \int_{\mathbb{R}} d \arg \Phi + \frac{1}{2} \quad \notin \quad \mathbb{Z} \,. \tag{23}$$

In this case Φ can be written as

$$\Phi \;=\; \zeta_k^\omega \, \Psi \quad , \quad \omega \;=\; \frac{1}{2\pi i} \int_{\mathbb{R}} d \log \Phi \tag{24}$$

with $\Psi \in \mathcal{G}C^\nu(\dot{\mathbb{R}})$ and $\operatorname{ind}\Psi = 0$. The factor ζ_k^ω can be replaced by ζ^ω (where $k = i$) which has the same behavior at infinity, however can be useful in applications concerning the Helmholtz equation. We write $\zeta_k^\omega = (\frac{\lambda_-}{\lambda_+})^\omega$ where, for $k, \omega \in \mathbb{C}$ and $\Im m k > 0$,

$$\lambda_\pm^\omega(\xi) \;=\; (\xi \pm k)^\omega \;=\; \exp\{\omega \log(\xi \pm k)\} \;\; , \;\; \xi \in \mathbb{C} \setminus \Gamma_k \, , \tag{25}$$

with vertical branch cuts $\Gamma_{\mp k}$ $(\Gamma_k = \Gamma_{+k} \cup \Gamma_{-k})$ taken from $\mp k$ to infinity not crossing the real line. Further put

$$\omega \;=\; \sigma + i\tau \;=\; \kappa + \varepsilon + i\tau \tag{26}$$

with $\sigma, \kappa, \varepsilon, \tau \in \mathbb{R}$, $\kappa \in \mathbb{Z}$, $\varepsilon \in]-1/2, +1/2[$ provided (23) holds [Dud79,MoST98]. Then we find the factorisation

$$\Phi \quad = \quad \Phi_- \, \zeta^\kappa \, \Phi_+ \quad = \quad (\Psi_- \lambda_-^{\varepsilon + i\tau}) \, \zeta^\kappa \, (\lambda_+^{\varepsilon + i\tau} \Psi_+) \tag{27}$$

where $\Psi_- \, \zeta^\kappa \, \Psi_+$ is a factorisation (22) of Ψ like Φ in the previous example.

It is possible to show that this is a a generalised factorisation in $L^2(\mathbb{R}, \lambda)$ in the sense of Simonenko. But is also not hard to verify that it represents a FIS through the weighted space $L^\infty(\mathbb{R}, \lambda^{-\varepsilon})$ which yields that the corresponding factorisation of A represents a FIS through the Sobolev space $H^{-\varepsilon}(\mathbb{R})$). This fact was efficiently used in [MoST98] for developing a normalisation method of such operators in the exceptional case of $\varepsilon = 1/2$.

Matrix functions with this kind of elements can lead to factors with logarithmic behavior at infinity and important applications in asymptotic analysis. Further applications appear when tackling WHOs in a half-space $\mathbb{R}^n_+$, see Example 6 later.

Ex 5: A glance at oscillating symbols

Consider $X = Y = L^p(\mathbb{R})^2$, $p \in [1, \infty]$, $P = \ell_0 r_+ \cdot$ as acting in X, i.e., $Pf = (\ell_0 r_+ f_1, \ell_0 r_+ f_2)^T$ for $f = (f_1, f_2)^T \in X$. Further let

$$\begin{aligned} A &= A_\Phi = \mathcal{F}^{-1}\,\Phi \cdot \mathcal{F} \\ \Phi &= \begin{pmatrix} \tau & 1 \\ 0 & \tau^{-1} \end{pmatrix} \end{aligned} \tag{28}$$

with $\tau(\xi) = e^{i\xi}$, $\xi \in \mathbb{R}$, where the right/left shift operators appear:

$$A_{\tau^{\pm 1}} = \mathcal{F}^{-1}\,\tau^{\pm 1} \cdot \mathcal{F} \quad , \quad A_{\tau^{\pm 1}} f_j(x) = f_j(x \mp 1) \quad , \quad x \in \mathbb{R}.$$

The factorisation $A = A_- A_+$ with $A_\pm = \mathcal{F}^{-1}\,\Phi_\pm \cdot \mathcal{F}$,

$$\Phi = \Phi_- \Phi_+ = \begin{pmatrix} 1 & 0 \\ \tau^{-1} & -1 \end{pmatrix} \begin{pmatrix} \tau & 1 \\ 1 & 0 \end{pmatrix} \tag{29}$$

represents a canonical cross factorisation and a canonical FIS, as well, with $Z = X$.

Remark If there is a function ϕ instead of 1 in (28), the factorisation problem is rather complicated in general, see Ilya Spitkovky's talk.

Considering another example, replacing (28) by (cf. [LitSpi87], p.57)

$$\Phi = \begin{pmatrix} \tau & 0 \\ 1 & \tau^{-1} \end{pmatrix} \tag{30}$$

we obtain, instead of (29) the factorisation

$$\Phi = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \tau & 0 \\ 0 & \tau^{-1} \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \tau & 1 \end{pmatrix} \tag{31}$$

which yields a cross factorisation (being also a FIS) $A = A_- C A_+$ with $A_- = I$ and a non-trival cross factor C generating in the diagram (6) the spaces

$$\begin{aligned} X_0 &= \{0\} \times L^p_{[0,1]}(\mathbb{R}) \\ Y_2 &= L^p_{[0,1]}(\mathbb{R}) \times \{0\} \end{aligned}$$

which are isomorphic to the infinite-dimensional defect spaces of W. Hence W is generalised invertible but neither Fredholm nor semi-Fredholm. A generalised inverse of W is simply given by formula (7).

In composition with Example 4 one obtains further examples:

Exercise. Determine suitable factorisations of A_Φ for

$$\Phi \quad = \quad \zeta^\omega \begin{pmatrix} \tau & 0 \\ 1 & \tau^{-1} \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} \tau & 0 \\ \phi & \tau^{-1} \end{pmatrix} \ , \ \phi \in C^\nu(\ddot{\mathbb{R}})$$

combining the ideas of the foregoing examples.

The first is a simple case where the factorisation is unbounded (if $\omega \notin \mathbb{Z}$), has a nice interpretation as FIS, W is not Fredholm but generalised invertible (with infinite dimensional defect spaces).

For the second just use the idea of full range factorisation.

OPEN FACTORISATION PROBLEM [CasS00,CasS05]
Matrices related to diffraction from a strip (2 D), $n = 1$ or $n = 2$:

$$\Phi \quad = \quad \zeta^r \begin{pmatrix} \tau_{-a} I_n & 0 \\ t_-^{s-r}\,\phi & \tau_a \zeta^{s-r} I_n \end{pmatrix}$$

with $\tau_a(\xi) = e^{ai\xi}, a > 0$. Especially $n = 1\,,\ r = \pm 1/2 = -s\,,\ \phi = t^{\pm 1}$.

Proof of Theorem 1

Sufficiency. As in many other cases, this step is just a verification of the formula for the inverse. If we have a canonical FIS, then

$$\begin{aligned} WW^{-} &= P_2A_-A_+|_{P_1X}\, A_+^{-1}PA_-^{-1}|_{P_2Y} = P_2A_-A_+P_1\, A_+^{-1}PA_-^{-1}|_{P_2Y} \\ &= P_2A_-\, PA_-^{-1}|_{P_2Y} = I|_{P_2Y}\,. \end{aligned} \tag{32}$$

Similarly we see that $W^-W = I|_{P_1X}$.

Necessity. Let W be invertible. We identify A with an equivalent operator matrix and factor this straightforwardly

$$\begin{aligned} A &\sim \tilde{A} = \begin{pmatrix} P_2A|_{P_1X} & P_2A|_{Q_1X} \\ Q_2A|_{P_1X} & Q_2A|_{Q_1X} \end{pmatrix} : P_1X \times Q_1X \to P_2Y \times Q_2Y \\ &= \begin{pmatrix} I|_{P_2X} & 0 \\ Q_1A^{-1}|_{P_2X} & Q_1A^{-1}|_{Q_2X} \end{pmatrix}^{-1} \begin{pmatrix} P_2A|_{P_1X} & P_2A|_{Q_1X} \\ 0 & I|_{Q_1X} \end{pmatrix} \\ &= \tilde{A}_-\, \tilde{A}_+ : P_1X \times Q_1X \to P_2Y \times Q_1X \to P_2Y \times Q_2Y. \end{aligned} \tag{33}$$

With the above-mentioned identification of the direct sum $P_1X\dot{+}Q_1X$ and the product space $P_1X \times Q_1X$ (in the algebraic and topological sense) we obtain a factorisation of $A = A_- A_+$ through $Z = P_2Y \times Q_1X$ because the invertibility of W implies that (dropping the tildes)

$$A_+ \quad = \quad \begin{pmatrix} I_{P_2Y} & P_2A|_{Q_1X} \\ 0 & I_{Q_1X} \end{pmatrix} \begin{pmatrix} I_{P_2Y}A|_{P_1X} & 0 \\ 0 & I_{Q_1X} \end{pmatrix} \tag{34}$$

is invertible. The calculation

$$\begin{aligned} A_-^{-1} \quad &= \quad A_+ \begin{pmatrix} P_1A^{-1}|_{P_2X} & P_1A^{-1}|_{Q_2X} \\ Q_1A^{-1}|_{P_2X} & Q_1A^{-1}|_{Q_2X} \end{pmatrix} \\ &= \begin{pmatrix} I|_{P_2X} & 0 \\ Q_1A^{-1}|_{P_2X} & Q_1A^{-1}|_{Q_2X} \end{pmatrix} \end{aligned}$$

shows that $A = A_-A_+$ where A_- is invertible, as well. Finally, the factor properties of $A_\pm$ are obvious from the foregoing formulas. □

This direct proof, say, has an alternative contained in the following proof, by reduction to a symmetric WHO.

Proof of Theorem 2

Sufficiency. If a FIS is given, we define

$$W^{-} = A_{+}^{-1} P C^{-1} P A_{-}^{-1}|_{P_2 Y} \quad : \quad P_2 Y \to P_1 X \,. \tag{35}$$

Now we verify $WW^{-}W = W$ by calculations similar to the previous and with the help of diagram (4) and by analogy to the calculations in case of a CFn, see [S85], p. 27-29.

The factor properties of A_{+} imply $P_1 \sim P$ and the factor properties of A_{-} imply $P_2 \sim P$, therefore $P_1 \sim P_2$ is necessarily satisfied.

Necessity. Since $P_1 \sim P_2$, we can confine ourselves to the symmetric case where $P_2 = P_1$ by splitting an isomorphism from A which maps P_2Y onto P_1X and Q_2Y onto Q_1X. Note that two bounded projectors in Banach spaces are equivalent if and only if their kernels are isomorphic and their co-kernels are isomorphic, as well [BT91]. Hence consider an operator of the form $W = P_1A|_{P_1X}$. This can be considered as an element of the form $w = pap$ in the unital ring $\mathcal{R} = \mathcal{L}(X)$ where p is idempotent and a invertible.

From [S85] we know that the regularity of w [Neu36], i.e., existence of an element $v \in \mathcal{R}$ with $pvp = v$ and $wvw = w$ implies a ring cross factorisation given by, e.g.,

$$\begin{aligned} a &= a_- \, c \, a_+ \\ &= [e + qav][a - ava + w + a(p - vw)a^{-1}(p - wv)a] \\ &\qquad \cdot [e + vaq - (p - vw)a^{-1}(p - wv)a] \,, \end{aligned} \tag{36}$$

see formula (6.7a) in [S85]. This can be interpreted as a cross factorisation of A which coincides with a FIS through the intermediate space $Z = X$ in this symmetric space setting ($X = Y$).

Non-redundance. We give an example where W is generalised invertible, but A does not admit a FIS in a case where $P_1 \sim P_2$ is violated. Let $X = Y = \mathbb{R}^3, P_1 x = (x_1, 0, 0), P_2 = (x_1, x_2, 0)$ for $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. If $A \in \mathcal{L}(\mathbb{R}^3)$ is any invertible operator (real linear transformation), W has finite rank (namely rank 0 or 1) and it is therefore generalised invertible, but obviously $P_1 \sim P_2$ is not satisfied, because their kernels are not isomorphic. □

Proof of Theorem 3

Starting with a canonical unbounded FIS $A = A_- A_+$ we put $Z_1 = \operatorname{im} A_+|_{\operatorname{dom} A_+}$ equipped with the norm induced by X:

$$\|z\|_{Z_1} = \|A_+^{-1} z\|_X \tag{37}$$

and define Z by taking the closure of Z_1 which is obviously isomorphic to the Banach space X. Now $Z_2 = \operatorname{im} A_-^{-1}|_{\operatorname{dom} A_-^{-1}}$ yields the same result since $X \cong Y$ because of the assumption that A is invertible. Namely:

$$\|z\|_{Z_1} = \|A_+^{-1} z\|_X \sim \|A A_+^{-1} z\|_Y = \|A_- z\|_Y \tag{38}$$

in the sense of equivalent norms. □

Proof of the Corollary

The only missing step is the conclusion that the generalised invertibility of T yields a full range factorisation of T. Hence, let T^- be a generalised inverse of $T \in \mathcal{L}(X,Y)$ and

$$\operatorname{Rst} T \quad : \quad X \ \to \ \operatorname{im} T \tag{39}$$

the image restricted operator, considered as an operator acting not into Y but onto $\operatorname{im} T = TX$. Then

$$\begin{array}{ccccccccc} T & = & T\,T^-\,T & = & & (T\,T^-)|_{\operatorname{im} T} & & \operatorname{Rst} T & \\ & & & & Y & \longleftarrow & \operatorname{im} T & \longleftarrow & X \end{array} \tag{40}$$

represents obviously a full range factorisation through $Z = \operatorname{im} T$. Another one would be

$$\begin{array}{ccccccccc} T & = & T\,T^-\,T & = & & T|_{X_1} & & \operatorname{Rst}(T^-\,T) & \\ & & & & Y & \longleftarrow & X_1 & \longleftarrow & X \end{array} \tag{41}$$

where the intermediate space $X_1 = \operatorname{im} T^-$ is a complement of the kernel of T. □

Ex 4: Two completion problems

Given a general WHO $W = P_2 A|_{P_1 X}$ one can ask if there is another underlying operator $\tilde{A} \in \mathcal{L}(\tilde{X}, \tilde{Y})$ and two projectors $\tilde{P}_1 \in \mathcal{L}(\tilde{X})$, $\tilde{P}_2 \in \mathcal{L}(\tilde{Y})$ in suitable Banach spaces such that

$$W \quad = \quad \tilde{P}_2 \tilde{A}|_{\tilde{P}_1 \tilde{X}} \quad = \quad \tilde{P}_1 \tilde{X} \to \tilde{P}_2 \tilde{Y}$$

and such that the new setting is profitable somehow. This question can obviously be seen as a completion problem for an operator matrix

$$\tilde{A} \quad \sim \quad \begin{pmatrix} W & \cdot \\ \cdot & \cdot \end{pmatrix} \quad : \quad \tilde{P}_1 \tilde{X} \times \tilde{Q}_1 \tilde{X} \longrightarrow \tilde{P}_2 \tilde{Y} \times \tilde{Q}_2 \tilde{Y}. \tag{42}$$

Problem 1. Given a WHO where A is not invertible, look for a setting $\tilde{X}, \tilde{Y}, \tilde{A}, \tilde{P}_1, \tilde{P}_2$ such that $W = \tilde{W} = \tilde{P}_2 \tilde{A}|_{\tilde{P}_1 \tilde{X}}$ where $\tilde{A}$ is boundedly invertible.

Problem 2. Given a WHO where A is invertible, look for a setting such that $W = \tilde{W} = \tilde{P}_2 \tilde{A}|_{\tilde{P}_1 \tilde{X}}$ where $\tilde{A}$ is a cross factor.

Solution of Problem 1

Any bounded linear operator acting in Banach spaces, $T \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, can be considered as a general WHO $T = P_2 A|_{P_1 X}$ where the underlying operator A is boundedly invertible.

Proof We consider the topological product spaces $X = \mathcal{X} \times \mathcal{Y}$, $Y = \mathcal{Y} \times \mathcal{X}$ as Banach spaces and

$$A \quad = \quad \begin{pmatrix} T & \mu I_{\mathcal{Y}} \\ \mu I_{\mathcal{X}} & 0 \end{pmatrix} \quad : \quad X \to Y \tag{43}$$

where $\mu \in \mathbb{C}, |\mu| > \|T\|$, and (interpreting the zeroes appropriately)

$$P_1 \, : \, X \to \mathcal{X} \times \{0\} \quad , \quad \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{pmatrix} x \\ 0 \end{pmatrix}$$

$$P_2 \, : \, Y \to \mathcal{Y} \times \{0\} \quad , \quad \begin{pmatrix} y \\ x \end{pmatrix} \mapsto \begin{pmatrix} y \\ 0 \end{pmatrix}.$$

Then we have an invertible A and T identified with

$$P_2 A|_{P_1 X} \quad : \quad \begin{pmatrix} x \\ 0 \end{pmatrix} \mapsto \begin{pmatrix} Tx \\ \mu x \end{pmatrix} \mapsto \begin{pmatrix} Tx \\ 0 \end{pmatrix} \quad \square$$

Solution of Problem 2

Let $W = P_2 A|_{P_1 X}$ be given as in Definition 1.1, A being invertible and W generalised invertible. Then W can be considered as a truncation of a cross factor, acting between the same spaces as A does:

$$W \quad = \quad P_2 \ C|_{P_1 X}. \tag{44}$$

Proof. In the symmetric case, this is an interpretation of formula (36). Namely, if v is a reflexive generalised inverse of w, one can verify in (36) that $pap = pcp$, since $pa_- p = p$ and $pa_+ p = p$, as well.

In the asymmetric case we modify Formula (36) in the sense of (33):

$$\tilde{A} \ = \ \tilde{A}_- \ C \ \tilde{A}_+ \quad : \quad P_1 X \times Q_1 X \ \rightarrow \ P_2 Y \times Q_2 Y \tag{45}$$

$$\tilde{A}_- \ = \begin{pmatrix} I|_{P_2 Y} & 0 \\ Q_2 A P_1 V|_{P_2 Y} & I|_{Q_2 Y} \end{pmatrix}$$

$$C \ = \begin{pmatrix} W & P_2(A - AVP_2A)|_{Q_1 X} \\ Q_2(A - AVP_2A)|_{P_1 X} & Q_2(A - AVP_2A + A(P_1 - VWP_1)A^{-1}(P_2 - WVP_2)A)|_{Q_1 X} \end{pmatrix}$$

$$\tilde{A}_+ \ = \begin{pmatrix} I|_{P_1 X} & (VP_2A - (P_1 - VWP_1)A^{-1}(P_2 - WVP_2)A)|_{Q_1 X} \\ 0 & I|_{Q_1 X} \end{pmatrix}$$

Verification is carried out by analogy to the symmetric case. □

Ex 6: Interface problems in $\mathbb{R}^2$

An important variant of Example 2 is the WHO

$$W \quad = \quad r_+ \, A|_{P_1 X} \quad : \quad H_+^{1/2} \times H_+^{-1/2} \to H^{1/2}(\mathbb{R}_+) \times H^{-1/2}(\mathbb{R}_+) \tag{46}$$

where $X = Y = H^{1/2}(\mathbb{R}) \times H^{-1/2}(\mathbb{R})$, $\Sigma = \mathbb{R}_+ =]0, \infty[$, $r_+ = r_{\mathbb{R}_+}$ and $A = \mathcal{F}^{-1}\sigma_\lambda \cdot \mathcal{F}$ with

$$\sigma_\lambda \quad = \quad \begin{pmatrix} 1 & t^{-1} \\ t & \lambda \end{pmatrix} \tag{47}$$

and $t(\xi) = (\xi^2 - k^2)^{1/2}$, $\xi \in \mathbb{R}$, $\lambda \in \mathbb{C} \setminus \{0, 1\}$.

A canonical generalised factorisation of $\sigma_\lambda = \sigma_{\lambda-} \sigma_{\lambda+}$ was derived in [S89] with the help of Khrapkov's formulas and Daniele's trick provided $\lambda \in \mathbb{C} \setminus [1, +\infty)$ (stimulated by the mixed Dirichlet-Neumann problem where $\lambda = -1$ [Raw81]):

$$
\begin{aligned}
\sigma_{\lambda+} &= (1-\lambda^{-1})^{-1/4}\begin{pmatrix} c_+ & -s_+\sqrt{\lambda}/t \\ -c_+\xi/\sqrt{\lambda} - s_+ t/\sqrt{\lambda} & s_+\xi/t + c_+ \end{pmatrix} \\
\sigma_{\lambda-} &= (1-\lambda^{-1})^{-1/4}\begin{pmatrix} c_- - s_-\xi/t & -s_-\sqrt{\lambda}/t \\ -s_- t/\sqrt{\lambda} + c_-\xi/\sqrt{\lambda} & c_- \end{pmatrix}
\end{aligned} \tag{48}
$$

where

$$
\begin{aligned}
c_\pm(\xi) &= \cosh[C \log \gamma_\pm(\xi)] \\
s_\pm(\xi) &= \sinh[C \log \gamma_\pm(\xi)] \\
\gamma_\pm(\xi) &= \frac{\sqrt{k \pm \xi} + i\sqrt{k \mp \xi}}{\sqrt{2k}} \quad , \quad \xi \in \mathbb{R} \\
C &= \frac{i}{\pi} \log \frac{\sqrt{\lambda}+1}{\sqrt{\lambda}-1} \quad .
\end{aligned}
$$

Because of the asymptotic behavior of $\sigma_{\lambda\pm}$ at infinity, the corresponding factorisation of $A = \mathcal{F}^{-1}\sigma_\lambda \cdot \mathcal{F}$ represents a canonical Wiener-Hopf factorisation through a vector Sobolev space:

$$
\begin{array}{ccccc}
A_\lambda & = & A_{\lambda-} \quad A_{\lambda+} & = & \mathcal{F}^{-1}\,\sigma_{\lambda-}\cdot\mathcal{F} \quad \mathcal{F}^{-1}\sigma_{\lambda+}\cdot\mathcal{F} \\
\multicolumn{5}{l}{\quad H^{1/2}\times H^{-1/2} \leftarrow \; Z \; \leftarrow H^{1/2}\times H^{-1/2}} \\
Z & = & H^{\vartheta}(\mathbb{R}) & , & \vartheta = (\vartheta_1, \vartheta_2) = (\frac{1}{2}(1-\delta), \frac{1}{2}(\delta-1))
\end{array}
\tag{49}
$$

where $\delta = \Re e\, C = \frac{-1}{\pi} \arg \frac{\sqrt{\lambda}+1}{\sqrt{\lambda}-1} \in]0,1]$.

Ex 7: Interface problems in $\mathbb{R}^n$, $n \geq 3$

Consider the higher-dimensional case ($m = n - 1 \geq 2$) where Σ is a half-space which is of particular interest in various applications:

$$X = Y = H^{1/2}(\mathbb{R}^m) \times H^{-1/2}(\mathbb{R}^m) \ , \ \Sigma = \mathbb{R}^m_+ = \mathbb{R}^{m-1} \times]0, \infty[$$

and $t(\xi) = (\xi_1^2 + ... + \xi_m^2 - k^2)^{1/2}$, $\xi = (\xi', \xi_m) \in \mathbb{R}^m$, we can consider the same factorisation given by (48) replacing k by $(k^2 - \xi'^2)^{1/2}$, i.e., the previous factorisation as to be parameter-dependent of $\xi' \in \mathbb{R}^{m-1}$. It turns out that the factorisation can be seen as a canonical FIS of A where the intermediate space is an unisotropic vector Sobolev space

$$\begin{aligned} Z &= H^{\vartheta}(\mathbb{R}^m) \times H^{-\vartheta}(\mathbb{R}^m) \qquad (50) \\ H^{\vartheta}(\mathbb{R}^m) &= \mathcal{F}(w_\vartheta L^2(\mathbb{R}^m)) \quad , \quad w_\vartheta(\xi) = (1 + |\xi'|^2)^{\vartheta_1/2} (1 + \xi_m^2)^{\vartheta_2/2}) \\ \vartheta &= (\vartheta_1, \vartheta_2) = (\frac{1}{2}(\delta - 1), \frac{1}{2}(1 - \delta)), \end{aligned}$$

see [S17] for more details.

Constructing a FRF of W from a FIS of A

We study the question: How can a FIS be employed to construct a FRF of W in a more direct (constructive) way than via a generalised inverse? In general the construction of a FRF of W is a difficult task and not much treated in the literature, see [S83] where a so-called weak factorisation was used and the complicated interaction between the two factors was pointed out.

Looking at the symmetric situation $W = PA|_{PX}$, $A \in \mathcal{GL}(X)$, $P^2 = P \in \mathcal{L}(X)$, a weak factorisation $A = B_- B_+$ is characterised by

$$B_\pm \in \mathcal{GL}(X) \quad , \quad B_+ P = P B_+ P \quad , \quad P B_- = P B_- P \, , \tag{51}$$

i.e., B_+ maps PX into PX and B_- maps the complement QX into QX. This yields

$$W = P B_- B_+|_{PX} = P B_-|_{PX} \, P B_+|_{PX} = W_- W_+ \tag{52}$$

where W_- is right invertible and W_+ is left invertible. I.e., we do not have a FRF and the consequences in general are poor.

However, in more special situations, the two operators W_- and W_+ commute. It happens typically in the case of classical Toeplitz and Wiener-Hopf operators. Looking again at Example 1 we observe that the (reduced) WHO

$$T \quad = \quad PC|_{PX} \quad = \quad P\,\mathrm{diag}\,(z^{\kappa_1}, ..., z^{\kappa_n})|_{PX} \tag{53}$$

has also this property: Writing

$$T \quad = \quad T_- T_+ \quad = \quad P\,\mathrm{diag}\,(z^{\kappa_1^-}, ..., z^{\kappa_n^-})|_{PX}\, P\,\mathrm{diag}\,(z^{\kappa_1^+}, ..., z^{\kappa_n^+})|_{PX} \tag{54}$$

where $\kappa_j^+ = \max\{\kappa_j, 0\}$, $\kappa_j^- = \min\{\kappa_j, 0\}$, we see that T_- and T_+ commute. So we arrive at the conclusion that any Φ-factorisation of a measurable matrix function can be easily transformed into a FRF of W, which can be also seen as a consequence of the general version:

Corollary. Let W be given as before and $A = A_- C A_+$ be a FIS where $PC|_{PX} = \mathrm{diag}\,(T_1, ..., T_n)$ and all T_j are one-sided invertible. Then a FRF of W is given by

$$W \quad = \quad (P_2 A_- C_+|_{PZ})\,(PC_- A_+|_{P_1 X} \tag{55}$$

with $PC_+|_{PZ}$ right invertible and $PC_+|_{PZ}$ left invertible.

Note that the knowledge of a CFn instead of a FIS does not suffice because the commutativity of the two middle factors is needed.

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